The EPRL intertwiners and correct partition function.

Wojciech Kamiński

Zakopane, 28.02.10

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- Is the EPRL map isometric, does it preserve the scalar product between the SU(2) intertwiners?
- If not, what is a complete, correct form of the partition function written directly in terms of the SU(2) intertwiners, the preimages of the EPRL map?

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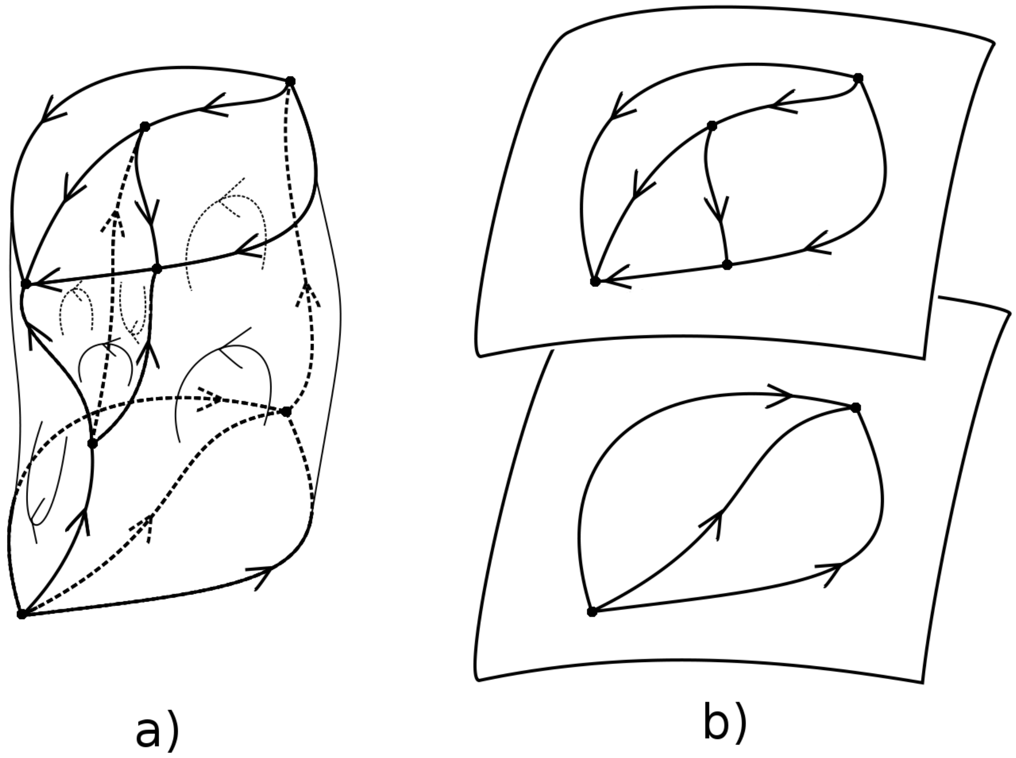

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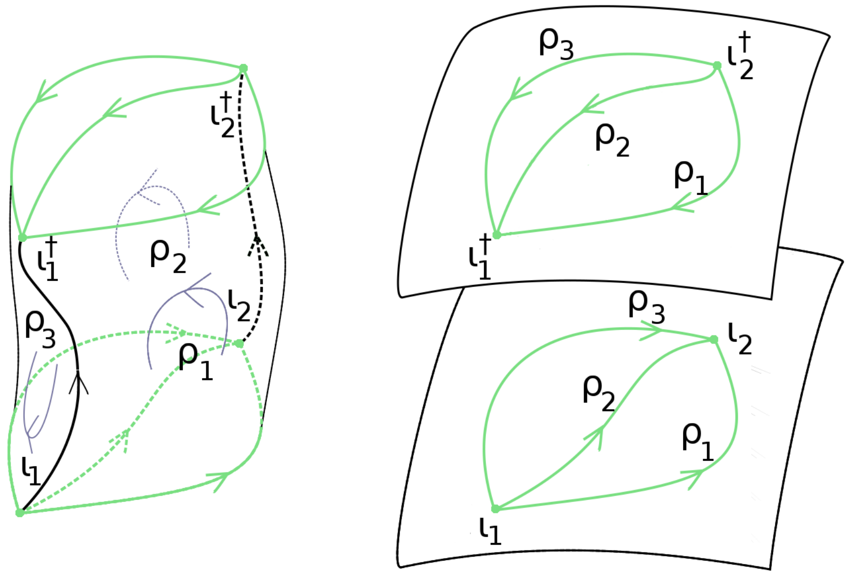

