The Free Polymer Quantum Scalar Field and its Classical Limit

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(work done in collaboration with Alok Laddha)

Plan

- Classical Parameterized Field Theory (PFT):Lagrangian, Hamiltonian
- Polymer Quantum Kinematics
- Quantum Dynamics
- Dirac Observables
- Physical States
- Spacetime Discreteness and Emergent Lattice Field Theory
- Fock from Polymer and the Continuum Limit
- Quantum Dynamics revisited

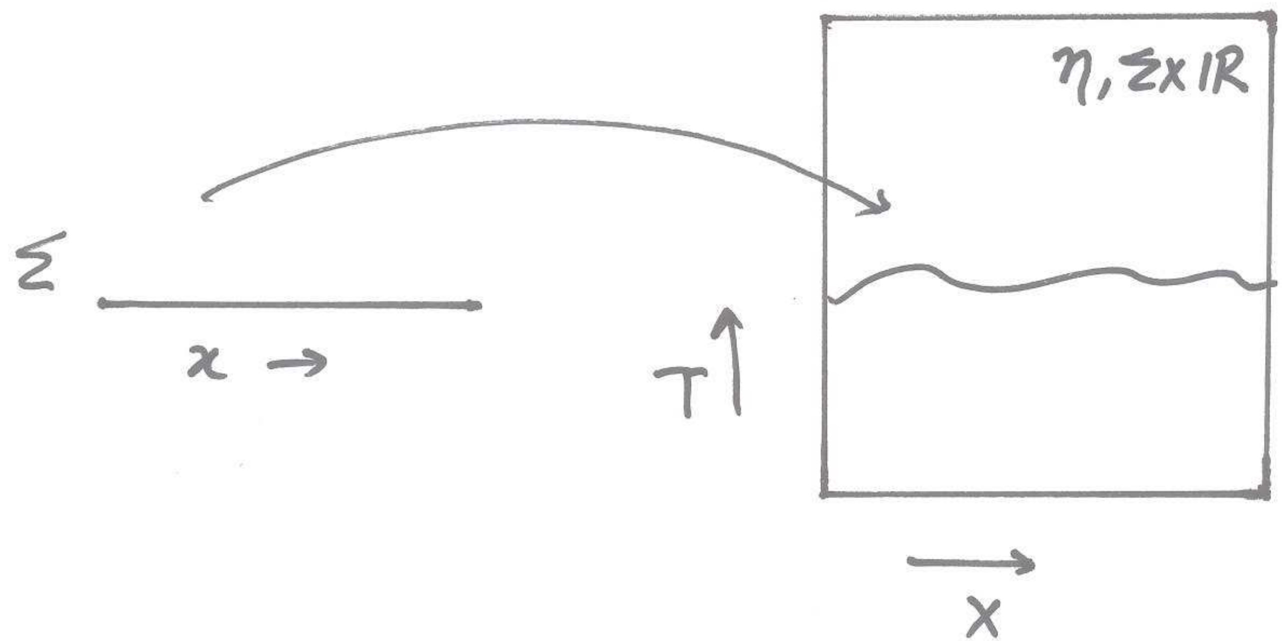
Classical PFT

- Free Scalar Field Action: $\mathbf{S}_0[f] = -\frac{1}{2}\int d^2X \eta^{AB}\partial_A f \partial_B f$
- Parametrize $X^A = (T, X) \rightarrow X^A(x^\alpha) = (T(x,t), X(x,t))$.
  $\Rightarrow \mathbf{S}_0[f] = -\frac{1}{2}\int d^2x \sqrt{\eta}\eta^{\alpha\beta}\partial_\alpha f \partial_\beta f$,
  $\eta_{\alpha\beta} = \eta_{AB}\partial_\alpha X^A \partial_\beta X^B$.
- Vary this action w.r.to f and 2 new scalar fields X^A:
  $S_{PFT}[f, X^A] = -\frac{1}{2}\int d^2x \sqrt{\eta(X)}\eta^{\alpha\beta}(X)\partial_\alpha f \partial_\beta f$
  δf: $\partial_\alpha(\sqrt{\eta}\eta^{\alpha\beta}\partial_\beta f) = 0 \equiv \eta^{AB}\partial_A\partial_B f = 0$
  δX^A: no new equations, $\Rightarrow X^A$ are undetermined functions of x, t, so 2 functions worth of Gauge!
- x, t arbitrary $\equiv$ general covariance
- So Hamiltonian theory has 2 constraints.
- **Remark:** Free sclar field solns are $f = f_+(T + X) + f_-(T - X)$
  Use split into "left movers + right movers" in Hamiltonian theory.

