DIRAC ALGEBROIDS IN ACTION

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GEOMETRICAL METHODS IN PHYSICS (IMPAN)

DIRAC STRUCTURE

N - a manifold
TN, T^*N - tangent and cotangent bundles
$T^*N \oplus TN$ - generalized tangent bundle, Pontryagin bundle...

$(\cdot|\cdot)$ - symmetric pairing :

$$\left(\alpha_1 + X_1 \,\middle|\, \alpha_2 + X_2\right) = \frac{1}{2}\left(\langle \alpha_1, X_2 \rangle + \langle \alpha_2, X_1 \rangle\right)$$

$$\alpha_i + X_i \in T^*N \oplus_N TN$$

$[\![\cdot,\cdot]\!]$ - Courant-Dorfman bracket on $Sec(TN \oplus_N T^*N)$

$$[\![\alpha_1 + X_1, \alpha_2 + X_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\alpha_2 - i_{X_2} d\alpha_1$$

An almost Dirac structure on N is a subbundle $D \subset T^*N \oplus_N TN$ maximally isotropic with respect to $(\cdot|\cdot)$. An almost Dirac structure is Dirac when $Sec(D)$ is closed with respect to $[\![\cdot,\cdot]\!]$.

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CANONICAL EXAMPLES:

Λ - a bivector on N

$$T^*N \xrightarrow{\tilde{\Lambda}} TN$$

$$D_\Lambda = \mathrm{graph}(\tilde{\Lambda})$$

D_Λ is an almost Dirac structure
D_Λ is a Dirac structure
if Λ is a Poisson bivector

ω - a two-form on N

$$T^*N \xleftarrow{\tilde{\omega}} TN$$

$$D_\omega = \mathrm{graph}(\tilde{\omega})$$

D_ω is an almost Dirac structure
D_ω is a Dirac structure
if ω is a presymplectic form

Δ - a distribution on N

$$D_\Delta = \Delta^\circ + \Delta$$

D_Δ is an almost Dirac structure.
D_Δ is a Dirac structure
if Δ is integrable

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DIRAC STRUCTURES

Geometric structures used to derive phase equations from Hamiltonian or Lagrangian can be considered within Dirac structure framework

symplectic, Poisson, algebroid...

RELATIONS

In some problems of differential geometry relations are used in place of maps

symplectic reductions, constrained systems implicit differential equations...

LINEARITY

Most structures used in geometric mechanics are linear. Linearity is best expressed in the language of double vector bundles.

DIRAC ALGEBROIDS IN MECHANICS

$E \xrightarrow{\tau} M$ vector bundle, $E^* \xrightarrow{\pi} M$ dual vector bundle

$$\begin{tikzcd}
& \mathsf{T}E^* \arrow[dl, "\tau_{E^*}"'] \arrow[dr, "\mathsf{T}\pi"] & \\
E^* \arrow[dr, "\pi"] & E^* \arrow[u, hook] \arrow[d, "\tau_0"] & \mathsf{T}M \arrow[dl, "\tau_M"'] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{T}^*E^* \arrow[dl, "\pi_{E^*}"'] \arrow[dr, "\zeta"] & \\
E^* \arrow[dr, "\pi"] & \mathsf{T}^*M \arrow[u, hook] \arrow[d, "\pi_M"] & E \arrow[dl, "\tau"'] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\
E^* \arrow[dr, "\pi"] & E^* \oplus_M \mathsf{T}^*M \arrow[u, hook] \arrow[d] & \mathsf{T}M \oplus_M E \arrow[dl] \\
& M &
\end{tikzcd}$$

Dirac algebroid is a linear almost Dirac structure on E^* which means that it is maximally isotropic double vector subbundle of the double vector bundle $\mathsf{T}E^* \oplus_{E^*} \mathsf{T}E^*$.