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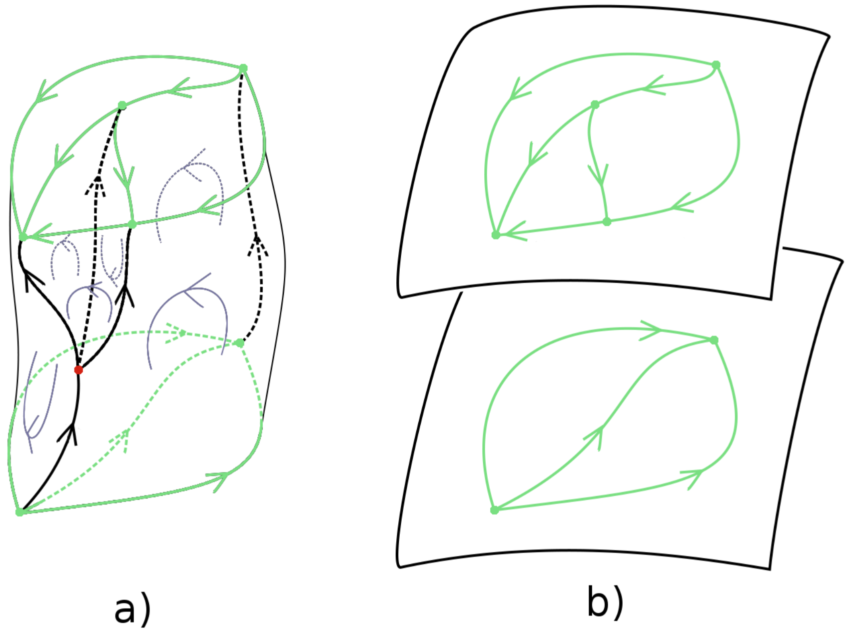

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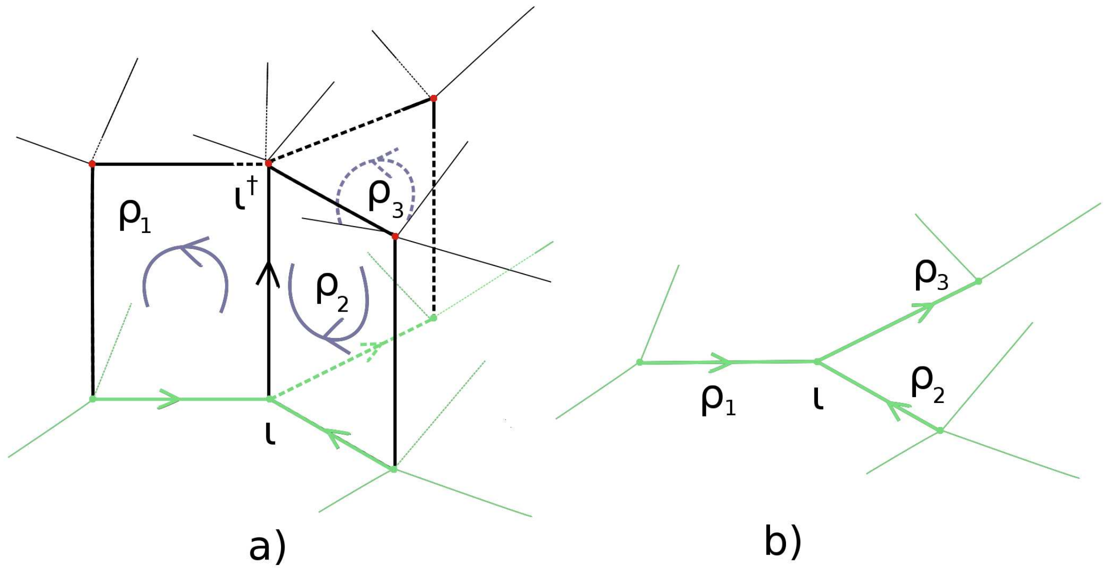
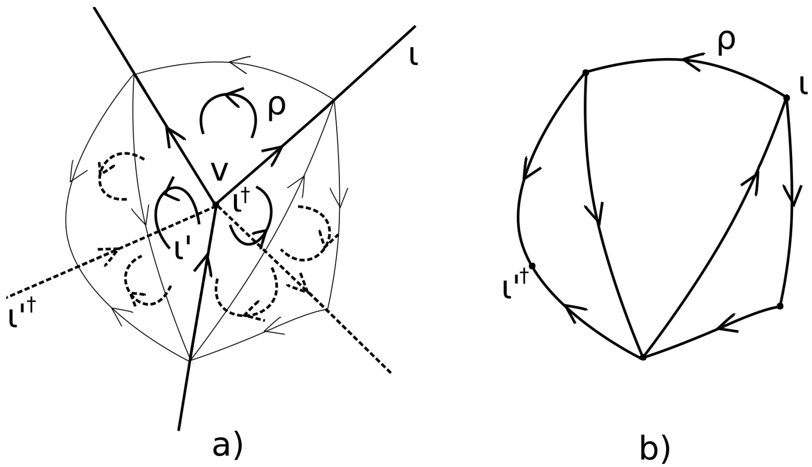


