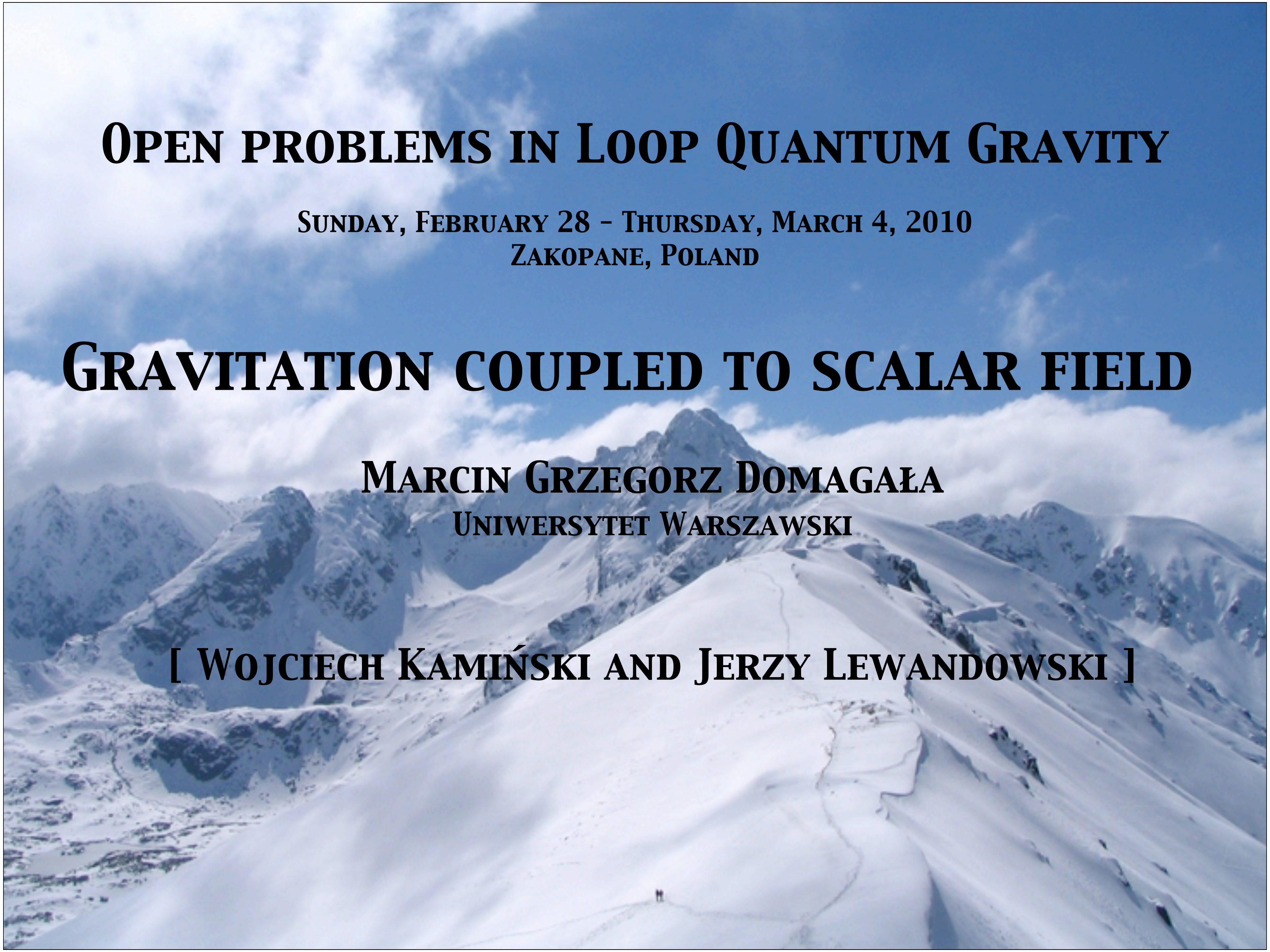

Open problems in Loop Quantum Gravity

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Gravitation coupled to scalar field

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[Wojciech Kamiński and Jerzy Lewandowski]

Gravitation coupled to scalar field

- GRAVITATIONAL PHASE SPACE $\left\{q_{ab}(x), p^{cd}(y)\right\} = \frac{1}{2}\left(\delta^c_a\delta^d_b + \delta^d_a\delta^c_b\right)\delta(x,y)$

Any other choice of gravitational variables is possible. It depends on the scheme of quantization.