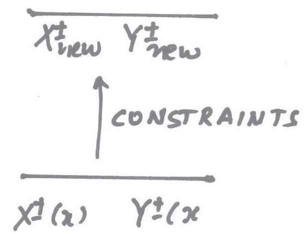
Hamiltonian description

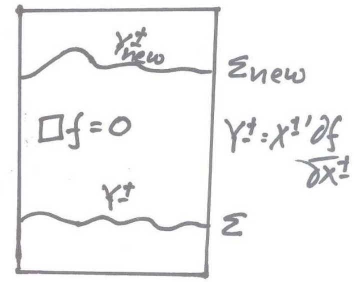
- $T(x), X(x)$ become canonical variables. Use light cone variables $T(x) \pm X(x) := X^{\pm}(x)$.
- Phase space: $(f, \pi_f), (X^+, \Pi_+), (X^-, \Pi_-)$
- Constraints:$H_{\pm}(x) \;=\; [\,\Pi_{\pm}(x) X^{\pm'}(x) \;\pm\; \frac{1}{4}(\pi_f \pm f')^2\,]$
- Define: $Y^{\pm} = \pi_f \pm f'$
  $\{Y^+, Y^-\} = 0$, $\{Y^{\pm}(x), Y^{\pm}(y)\}$= derivative of delta function
- Gauge fix: $X^{\pm} = t \pm x$ "deparameterize".
  get back standard flat spacetime free scalar field action.

Embeddings: $X(x), T(x) \equiv X^+(x), X^-(x)$

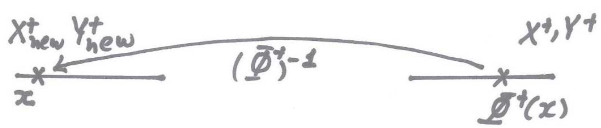


ACTION OF CONSTRAINTS:

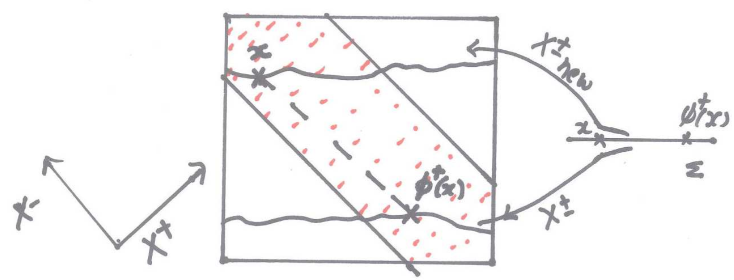




Y^{+}_{new}, X^{+}_{new} : Action of diffeo $\underline{\Phi}^{+}$ on Y^{+}, X^{+}
Y^{-}_{new}, X^{-}_{new} : Action of diff-eo $\underline{\Phi}^{-}$ on Y^{-}, X^{-}.



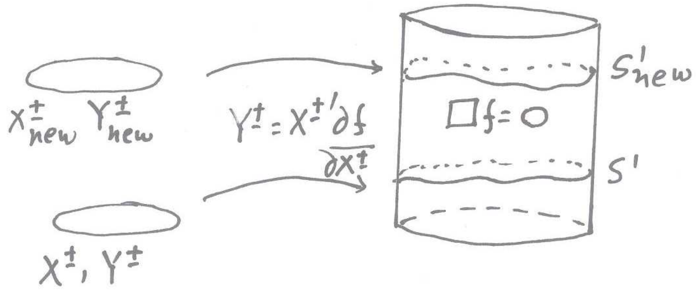
PLAUSIBLE... $f = f_{+}(x^{+}) + f_{-}(x^{-})$



Caution:

- We set Spacetime Topology = $S^1 \times R$
- Not much known re:Polymer repn when space is non-compact. Hence choose space= circle.
- There are complications coming from using "single angular coordinate chart" x on embedded circle. Also from using single spatial angular inertial coordinate X on the flat spacetime. Identifications of x and X "mod 2π" are needed. These can be taken care of. Will mention subtelities as and when dictated by pedagogy.