$$\begin{tikzcd}
& \mathsf{T}E^* \oplus_{E^*} \mathsf{T}^*E^* \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\
E^* \arrow[dr, "\pi"] & E^* \oplus_M \mathsf{T}^*M \arrow[u, hook] \arrow[d] & \mathsf{T}M \oplus_M E \arrow[dl] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& D \arrow[dl, "\tau_1^D"'] \arrow[dr, "\tau_2^D"] & \\
Ph_D \arrow[dr, "\pi^D"] & \mathcal{C}_D \arrow[u, hook] \arrow[d] & Vel_D \arrow[dl] \\
& M_D &
\end{tikzcd}$$

may be seen as a relation from T*E* to T*E

$$\begin{tikzcd}
& D \arrow[dl, "\tau_1^D"'] \arrow[dr, "\tau_2^D"] & \\
Ph_D \arrow[dr, "\pi^D"] & \mathcal{C}_D \arrow[u, hook] \arrow[d] & Vel_D \arrow[dl] \\
& M_D &
\end{tikzcd}$$

these two annihilate one another

subbundle may be over a submanifold

In mechanics Dirac algebroid replaces algebroid structure, symplectic structure Tulczyjew α_M map in generating phase equations from $\mathcal{L}$ or $\mathcal{H}$.

SYMPLECTIC SETTING

$$
\begin{tikzcd}
& \mathcal{D}_H \arrow[d, hook] & & & {\color{red}\mathcal{D}} \arrow[d, hook] & & & \mathcal{D}_L \arrow[d, hook] & \\
& T^*T^*M \arrow[dr] \arrow[ddl] & & & TT^*M \arrow[lll, "\beta_M"'] \arrow[rrr, "\alpha_M"] \arrow[dr] \arrow[ddl] & & & T^*TM \arrow[dr] \arrow[ddl] & \\
& & TM \arrow[rrr, dotted, no head] & & & TM \arrow[rrr, dotted, no head] & & & TM \arrow[ul, bend right, "{\color{red}dL}"'] \arrow[dll, "{\color{red}\lambda}"'] \\
T^*M \arrow[uur, bend left] \arrow[rrr, dotted, no head] & & & T^*M \arrow[rrr, dotted, no head] & & & T^*M & &
\end{tikzcd}
$$

ALGEBROID SETTING

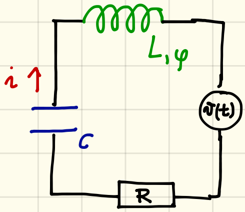
$$
\begin{tikzcd}
& \mathcal{D}_H \arrow[d, hook] & & & {\color{red}\mathcal{D}} \arrow[d, hook] & & & \mathcal{D}_L \arrow[d, hook] & \\
& T^*E^* \arrow[rrr, "\tilde{\pi}"] \arrow[dr] \arrow[ddl] & & & TE^* \arrow[dr] \arrow[ddl] & & & T^*E \arrow[lll, "\varepsilon"'] \arrow[dr] \arrow[ddl] & \\
& & E \arrow[rrr, "\rho"] \arrow[ddl] & & & TM \arrow[ddl] & & & E \arrow[lll, "\rho"'] \arrow[ul, bend right, "{\color{red}dL}"'] \arrow[dll, "{\color{red}\lambda}"'] \arrow[ddl] \\
E^* \arrow[uur, bend left, "{\color{red}dH}"] \arrow[rrr] \arrow[dr] & & & E^* \arrow[dr] & & & E^* \arrow[lll] \arrow[dr] & & \\
& M \arrow[rrr] & & & M & & & M \arrow[lll] &
\end{tikzcd}
$$

DIRAC ALGEBROID SETTING

$$
\begin{tikzcd}
& \mathcal{D}_H \arrow[d, hook] & & & {\color{red}\mathcal{D}} \arrow[d, hook] & & & \mathcal{D}_L \arrow[d, hook] & \\
& T^*E^* \arrow[rrr, "\beta_D"] \arrow[dr] \arrow[ddl] & & & TE^* \arrow[dr] \arrow[ddl] & & & T^*E \arrow[lll, "\varepsilon_D"'] \arrow[dr] \arrow[ddl] & \\
& & E \arrow[rrr, "Vel_D"] & & & TM & & & E \arrow[lll, "Vel_D"'] \arrow[ul, bend right, "{\color{red}dL}"'] \arrow[dll, "{\color{red}\lambda}"'] \\
E^* \arrow[uur, bend left, "{\color{red}dH}"] \arrow[rrr, "Ph_D"] & & & E^* & & & E^* \arrow[lll, "Ph_D"'] & &
\end{tikzcd}
$$