Euclidean 4d QG as spin-foam

- 4d QG is regarded as a BF theory with constraints.
- Spin-networks consists of gauge invariant functions.

Euclidean 4d QG as spin-foam



- In each step we add one vertex with evolution of BF theory, obtaining a transition amplitude between spin networks.

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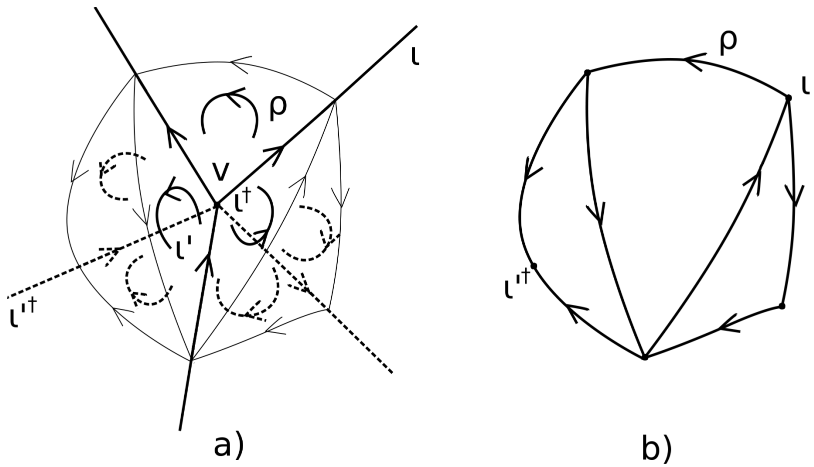

- In each step we add one vertex with evolution of BF theory, obtaining a transition amplitude between spin networks.
- Constraints are imposed as projections on edges (nodes of spin-network).

Details

- The simplicity constraints are imposed on the elements of $\mathcal{H}_\Sigma$, locally at each vertex.
  $\mathrm{Inv}_{\mathrm{Simp}}(\rho_1 \otimes \ldots \rho_k \otimes \rho^*_{k+1} \otimes \ldots \otimes \rho^*_N)$

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- In each subsequent spin-network intertwiners ι should be in this subspace,
- To sum with respect to the spin-network histories with the amplitude as a weight, one fixes an *orthonormal basis* in each space of simple intertwiners.

Simple intertwiners

There are three main proposals for the simple intertwiners:

1. that of Barrett-Crane (BC) corresponding to the Palatini action,
2. that of Engle-Pereira-Rovelli-Livine (EPRL) corresponding the Holst action with the value of the Barbero-Immirzi parameter $\gamma \neq \pm 1$,
3. that of Freidel-Krasnov (FK) also corresponding to the Holst action with the value of the Barbero-Immirzi parameter $\gamma \neq \pm 1$,

EPRL intertwiner

Given intertwiner $\eta \in Inv(\rho_{k_1} \otimes \ldots)$ with the spins k_I

$$j_I^{\pm} = \left| \frac{1 \pm \gamma}{2} \right| k_I$$

$k_I = j^+ + j^-$, if $|\gamma| < 1$ and $k_I = |j^+ - j^-|$, if $|\gamma| > 1$. This follows from adjusted/improved constraints.