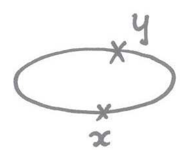
ACTION OF CONSTRAINTS



$$(X^\pm_{new}, Y^\pm_{new}) \equiv \Phi^{\pm *}(X^\pm, Y^\pm)$$

$\Phi^\pm$ = diffeo of S^1 + # of windings 'm'

$$x_{\in[0,2\pi]} \longrightarrow y_{\in[0,2\pi]} + 2\pi m$$



$$Y^\pm_{new}(x) = Y^\pm(y)$$

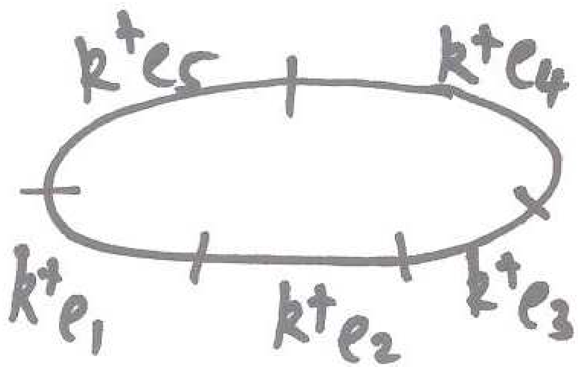
$$X^\pm_{new}(x) = X^\pm(y) \pm 2\pi m$$

"$\Phi^\pm$ ~ periodic diffeos of $\mathbb{R}$"

Quantum Kinematics: Embedding Sector

- Holonomies:
  "Graph": set of edges which cover the circle.
  "Spins": a label k_e for each edge e.
  "Holonomies":$e^{i\sum_e k_e \int_e \Pi_+}$
- "Electric Field": $X^+(x)$
- Poisson Brkts: $\{X^+(x), e^{i\sum_e k_e \int_e \Pi_+}\} = ik_e e^{i\sum_e k_e \int_e \Pi_+}$
  (for x inside e)

- Charge Networks: $|\gamma, \vec{k}\rangle$



$\hat{X}^+(x)|\gamma, \vec{k}\rangle = \hbar k_e^+|\gamma, \vec{k}\rangle$, x inside e.

$e^{i\sum_e \widehat{k'^+_e \int_e \Pi_+}}|\gamma, \vec{k}\rangle = |\gamma, \vec{k}+\vec{k}'\rangle$

Inner Product: $\langle\gamma', \vec{k}'|\gamma, \vec{k}\rangle = \delta_{\gamma',\gamma}\delta_{\vec{k},\vec{k}'}$

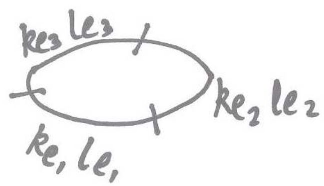
- Range of k_e: $\hbar k_e \in \mathbf{Z}a$, a is a Barbero- Immirzi parameter with dimensions of length.
- Similarly for $-$ sector.

Quantum Kinematics: Matter Sector

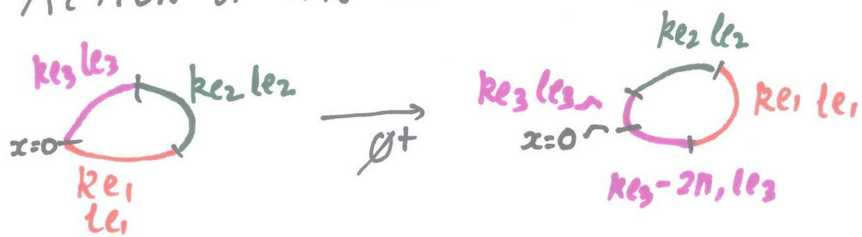
- Holonomy: $e^{i\sum_e l_e \int_e Y^+}$
- Recall that $\{Y^+(x), Y^+(y)\}$ is non- vanishing (=derivative of delta function).
  $\Rightarrow$ "Weyl Algebra":
  $$\widehat{e^{i\sum_e l'_e \int_e Y^+}}\ \widehat{e^{i\sum_e l_e \int_e Y^+}} = \exp[-\tfrac{i\hbar}{2}\alpha(\vec{l'},\vec{l})]\widehat{e^{i\sum_e (l_e+l'_e)\int_e Y^+}}$$
- Charge network: $|\gamma, \vec{l}\rangle$
- $\widehat{e^{i\sum_e l'_e \int_e Y^+}}\ |\gamma,\vec{l}\rangle\ =\ \exp[-\frac{i\hbar}{2}\alpha(\vec{l'},\vec{l})]\ \ |\gamma,\vec{l}+\vec{l'}\rangle$
- Range of l_e: Modulo some technicalities, $l_e \in \mathbf{Z}\epsilon$, ϵ is another Barbero- Immirizi like parameter with dimensions $(ML)^{-\frac{1}{2}}$.