EXAMPLE: LC circuit (or RLC) (A. van der Schaft)

Let us start with the simplest case where no Dirac structure is needed:



In the literature we find that

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = v(t)$$

$M = \mathbb{R} \ni Q$, $\quad TM = \mathbb{R}\times\mathbb{R} \ni (Q, i) \quad T^*M = \mathbb{R}\times\mathbb{R}^* \ni (Q, \varphi)$

charge — current — magnetic flux

1) No voltage source and resistance

$$\begin{array}{ccccc} T^*T^*M & \xleftarrow{\beta_M} & TT^*M & \xrightarrow{\alpha_M} & T^*TM \\ (Q,\varphi,-\dot{\varphi},\dot{Q}) & \longleftarrow\!\shortmid & (Q,\varphi,\dot{Q},\dot{\varphi}) & \longmapsto & (Q,\underset{\overset{\|}{i}}{\dot{Q}},\dot{\varphi},\varphi) \end{array}$$

$$\mathcal{H}(Q,\varphi) = \frac{\varphi^2}{2L} + \frac{Q^2}{2C}$$

$$d\mathcal{H} = \frac{\varphi}{L}d\varphi + \frac{Q}{C}dQ$$

$$\dot{\varphi} = -\frac{Q}{C}$$

$$\dot{Q} = i = \frac{\varphi}{L}$$

$$\varphi = Li$$

$$L\ddot{Q} + \frac{Q}{C} = 0$$

$$\mathcal{L}(Q,i) = \frac{Li^2}{2} - \frac{Q^2}{2C}$$

$$d\mathcal{L} = -\frac{Q}{C}dQ + Li\,di$$

$$\varphi = \frac{\partial\mathcal{L}}{\partial i} = Li$$

2) With voltage source

Voltage source is like an external force in mechanics. It is an element of $T^*M \ni (Q, v)$. In the middle of the triple we have $TT^*M \times_M T^*M$

EXTERNAL FORCES IN MECHANICS

$p, f \in T^*M$

momentum force

$f_p^v \in T_p T^*M \quad f_p^v = \frac{d}{dt}(p + tf)$

$\tilde{\lambda}_M(\dot{p}, f) = \alpha_M(\dot{p} - f^v)$

$(Q, v) \in T^*M$,

$$(Q, \varphi, \dot{Q}, \dot{\varphi}) - (Q, v)^v_{(Q,\varphi)} = (Q, \varphi, \dot{Q}, \dot{\varphi} - v)$$

$\dot{Q} = i = \frac{\varphi}{L}$

$\varphi = Li$

$\dot{\varphi} - v = -\frac{Q}{C}$

$\longrightarrow \quad L\ddot{Q} + \frac{Q}{C} = v$

$$\tilde{\alpha} : TT^*M \times_M T^*M \longrightarrow$$

3) With resistance

Resistance produces dissipation of energy. Instead of Lagrangian and Hamiltonian and their differentials we have non-closed one forms:

$$d\mathcal{L} \rightsquigarrow d\mathcal{L} - Ri\,dQ = Li\,di - \left(\frac{Q}{C} + Ri\right)dQ$$

$$d\mathcal{H} \rightsquigarrow d\mathcal{H} + Ri\,dQ = \frac{\varphi}{L}d\varphi + \left(\frac{Q}{C} + R\frac{\varphi}{L}\right)dQ$$

$\dot{Q} = i = \frac{\varphi}{L}$

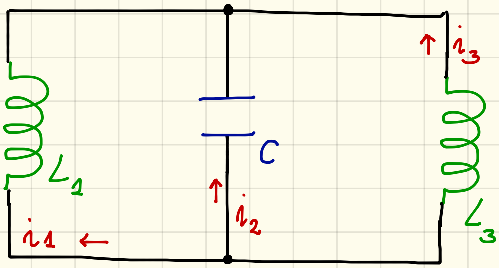
$\dot{\varphi} - v = -\left(\frac{Q}{C} + R\frac{\varphi}{L}\right)$