- SCALAR FIELD PHASE SPACE $\{\phi(x), \pi(y)\} = \delta(x,y)$

- CONSTRAINTS

$$C(x) = C^{GR}(x) + C^{\phi}(x) = C^{GR}(x) + \frac{1}{2}\frac{\pi(x)^2}{\sqrt{q(x)}} + \frac{1}{2}\sqrt{q(x)}q^{ab}(x)\phi_{,a}(x)\phi_{,b}(x) + \sqrt{q(x)}V\left[\phi(x)\right] = 0$$

$$C_a(x) = C^{GR}_a(x) + C^{\phi}_a(x) = C^{GR}_a(x) + \pi(x)\phi_{,a}(x) = 0$$

Solving the constraints with respect to $\pi(x)$

One can solve for $\phi_{,a}(x)$ using diffeomorphism constraint and insert it into scalar constraint.

We get 4th order equation

$$\frac{1}{2\sqrt{q(x)}}\pi(x)^4 + \left(C^{GR}(x) + \sqrt{q(x)}V[\phi(x)]\right)\pi(x)^2 + \frac{1}{2}\sqrt{q(x)}q^{ab}(x)C_a^{GR}(x)C_b^{GR}(x) = 0$$

The solution has sign ambiguities

$$\pi(x) = \pm q(x)^{\frac{1}{4}}\left[-\left(C^{GR}(x) + \sqrt{q(x)}V[\phi(x)]\right) \pm \sqrt{\left(C^{GR}(x) + \sqrt{q(x)}V[\phi(x)]\right)^2 - q^{ab}(x)C_a^{GR}(x)C_b^{GR}(x)}\right]^{\frac{1}{2}}$$

Comparing to FRW solution we can make the choice in the second ambiguity (plus).

The first one depends on the choice of the direction of the evolution.

We can also understand the meaning of this choice by inspection of the transformation given by the new constraints above.

NEW Constraints

Thanks to using above results we get new constraints (introducing new notation)

$$C^{NEW}(x) = \pi(x) + h\left[q_{ab}, p^{ab}, \phi\right](x) = 0$$

$$C_a(x) = C_a^{GR} + C_a^{\phi}(x) = 0$$

This new constraints have nice property

$$\left\{h\left[q_{ab}, p^{ab}, \phi\right](x), h\left[q_{ab}, p^{ab}, \phi\right](y)\right\} = 0$$

$$\left\{C^{NEW}(x), C^{NEW}(y)\right\} = 0$$

This result is a generalization of the known result by Kuchar and Romano

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QUANTIZATION

FROM NOW ON LET US ASSUME THAT $V[\phi(x)] = 0$ $\quad h\left[q_{ab}, p^{ab}, \phi\right](x) = h\left[q_{ab}, p^{ab}\right](x)$

- QUANTUM STATE $\quad \Psi[q_{ab}, \phi] \in \mathcal{H}^{GR} \otimes \mathcal{H}^{\phi}$

- BASIC OPERATORS

$$\widehat{\phi}(x)\Psi[q_{ab}, \phi] = \phi(x)\Psi[q_{ab}, \phi]$$

$$\widehat{\pi}(x)\Psi[q_{ab}, \phi] = -i\hbar\frac{\delta}{\delta\phi(x)}\Psi[q_{ab}, \phi]$$

$$\widehat{q_{ab}}(x)\Psi[q_{ab}, \phi] = q_{ab}(x)\Psi[q_{ab}, \phi]$$

$$\widehat{p^{ab}}(x)\Psi[q_{ab}, \phi] = -i\hbar\frac{\delta}{\delta q_{ab}(x)}\Psi[q_{ab}, \phi]$$