- $\mathcal{H}_{kin} = \mathcal{H}^{+}_{kin} \otimes \mathcal{H}^{-}_{kin}$

  $\mathcal{H}^{\pm}_{kin} : |\gamma^{\pm}, \vec{k}^{\pm}, \vec{l}^{\pm}\rangle$



- ACTION OF GGE TRANSF: $\hat{U}(\phi^{+})|\gamma^{+}, \vec{k}^{+}, \vec{l}^{+}\rangle$



ϕ^{+} = diffeo + winding

l's move by diffeo along with edges

k's move with edges by diffeo but are augmented by "$2m\pi$"

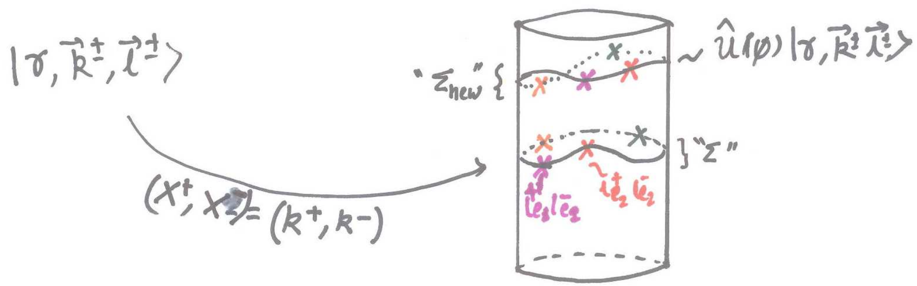
Dirac Observables

- Since gge transf act as diffeos, the integral over $x \in [0, 2\pi]$ of any periodic scalar density constructed solely from the phase space variables, is an observable so that $O^{\pm}_{f} := \int_{S^1} dx Y^{\pm}(x) f(X^{\pm}(x))$ is an observable for real periodic f.
- In polymer repn $Y^{\pm}$ are not good operators only their exponentials are. So cant construct a "triangulation indep" $\widehat{O^{\pm}_{f}}$.
- But $(\widehat{\exp iO^{\pm}_{f}})$ can be constructed!
- $\exp \int dx \widehat{Y^{+}(x)} f(X^{+}(x)) |\gamma, \vec{k}, \vec{l}\rangle := e^{-\frac{i\hbar}{2}\alpha(\vec{f},\vec{l})} |\gamma, \vec{k}, (\vec{l} + \vec{f})\rangle$ where $f_e = f(\hbar k_e)$.

Physical Hilbert Space by Group Averaging

- Gge inv states satisfy $\hat{U}^{\pm}(\phi^{\pm})\Psi = \Psi\ \forall\phi^{\pm}$.
- Formally:$\Psi = \sum|\psi'\rangle$, sum over all distinct $|\psi'\rangle$ gge related tp $|\psi\rangle$. Sum not normalizable in kinematic Hilbert space. Better to think of sum of bras.
- $\sum\langle\gamma_\phi, \vec{k}_\phi, \vec{l}_\phi|$ lives in space of distributions:
- Distributions are linear maps from finite span of chrge nets into complexes. Sum of inner prodt of each bra in sum on a given chrge network ket is finite (most terms are zero).
- Grp averaging technique yields correct physical inner product on space of Grp Averages of charge nets.

POLYMER STATES AND DISCRETE SPTIME:

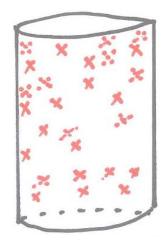


. GRP AVG $\Rightarrow$ $\quad \sum \langle \gamma_\phi, \vec{k}_\phi, \vec{l}_\phi |$

$$\downarrow$$

$$\{(k_1^+ k_1^- \; l_1^+ l_1^-), \; (k_2^+ k_2^-, l_2^+, l_2^-) \ldots \}$$

$$\downarrow$$



DISCRETE SPTIME each point

$(X^+, X^-) = (k^+_i, k^-_i)$ labelled by (l^+_i, l^-_i)

Finest Lattice

- Turns out that Grp Average of a certain family of charge nets yields a superselected sector corresponding to a regular spacetime lattice a.
- Roughly speaking the family is that of all charge nets such that the difference of the embedding charges on successive edges is the minimal possible i.e. on $\gamma^{\pm}$, $\hbar k^{\pm}_{e_{I+1}} - \hbar k^{\pm}_{e_I} = \pm a$.
- The span of Grp Averages of all charge nets in this family is left invariant by the action of all the Dirac observables.
- Since the minimal possible increment, a, in the embedding charges is used to define this sector, there is no state outside this sector corresponds to any finer discrete spacetime structure. Hence the name "Finest Lattice".