$\longrightarrow \quad L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = v$

The advantage of this toy model is that we can connect RLC circuits in different ways producing constraints and singular lagrangians. Than we have to replace α_M, β_M with the appropriate Dirac structure.

$$M = \mathbb{R}^3 \ni (Q^1, Q^2, Q^3)$$

$$TM = \mathbb{R}^3 \times \mathbb{R}^3 \ni (Q^A, i^A)$$



$$\mathcal{L}(Q^A, i^A) = \frac{L_1 (i^1)^2}{2} + \frac{L_2 (i^3)^2}{2} - \frac{(Q^2)^2}{2C}$$

$$\Delta : \quad i^1 + i^2 + i^3 = 0$$

constraints of nonholonomic type

Lagrangian does not depend on Q^1, Q^3 (reduction) and i^2 (singularity).

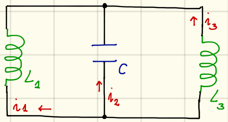
First we reduce the whole problem with respect to Q^1, Q^3. Instead of Lagrangian system on the tangent bundle TM we get the Lagrangian system on algebroid

$$E \xrightarrow{\tau} N$$

$$N \times V = \mathbb{R} \times (\mathbb{R}^3) \ni (Q, i^1, i^2, i^3) \qquad \mathbb{R} \ni Q = Q^2$$

$$E^* = N \times V^* \ni (Q, \varphi_1, \varphi_2, \varphi_3)$$

Initial (without constraints) Dirac structure is associated to the algebroid structure on $E \to N$



$$E = N \times V \,,\quad E^* = N \times V^*$$

$$T^*E = N \times V \times N^* \times V^*$$

$$\begin{cases} TE^* = N \times V^* \times N \times V^* \\ T^*E^* = N \times V^* \times N^* \times V \end{cases}$$

$$\begin{tikzcd}
& TE^* \oplus_{E^*} T^*E^* \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\
E^* \arrow[dr, "\pi"'] & E^* \oplus_M T^*M \arrow[u, hook] \arrow[d] & TM \oplus_M E \arrow[dl] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& N \times V^* \times N \times V^* \times N^* \times V \arrow[dl] \arrow[dr] & \\
N \times V^* \arrow[dr] & N \times V^* \times N^* \arrow[u, hook] \arrow[d] & N \times N \times V \arrow[dl] \\
& N &
\end{tikzcd}$$

$$(Q,\ \varphi_A,\ \dot{Q},\ \dot{\varphi}_A,\ a,\ i^A)$$

$$\mathcal{D} = \{ (Q, \varphi_A, \dot{Q} = i^2, \dot{\varphi}_1 = 0, \dot{\varphi}_2 = -a, \dot{\varphi}_3 = 0, a, i^A) \}$$

$$\begin{tikzcd}
& \mathcal{D} \arrow[dl] \arrow[dr] & \\
Ph_{\mathcal{D}} = N \times V^* & C_{\mathcal{D}} = \{ (Q, 0, -a, 0, a) \} \arrow[u, hook] & Vel_{\mathcal{D}} = \{ (Q, i^2, i^A) \}
\end{tikzcd}$$

$$\mathcal{D}=\left\{\left(Q,\varphi_A,\dot{Q}=i^2,\ \dot{\varphi}_1=0,\ \dot{\varphi}_2=-a,\ \dot{\varphi}_3=0,\ a,\ i^1\right)\right\}$$

$$Ph_D = N\times V^* \qquad C_D=\left\{\left(Q,0,-a,0,a\right)\right\} \qquad Vel_D=\left\{\left(Q,i^2,i^1\right)\right\}$$