OR ANY OTHER SCHEME OF QUANTIZATION IS POSSIBLE

- QUANTUM CONSTRAINTS

$$\widehat{C}^{NEW}(x)\Psi[q_{ab}, \phi] = 0 \qquad \widehat{C_a}(x)\Psi[q_{ab}, \phi] = 0$$

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SOLVING $\widehat{C}^{NEW}(x)\Psi\left[q_{ab},\phi\right]=0$

THIS EQUATION CAN BE WRITTEN AS

$$\widehat{\pi}(x)\Psi\left[q_{ab},\phi\right]+\widehat{h}\left[q_{ab},p^{ab}\right](x)\Psi\left[q_{ab},\phi\right]=0$$

REMEMBERING THE DEFINITION OF $\widehat{\pi}(x)$, AND QUANTIZING $\widehat{h}\left[q_{ab},p^{ab}\right](x)$ IN “PURE GRAVITATIONAL MANNER”.

WHATEVER QUANTIZATION SCHEME WE USE TO QUANTIZE $\widehat{h}\left[q_{ab},p^{ab}\right](x)$ ITS DEFINITION MUST BE SUCH, THAT COMMUTATOR $\left[\widehat{h}\left[q_{ab},p^{ab}\right](x),\widehat{h}\left[q_{ab},p^{ab}\right](y)\right]=0$ CORRESPONDINGLY TO THE POISSON BRACKET.

WE GET SCHROEDINGER EQUATION

$$\frac{\delta}{\delta\phi(x)}\Psi\left[q_{ab},\phi\right]=-\frac{i}{\hbar}\widehat{h}\left[q_{ab},p^{ab}\right](x)\Psi\left[q_{ab},\phi\right]$$

Construction of the solution

Assuming that we defined $\widehat{h}\left[q_{ab}, p^{ab}\right](x)$ acting in $\mathcal{H}^{GR}$ we can construct

$$\Psi\left[q_{ab}, \phi\right] = e^{-\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}\left[q_{ab},p^{ab}\right](x)}\psi\left[q_{ab}\right]$$

Where $\psi\left[q_{ab}\right]$ belongs to some subspace $\mathcal{H}^{GR}_{+}$ of $\mathcal{H}^{GR}$ on which $\widehat{h}\left[q_{ab}, p^{ab}\right](x)$ is well defined ($h\left[q_{ab}, p^{ab}\right](x) = \sqrt{\ldots + \sqrt{\ldots}}$)

Therefore we have constructed the map that assigns to $\phi(x)$ unique element $\Psi \in \mathcal{H}^{GR}_{+}$

$$\phi \longrightarrow \Psi\left[\ .\ , \phi\right] \in \mathcal{H}^{GR}_{+}$$

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OBSERVABLES

- FOR EVERY $\widehat{A} \in \mathfrak{L}\left[\mathcal{H}_{+}^{GR}\right]$ WE DEFINE THE DIRAC OBSERVABLE

$$\widehat{\mathcal{O}}_{\widehat{A}} = e^{-\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)} \quad \widehat{A} \quad e^{\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)}$$

REMEMBERING THAT $\Psi\left[q_{ab},\phi\right] = e^{-\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}\left[q_{ab},p^{ab}\right](x)}\psi\left[q_{ab}\right]$ WE GET

$$\widehat{\mathcal{O}}_{\widehat{A}}\Psi\left[q_{ab},\phi\right] = e^{\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)} \quad \widehat{A}\psi[q_{ab}]$$

$$= e^{\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)}\psi'[q_{ab}]$$

WE CAN SEE THE RIGHT HAND SIDE HAS A FORM OF SOLUTION OF $C^{NEW}(x)$

- IN PARTICULAR $f(x) \in \mathfrak{Fun}[M]$ WE HAVE $\widehat{\pi}\left[f\right] = \int d^3x f(x)\widehat{\pi}(x)$ WHICH IS

$$\widehat{\pi}\left[f\right] = \int d^3x f(x)\widehat{h}\left[q_{ab},p^{ab}\right](x)$$