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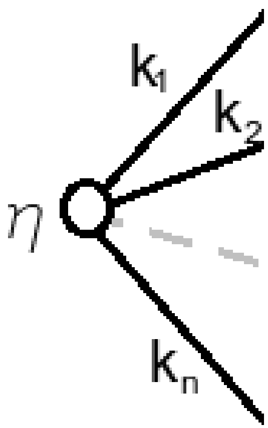

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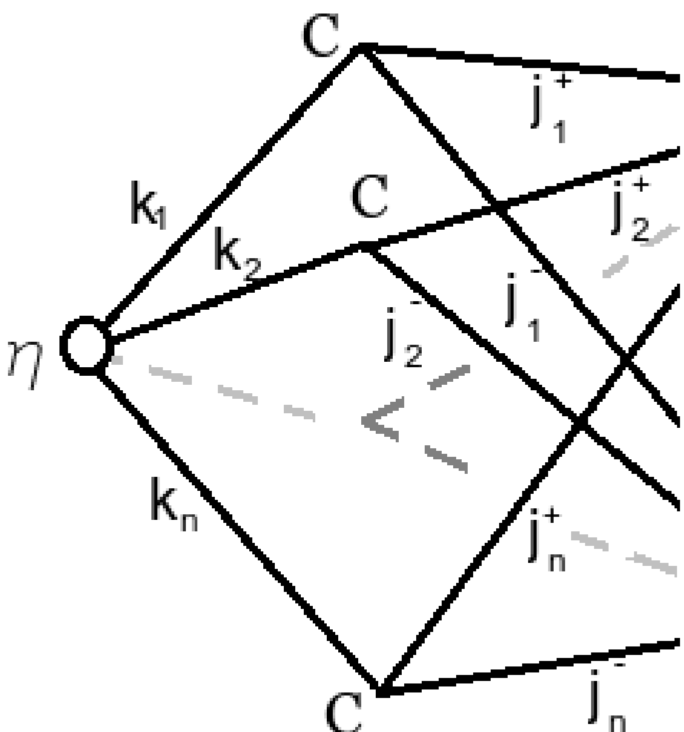

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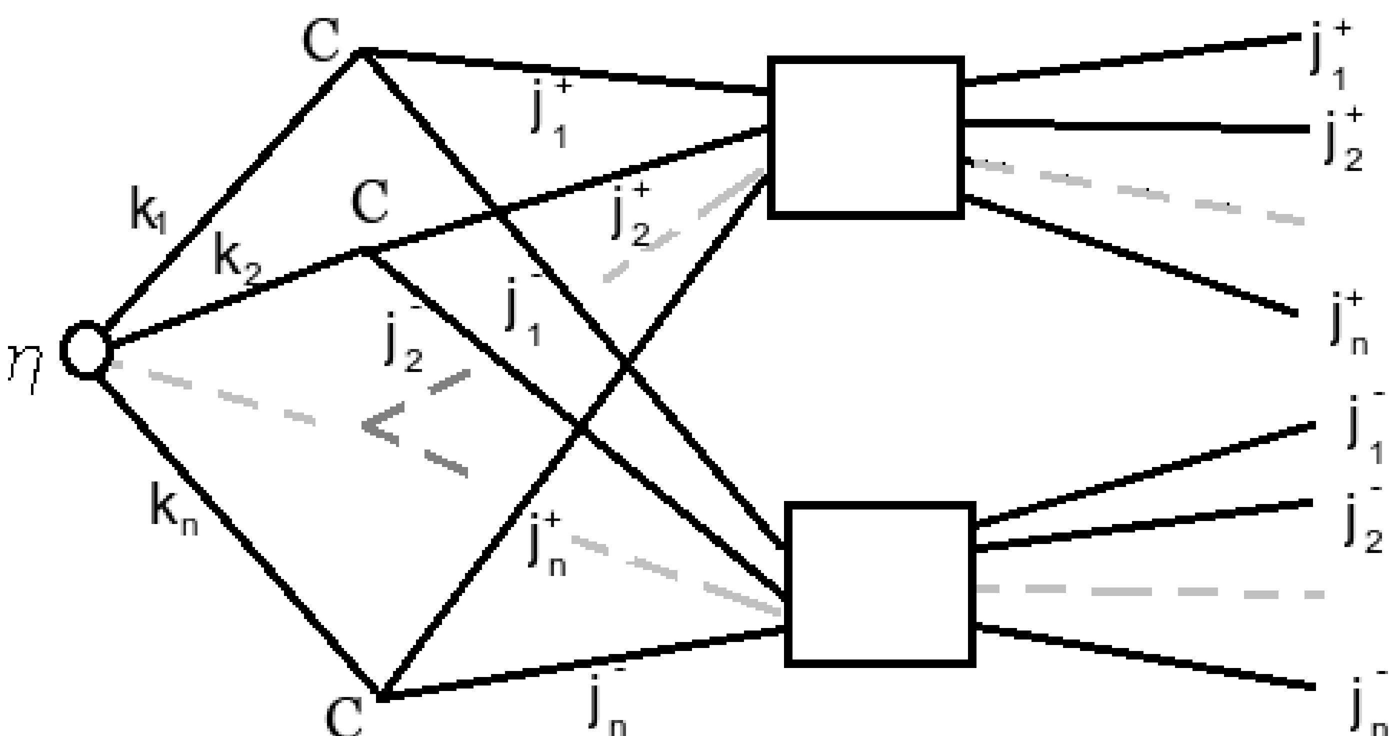