Now we translate it into relations that appear in the triple :

$$N\times V^*\times N^*\times V \longrightarrow N\times V^*\times N\times V^* \longleftarrow N\times V\times N^*\times V^*$$

$$(Q,\varphi_A,a,i^1)\longmapsto (Q,\varphi_A,i^2,0,-a,0) \longleftarrow\!\shortmid (Q,i^1,-a,\varphi_A)$$

$$\mathcal{L}(Q,i^1)=\frac{L_1(i^1)^2}{2}+\frac{L_3(i^3)^2}{2}-\frac{Q^2}{2C}$$

$$\varphi_1=\frac{\partial\mathcal{L}}{\partial i^1}=L_1 i^1$$

$$\varphi_2=0$$

$$\varphi_3=\frac{\partial\mathcal{L}}{\partial i^3}=L_3 i^3$$

On the Hamiltonian side we get

$$S_H\subset T^*E^* \qquad S_H=\left\{\left(Q,\varphi_1,0,\varphi_2,\frac{Q}{C},\frac{\varphi_1}{L_1},i^2,\frac{\varphi_3}{L_3}\right)\right\}$$

arbitrary

Submanifold S_H is generated by $H(Q,\varphi_1,\varphi_3)=\dfrac{Q^2}{2C}+\dfrac{\varphi_1^2}{2L_1}+\dfrac{\varphi_2^2}{2L_2}$

defined on $C\subset N\times V^*$

$C=\left\{\left(Q,\varphi_1,0,\varphi_3\right)\right\}$

Time to add constraints. There is a recipe how to modify the given Dirac structure with respect to constraints of nonholonomic type.

$$\begin{tikzcd}
 & D \arrow[dl, "\tau_1^D"'] \arrow[dr, "\tau_2^D"] & \\
Ph_D \arrow[dr, "\pi^D"'] & C_D \arrow[u, hook] \arrow[d] & Vel_D \arrow[dl] \\
 & M_D &
\end{tikzcd}$$

$\Delta \subset Vel_D$

$\tilde{\Delta} = (\tau_2^D)^{-1}(\Delta) \subset D$

$E^* \oplus T^*M \supset \Delta^\circ \supset C_D$

$D^\Delta = \tilde{\Delta} + \Delta^\circ$

$$\begin{tikzcd}
 & \mathcal{D}^\Delta \arrow[dl] \arrow[dr] & \\
Ph_{\mathcal{D}^\Delta} \arrow[dr] & \Delta \arrow[u, hook] \arrow[d] & \Delta \arrow[dl] \\
 & M_{\mathcal{D}^\Delta} &
\end{tikzcd}$$

Dirac algebroid $\mathcal{D}^\Delta$ is called induced from $\mathcal{D}$ by the constraints Δ.

We apply the recipe to our example (Linear Algebra, semester I)

$$\Delta \subset Vel_D = \{(Q, \dot{i}^2, \dot{i}^A)\} \quad : \quad \dot{i}^1 + \dot{i}^2 + \dot{i}^3 = 0$$

It is convenient to change basis in $V = \mathbb{R}^3$. Instead of canonical basis and canonical coordinates we will use $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and coordinates (j^A) i.e.

$$\dot{i}^1 = -j^1$$
$$\dot{i}^2 = j^1 + j^2 + j^3 \qquad \text{constraint condition: } \dot{i}^1 + \dot{i}^2 + \dot{i}^3 = 0 = j^3$$
$$\dot{i}^3 = -j^2$$

$$N \times N \times V \supset \Delta = \{(Q ; j^1 + j^2, j^1, j^2, 0)\}$$

$C_D \subset \Delta^\circ \subseteq N \times V^* \times N^*$ $\quad \Delta^\circ$ in new (dual) coordinates ψ_A in V^* reads

$$\Delta^\circ = \{(Q, -a, -a, \dot{\psi}_3, a)\}$$

arbitrary

$$N \times V^* \times N \times V^* \times N^* \times V$$

$$(Q, \psi_A, \dot{Q}, \dot{\psi}_A, a, j^A)$$

$$D^\Delta = \{(Q, \psi_A, j^1 + j^2, -a, -a, \dot{\psi}_3, a, j^1, j^2, 0)\}$$

What to do now? Translate it into relations and check what equations we get.