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Summary

The theory defined by the scalar constraint amounts to

- Algebra of the Dirac observables isomorphic to the operators in $\mathcal{H}_{+}^{GR}$:

$$\widehat{\mathcal{O}}_{\widehat{A}} \longmapsto \widehat{A}$$

- Group of automorphisms

$$\widehat{A} \ \mapsto \ e^{\frac{i}{\hbar}\int d^3x\phi_0(x)\widehat{h}(x)} \ \widehat{A} \ e^{-\frac{i}{\hbar}\int d^3x\phi_0(x)\widehat{h}(x)}$$

- The generators of automorphisms

$$\widehat{\pi}\left[\phi_0\right] = \int d^3x \ \phi_0(x) \ \widehat{h}\left[q_{ab}, p^{ab}\right](x)$$

Imposing diffeomorphism constraints

- Now $\Psi\left[q_{ab},\phi\right]=e^{-\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}\left[q_{ab},p^{ab}\right](x)}\psi\left[q_{ab}\right]$ is additionally diffeomorphism invariant

$\Longrightarrow \psi \in \mathcal{H}^{GR}_{diff\ inv}$ defined by diffeomorphism rigging map

- Dirac observables

Now we have isomorphism $\widehat{A} \mapsto \widehat{\mathcal{O}}_{\widehat{A}}$ with the algebra $\mathfrak{L}\left[\mathcal{H}^{GR}_{diff\ inv}\right]$:

$$\widehat{\mathcal{O}}_{\widehat{A}}=e^{-\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)}\ \widehat{A}\ e^{\frac{i}{\hbar}\int d^3x\phi(x)\widehat{h}(x)} \quad \text{but now} \quad \widehat{A}\in\mathfrak{L}\left[\mathcal{H}^{GR}_{diff\ inv}\right]$$

The generator of the automorphisms:

$$\int d^3x\ \widehat{h}\left[q_{ab},p^{ab}\right](x)$$

And the group of automorphisms is restricted to

$$\widehat{A}\mapsto e^{\frac{i}{\hbar}\int d^3x\widehat{h}(x)\tau}\ \widehat{A}\ e^{-\frac{i}{\hbar}\int d^3x\widehat{h}(x)\tau} \quad \text{where } \tau \text{ is a number}$$

Summary

- The algebra of the Dirac observables is isomorphic to $\mathfrak{L}\left[\mathcal{H}^{GR}_{diff\ inv}\right]$
- Group of automorphisms

$$\widehat{A} \mapsto e^{\frac{i}{\hbar}\int d^3x\widehat{h}(x)\tau}\ \widehat{A}\ e^{-\frac{i}{\hbar}\int d^3x\widehat{h}(x)\tau}$$

where τ is a number

- The generators of automorphisms

$$\widehat{\pi} = \int d^3x\ \widehat{h}\left[q_{ab}, p^{ab}\right](x)$$

This is the exact LQC with all the local degrees of freedom

Closing remarks

- What is well defined

  - $\mathcal{H}^{GR}_{diff\ inv}$

  - $\widehat{h}\left[q_{ab}, p^{ab}\right](x) \ \in \ \mathfrak{L}\left[\mathcal{H}^{GR}_{diff\ inv}\right]$

  - Upon suitable choice of ordering

$$\widehat{h} = -q(x)^{\frac{1}{4}}\left[-C^{GR}(x) + \sqrt{\widehat{C^{GR}(x) - q^{ab}(x)C^{GR}_a(x)C^{GR}_b(x)}}\right]^{\frac{1}{2}}$$

becomes

$$\widehat{h} = -\sqrt{2\widehat{\sqrt{q(x)}}\ \ |C^{GR}(x)|}$$

what agrees Rovelli - Smolin (1994)