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The map EPRL

$$\iota_{\text{EPRL}}(\eta) \ := \ (P^+ \otimes P^-) c_1 \otimes \ \ldots \ \otimes c_n \lrcorner \eta.$$

- $P^{\pm}$ projections onto $SU(2)_{\pm}$ invariants,
- c_i Clebsch-Gordon coefficients $\rho_{k_i} \longrightarrow \rho_{j_i^+} \otimes \rho_{j_i^-}$.

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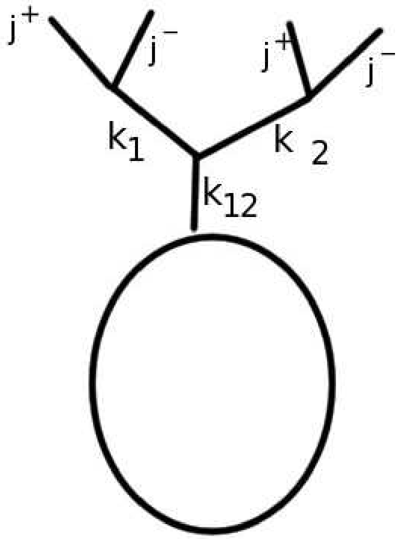
- One can show that the map $\eta \longrightarrow \iota_{EPRL}(\eta)$ is injective, so one can labelled EPRL intertwiners by the $SU(2)$ intertwiners.
- The proof is based on the observation that in suitable basis matrix of the ι_{EPRL} is "upper" triangular and injective on "diagonal" entries.

Proof 1

Suitable basis is given by an intermediate spin k_{12}



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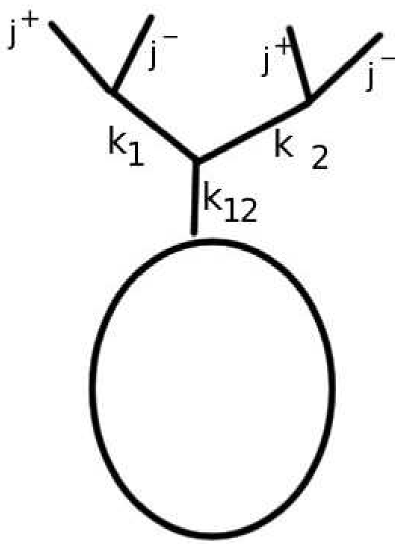

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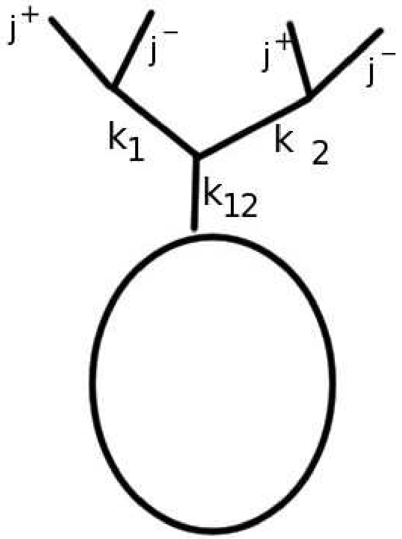