$$N \times V^* \times N^* \times V \longrightarrow N \times V^* \times N \times V^* \longleftarrow N \times V \times N^* \times V^*$$

$$(Q, \psi_A, a, j^1, j^2, 0) \longmapsto (Q, \psi_A, j^1+j^2, -a, -a, \dot\psi_3) \longleftarrow (Q, j^1, j^2, 0, -a, \psi_A)$$

$$\mathcal{L}(Q, j^A) = \frac{L_1 (j^1)^2}{2} + \frac{L_2 (j^2)^2}{2} - \frac{Q^2}{2C}$$

$$a = \frac{Q}{C}$$

$$\psi_1 = L_1 j^1$$

$$\psi_2 = L_2 j^2$$

$$\psi_3 = 0$$

$$\longleftarrow$$

$$\begin{cases} \dot Q = j^1 + j^2 = \frac{\psi_1}{L_1} + \frac{\psi_2}{L_2} \\ \dot\psi^1 = \frac{Q}{C} \\ \dot\psi^2 = \frac{Q}{C} \\ \psi_3 = 0 \quad \dot\psi_3 \text{ arbitrary} \end{cases}$$

Back to old variables

$$\varphi_1 = \psi_3 - \psi_1 \qquad \varphi_2 = \psi_3 \qquad \varphi_3 = \psi_3 - \psi_2$$

$$\varphi_2 = 0$$

$$\dot Q = -\frac{\varphi_1}{L_1} - \frac{\varphi_3}{L_3} + \cancel{\frac{\varphi_2}{L_1} + \frac{\varphi_2}{L_3}}$$

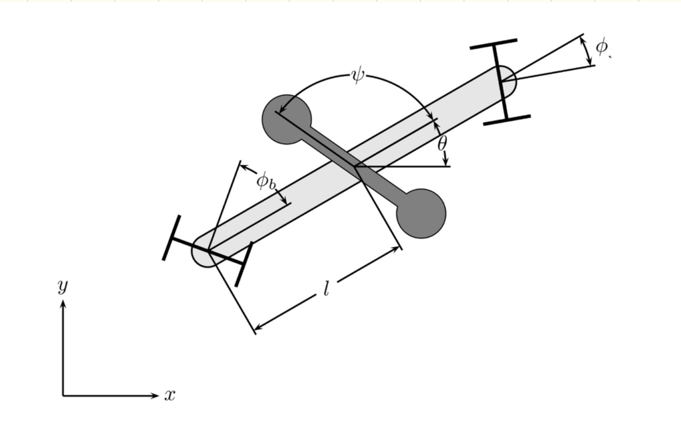
$$\dot\varphi_1 = -\frac{Q}{C} + \cancel{\dot\varphi_2}$$

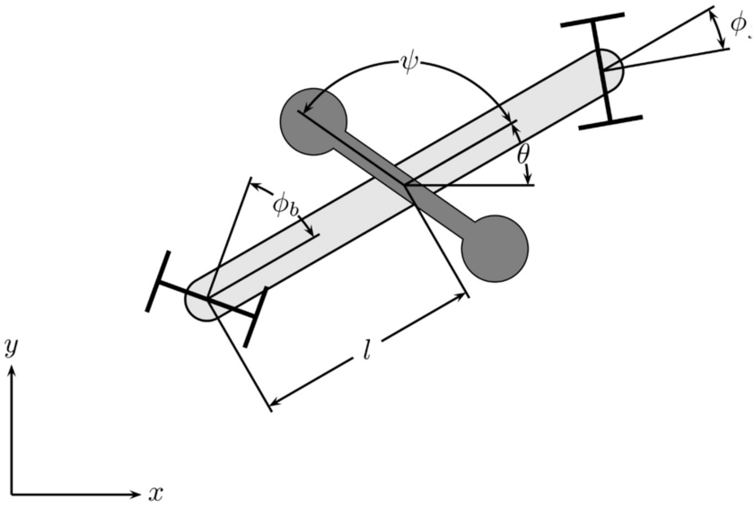
$$\dot\varphi_3 = -\frac{Q}{C} + \cancel{\dot\varphi_2}$$

integrability conditions

EXAMPLE mechanical nonholonomic systems.