Emergence of Lattice Field Theory

- Restrict attention to Superselected "Finest Lattice" Sector
- True degrees of freedom encoded in Dirac Observables.
- Classical sympl structure is represented by commutators of basic kinematic operators. Dirac Observables are composite opertrs. So do not expect (do not get!) repn of classical continuum sympl structure for Dirac Observables i.e. of true d.o.f.
- Recall:$e^{iO_f^+} = e^{i\int_{S^1} dxY^+(x)f(X^+(x))}$. Do not get repn of classical P.B. for all choices of f.
- However: Do get repn of classical sympl structure for those continuum functions $f(X^\pm)$ which are piecewise constant - i.e. constant on edges of dual lattice.
- Thus emergent sympl structure of true degrees of freedom that of lattice field theory.

- Can also show that dynamics of true degrees of freedom is that of lattice field theory.
- Remark: localising support of f to one dual lattice edge can get lattice approximant to local field operator. Discrete (lattice) Fourier transform of this yields approximant to the creation-ann modes.

Fock vacuum 2 point function

- Want polymer state which approximates behaviour of Fock 2 pt function. Precise nature of approx is thru defn of continuum limit.
- Note that the B-I parameters a, ϵ dictate the smallest increment of embedding,matter chrges. Also a is the lattice spacing. Hence continuum limit is $a \rightarrow 0, \epsilon \rightarrow 0$.
- Accordingly we consider 2 parameter family of polymer quantizations with 2 parameter family of states and 2 parameter family of Dirac observables which are lattice approximant to the annihilation- creation modes.
- Require exp value of quadratic combinations of mode operators for wavelengths $>>$ lattice spacing to approach Fock vacuum exp value in the continuum limit.
- We explicitly construct such a set of states.

Summary

We constructed Polymer quantization of PFT with following features:

- **Absence of Ad-hoc Triangulations in oprtr actions**
- **Correct implementation of constraint algebra:** Classical constraints ensure $foliation$ indep of the dynamics. Foliation indep implies a consistent spacetime dynamics. Correctly implemented, quantum constraints play similar role (Kuchař). Quantum sptime covariance is tied to our faithful repn of grp of gge transformations.
- **Continuum Limit:** Crucial that limit is indep of $\hbar$ so seperation of notions of $quantum$ and $continuum$. Limit not of 1 parameter set of ad- hoc triangulations in single quantiztn; rather, limit of 1 (B-I) parameter family of unitarily inequivalent quantizations.

Further research...

- **Thiemann Quantzn and Sptime Covariance:**We used density weight 2 constraints. Classically equiv to 1 sptl diffeo and 1 Hamiltonian constr. As in LQG solve sptl diffeo by averaging, impose Ham constr on diff inv distributions (more on this if time...)
- **Non- comp sptl toplgy:** Related to CGHS model.
- **Breaking of Local Lorentz Invariance**: No dispersion due to $\pm$ seperation. Lattice breaks LLI. Eff theory?
- **Lorentzian to Euclidean PFT?:**
- etc., etc...

Thiemann type quantization

- We use density wt 2 constraints:
$H_{\pm}(x) \;=\; [\,\Pi_{\pm}(x)X^{\pm'}(x) \;\pm\; \frac{1}{4}(\pi_f \pm f')^2\,]$.
- Constraint algebra is Lie algebra
- Define $C_{diff}(x) = H^{+} + H^{-}$,
$C_{diff}(x) \;=\; \Big[\Pi_{+}(x)X^{+'}(x) \;+\; \Pi_{-}(x)X^{-'}(x) \;+\; \pi_f(x)f'(x)\Big]$.
- Define $C_{ham} = \frac{1}{\sqrt{X^{+'}(x)X^{-'}(x)}}(H^{+} - H^{-})$
- Constraint algebra is Dirac algebra, same algebra as for gravity with sptl metric = pull back of flat sptime metric to the Cauchy slice.
- First impose sptl diffeo constr then Ham constr on sptl diff inv distrbtns.

Results

- Can write $\frac{1}{\sqrt{X^{+'}(x)X^{-'}(x)}}$ as opertr thru Thiemann- like tricks.
- Solns obtained here, trivially solve sptl diff constraint
- If straightfwd regulrztn done for Ham constr ala LQG, *no* soln of ours is soln to Ham constr!
- There is non trivial reg for which our solns are solns of Ham constr!