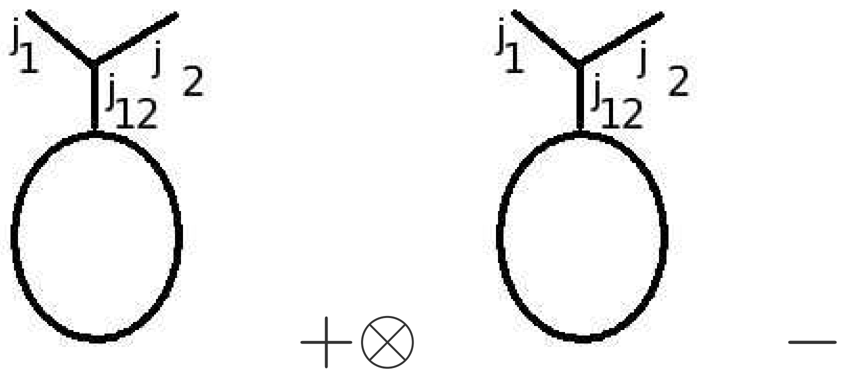

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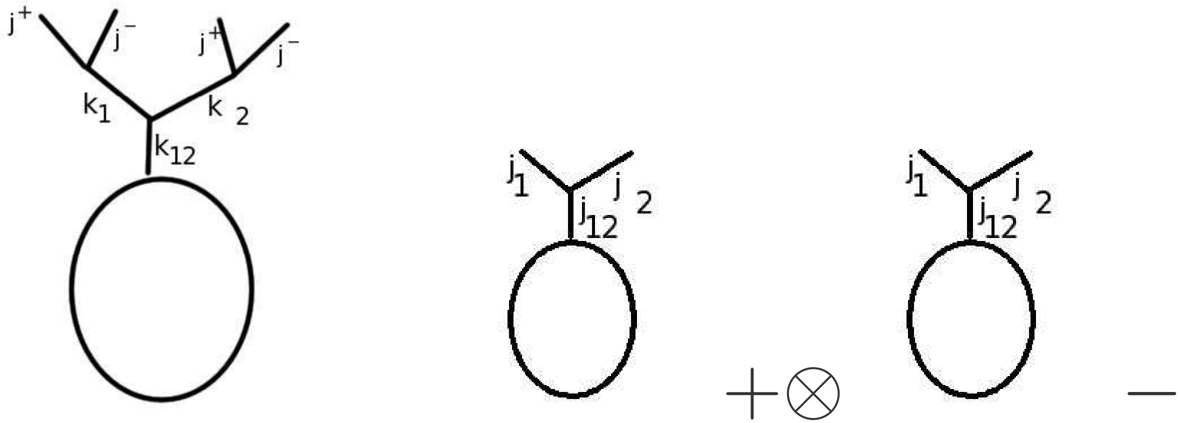
Proof 2

Such a basis is graded by k_{12} and $j^+_{12} + j^-_{12}$ respectively,



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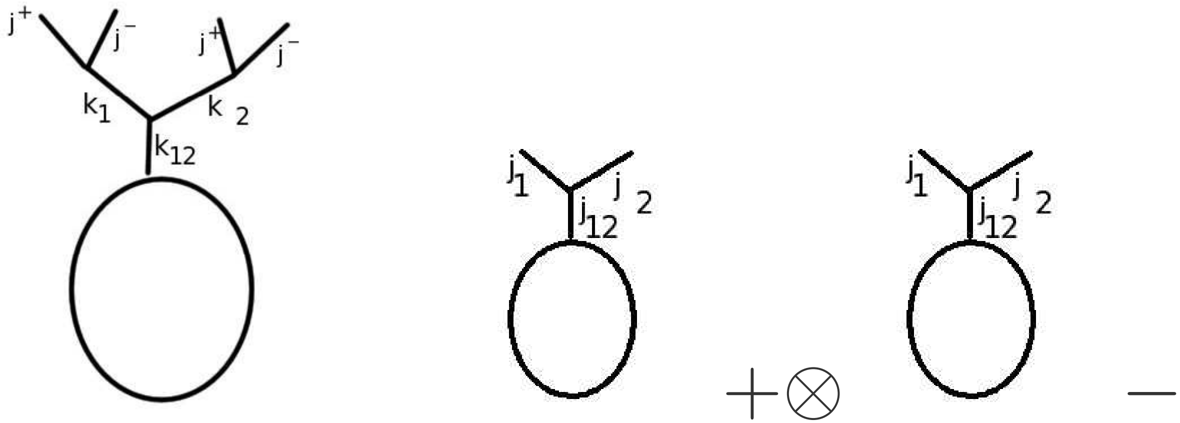