The system that I would like to present





$$M = \underset{x,y,\vartheta}{SE(2)} \times \underset{\psi}{S^1} \times \underset{\varphi_f}{S^1} \times \underset{\varphi_b}{S^1}$$

position of the board

rider

front and back wheels

The constraint distribution is given as the kernel of the following forms:

$$\omega^1 = -\sin(\phi_b + \theta)\mathrm{d}x + \cos(\phi_b + \theta)\mathrm{d}y - l\cos(\phi_b)\mathrm{d}\theta$$
$$\omega^2 = -\sin(\phi_f + \theta)\mathrm{d}x + \cos(\phi_f + \theta)\mathrm{d}y + l\cos(\phi_f)\mathrm{d}\theta$$

Nonholonomic mechanics and locomotion: the Snakeboard example

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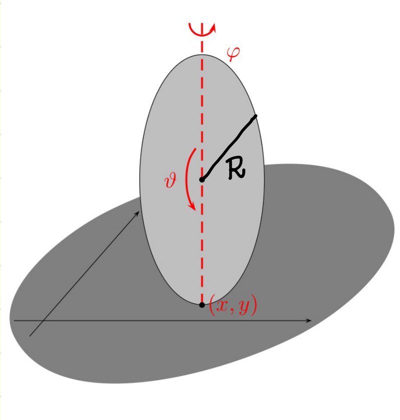
01/03/1994
Last updated: 28/07/1998

Example that I will present

$$N = \mathbb{R}^2 \times S^1 \times S^1$$
$$\quad x, y \quad \varphi \quad \vartheta$$



$$\begin{cases} \dot{x} = R\,\dot{\vartheta}\cos\varphi \\ \dot{y} = R\,\dot{\vartheta}\sin\varphi \end{cases}$$ constraint distribution

$$L(\dots) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{J_1}{2}\dot{\varphi}^2 + \frac{J_2}{2}\dot{\vartheta}^2$$

L does not depend on positions, but constraints depend on φ therefore we reduce with respect to x, y, ϑ and get

$$E = TS^1 \times \mathbb{R}^3 \ni (\varphi, \dot{\varphi}, x, y, \vartheta)$$
$$\downarrow$$
$$M = S^1 \qquad \ni \qquad \dot{\varphi}$$

For the purpose of calculation it is again convenient to change coordinates (basis)

$$\partial_\varphi, \partial_\vartheta, \partial_x\ \partial_y \rightsquigarrow \quad e_1 = \partial_\varphi \quad e_2 = \partial_\vartheta + R\cos\varphi\,\partial_x + R\sin\varphi\,\partial_y$$
$$e_3 = \partial_x \quad e_4 = \partial_y$$

$[e_1, e_2] = R\cos\varphi\, e_4 - R\sin\varphi\, e_3$

$$(\dot{\varphi}, \dot{\vartheta}, \dot{x}, \dot{y}) \rightsquigarrow \quad y^1, y^2, y^3, y^4$$

constraint distribution is spanned by e_1, e_2.

$$E = TS^1 \times \mathbb{R}^3 \longrightarrow M = S^1$$

Canonical Dirac structure of the Lie algebroid

$$D \subset TE^* \oplus_{E^*} T^*E^*$$

$$(\varphi, \xi_i, \dot\varphi, \dot\xi_k, p, y^k)$$

$$\begin{tikzcd}
 & D \arrow[dl] \arrow[dr] & \\
Ph_D = E^* & C_D \arrow[u, hook] & Vel_D = \operatorname{graph} \rho
\end{tikzcd}$$

the anchor

φ, ξ_i

$$\dot\varphi = y^1$$

$$\dot\xi_3 = 0$$

$$\dot\xi_4 = 0$$

$$\dot\xi_1 = y^2 (R\,\xi_3 \sin\varphi - R\,\xi_4 \cos\varphi) - p$$

$$\dot\xi_2 = -R\,\xi_3 \sin\varphi + R\,\xi_4 \cos\varphi$$

$$Vel_D \subset TM \oplus_M E$$

$$C_D \subset T^*M \oplus_M E^*$$

The constraint distribution $W = \langle e_1, e_2 \rangle \subset E$ has to be transformed into the subset of Vel_D

$$W = \{ y^3 = 0 = y^4 \} \qquad \Delta \subset Vel_D = \{ (\varphi, \dot\varphi = y^1, y^1, y^2, 0, 0) \}$$