then

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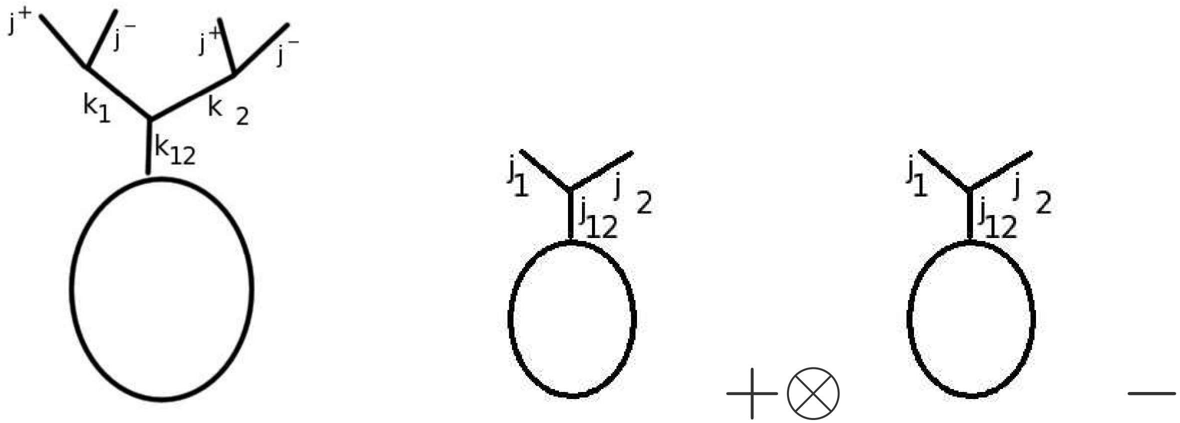

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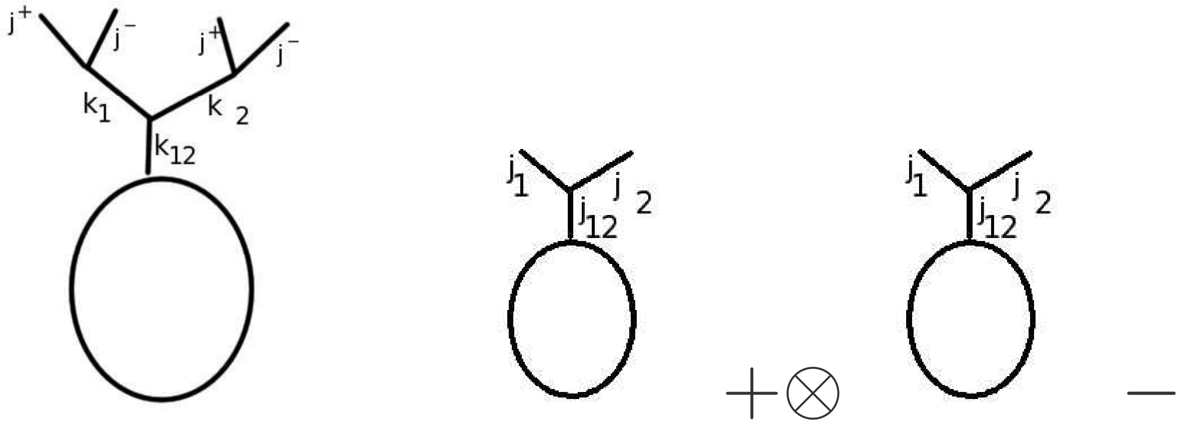
We can restrict our attention to $k_{12} = j^+_{12} + j^-_{12}$

Proof 3

The contraction of



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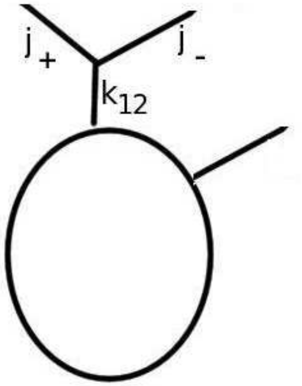
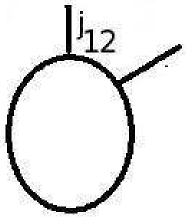
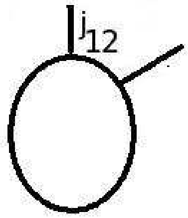




$+ \otimes$



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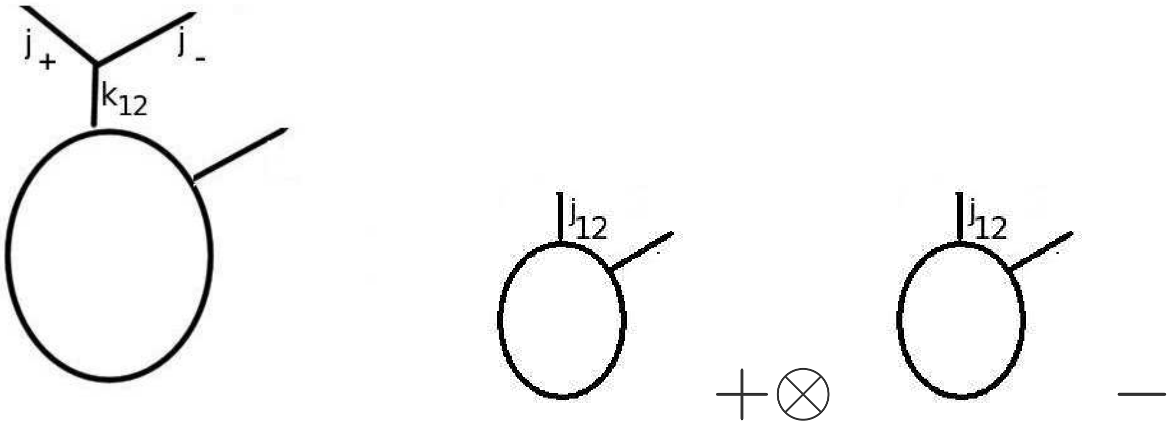

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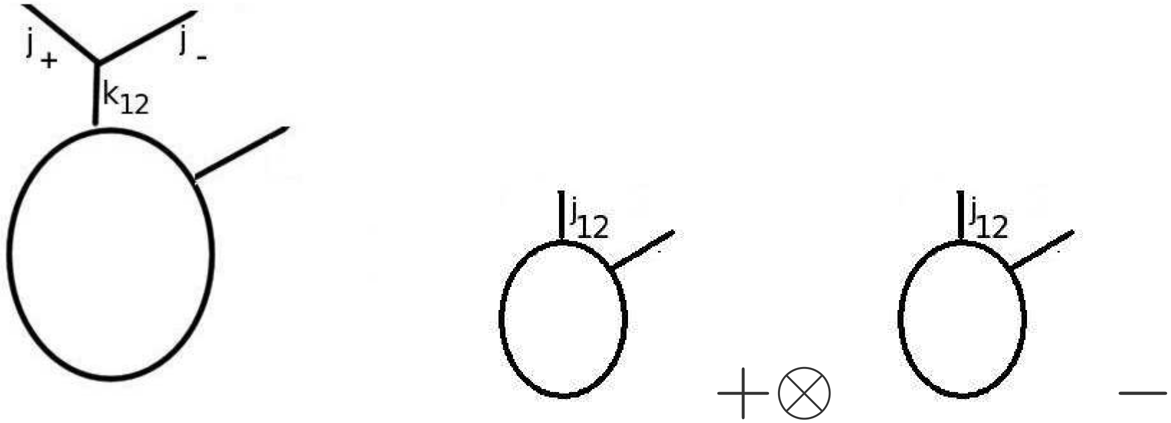

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There are some technical details...

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$A(\eta_1, \eta_2)$ is the inverse of the matrix $\langle \iota(\eta_1), \iota(\eta_2) \rangle$ so

$$P_{EPRL} = \sum_{\eta_{in/out}} A_{in,out} |\iota_{EPRL}(\eta_{out})\rangle\langle\iota_{EPRL}(\eta_{in})|$$

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$\gamma = \frac{1}{2}, j_1 = 2, j_2 = 4, j_3 = 4, j_1 = 2; a, b \in \{2, \ldots, 6\}$:

$$\begin{pmatrix}
\frac{53723}{175616} & -\frac{2265\sqrt{\frac{5}{7}}}{50176} & \frac{5093\sqrt{5}}{1053696} & -\frac{3\sqrt{55}}{25088} & 0 \\
-\frac{2265\sqrt{\frac{5}{7}}}{50176} & \frac{117853}{501760} & -\frac{12805}{301056\sqrt{7}} & \frac{45\sqrt{\frac{11}{7}}}{7168} & -\frac{3\sqrt{\frac{13}{7}}}{8960} \\
\frac{5093\sqrt{5}}{1053696} & -\frac{12805}{301056\sqrt{7}} & \frac{741949}{3512320} & -\frac{781\sqrt{11}}{752640} & \frac{5\sqrt{13}}{5376} \\
-\frac{3\sqrt{55}}{25088} & \frac{45\sqrt{\frac{11}{7}}}{7168} & -\frac{781\sqrt{11}}{752640} & \frac{583}{2560} & 0 \\
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\end{pmatrix}$$

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- Examples of low j indicate that matrix A is aproximately diagonal (conjecture).
- If this is true A does not change (spoil) asymptotic behavior of the spin foam amplitude.

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is injective for all γ,
- Simple examples shows that it is not unitary for $|\gamma| \neq 1$.
- The basis labelled by $\mathrm{SU}(2)$ intertwiners is not orthonormal and we should introduce an additional factor A in the spin-foam amplitude
$$\langle \eta_1 | A^{-1} \eta_2 \rangle = \langle \iota_{EPRL}(\eta_1) | \iota_{EPRL}(\eta_2) \rangle$$