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ Ph_D = E^* & C_D \arrow[u, hook] & Vel_D = \text{graph}\, \varphi \end{tikzcd}$$

the anchor

φ, ξ_i

$\dot\varphi = y^1$

$$\dot\xi_3 = 0$$
$$\dot\xi_4 = 0$$
$$\dot\xi_1 = y^2 (R\xi_3 \sin\varphi - R\xi_4 \cos\varphi) - p$$
$$\dot\xi_2 = -R\xi_3 \sin\varphi + R\xi_4 \cos\varphi$$

$Vel_D \subset TM \oplus_M E$

$C_D \subset T^*M \oplus_M E^*$

$$\begin{tikzcd} & D^\Delta \arrow[dl] \arrow[dr] & \\ Ph_D = E^* & C_{D^\Delta} \arrow[u, hook] & Vel_{D^\Delta} = \Delta = \{(\varphi, \dot\varphi = y^1, y^1, y^2, 0, 0)\} \end{tikzcd}$$

$C_{D^\Delta} = \Delta^o$

$$\{(\varphi, -\dot\xi_1, \dot\xi_1, \dot\xi_2 = 0, \underbrace{\dot\xi_3, \dot\xi_4}_{\text{arbitrary}})\}$$

$$
\begin{tikzcd}
 & \mathcal{D}_H \arrow[d, hook] & & & \mathcal{D} \arrow[d, hook] & & & \mathcal{D}_L \arrow[d, hook] & \\
 & \mathsf{T}^*E^* \arrow[rrr, "\beta_{DV}"] \arrow[dr] \arrow[ddl] & & & \mathsf{T}E^* \arrow[dr] \arrow[ddl] & & & \mathsf{T}^*E \arrow[lll, "\varepsilon_{DV}"'] \arrow[dr] \arrow[ddl] & \\
 & & E \arrow[rrr, "Vel_{DV}"] & & & \mathsf{T}S^1 & & & E \arrow[lll, "Vel_{DV}"'] \arrow[ul, bend right, "dL"'] \arrow[dlllll, dotted, "\lambda"] \\
E^* \arrow[uur, bend left, "dH"] \arrow[rrr] & & & E^* & & & E^* \arrow[lll] & &
\end{tikzcd}
$$

The dynamics $\mathcal{D}$

$$
\begin{aligned}
&\{(\varphi, \xi_i, \dot{\varphi}, \dot{\xi}_j) : \\
&\xi_3 = \frac{mR}{mR^2 + J_2}\xi_2 \cos\varphi, \quad \dot{\varphi} = \tfrac{1}{J_1}\xi_1, \\
&\xi_4 = \frac{mR}{mR^2 + J_2}\xi_2 \sin\varphi, \quad \dot{\xi}_1 = \dot{\xi}_2 = 0, \quad \dot{\xi}_3,\ \dot{\xi}_4 \text{ arbitrary}\}.
\end{aligned}
$$

integrability conditions...

$$\dot{\xi}_3 = \frac{mR}{mR^2+J_2}\left(\cancel{\dot{\xi}_2} - \sin\varphi\cdot\dot{\varphi}\right)$$

$$\dot{\xi}_4 = \frac{mR}{mR^2+J_2}\left(\cancel{\dot{\xi}_2} + \cos\varphi\cdot\dot{\varphi}\right)$$

OBSERVATION The equations we obtain using Dirac algebroids are usualy implicit, there appear integrability questions.

Several procedures for extracting integrable part of an implicit equations are present in the literature. They are however of pure differential character ...

THE END