Short and biased introduction to groupoids.

Piotr Stachura

Katedra Zastosowań Matematyki, SGGW

February 2, 2015

A relation $r : X \longrightarrow\!\triangleright Y$ is a triple $(X, Y; Gr(r))$ where X, Y are sets and $Gr(r) \subset Y \times X$.
If $r : X \longrightarrow\!\triangleright Y$ then $r^T : Y \longrightarrow\!\triangleright X$ is defined by

$$(x, y) \in Gr(r^T) \iff (y, x) \in Gr(r).$$

A domain of r is a set $D(r) := \{x \in X : \exists y \in Y \, (y, x) \in Gr(r)\}$
An image of r is a set $Im(r) := \{y \in Y : \exists x \in X \, (y, x) \in Gr(r)\}$
A composition $r : X \longrightarrow\!\triangleright Y$, $s : Y \longrightarrow\!\triangleright Z$, $sr : X \longrightarrow\!\triangleright Z$:

$$Gr(sr) := \{(z, x) : \exists \, y \in Y : (z, y) \in s \, , \, (y, x) \in r\}$$

Definition

Groupoid $\Gamma \rightrightarrows E$ *consists of a set* Γ, *two* *relations* $m : \Gamma \times \Gamma \longrightarrow\!\!\triangleright\, \Gamma$, $e : \{1\} \longrightarrow\!\!\triangleright\, \Gamma$, $E := Im(e) \subset \Gamma$ *satisfying conditions:*

$$m(m \times id) = m(id \times m) \tag{1}$$

$$m(e \times id) = m(id \times e) = id \tag{2}$$

and such that $m^T(E) \subset \Gamma \times \Gamma$ *is a graph of an involution* $s : \Gamma \to \Gamma$

Groupoids – definition

From (1) and (2) it follows:

- $(e_1, e_2) \in D(m) \iff e_1 = e_2$ and then $e = m(e, e)$
- There exist unique mapping $e_L, e_R : \Gamma \longrightarrow E$ defined by the conditions $(g, e_R(g)) \in D(m)$ and $(e_L(g), g) \in D(m)$ and then $m(g, e_R(g)) = m(e_L(g), g) = g$ and $e = e_L(e) = e_R(e)$.
- $(g_1, g_2) \in D(m) \Rightarrow$ $[e_R(g_1) = e_L(g_2)\,,\ e_L(m(g_1, g_2)) = e_L(g_1)\,,\ e_R(m(g_1, g_2)) = e_R(g_2)]$

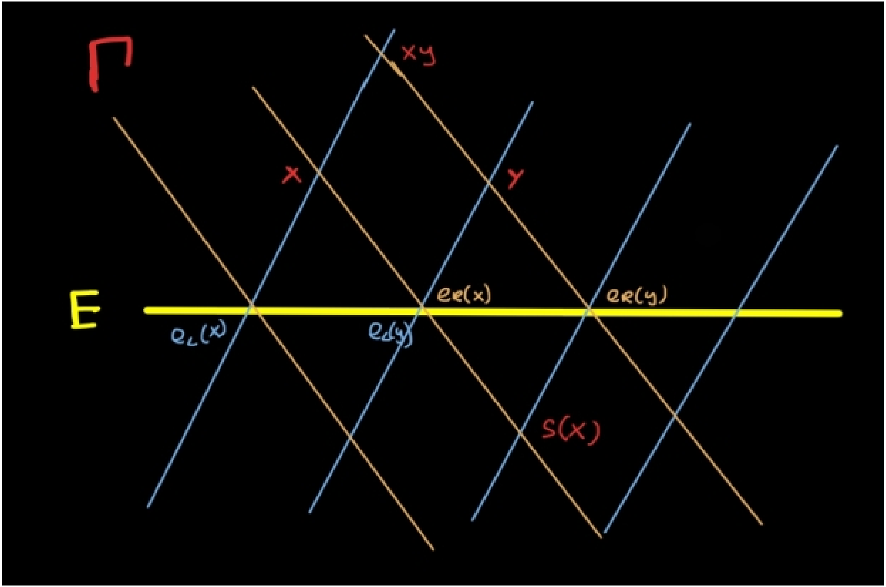
e_R is source or domain and e_L is target or range.

The existence of s gives additionally:

- $e_L(s(g)) = e_R(g)\,,\ e_R(s(g)) = e_L(g)$,
- $m(g, s(g)) = e_L(g)\,,\ m(s(g), g) = e_R(g)$
- $e_R(g_1) = e_L(g_2) \Rightarrow (g_1, g_2) \in D(m)$ i.e.
  $D(m) = \{(g_1, g_2) : e_R(g_1) = e_L(g_2)\}$
- $(s(g_3); g_1, g_2) \in m \iff (g_3; s(g_2), s(g_1)) \in m$
  (i.e. $s(g_1 g_2) = s(g_2)s(g_1)$)
- m is a mapping $D(m) \to \Gamma$

If E consists of one point then $D(m) = \Gamma \times \Gamma$, m is a mapping and Γ is a group.

Groupoids – definition



Operations on groupoids

- Cartesian product: $\Gamma_1 \rightrightarrows E_1$, $\Gamma_2 \rightrightarrows E_2$ then $\Gamma_1 \times \Gamma_2 \rightrightarrows E_1 \times E_2$ with operations defined "coordinatewise".
  But this is not a categorical product.
- Disjoint union of groupoids is a groupoid.
- Restriction: for a subset $F \subset E$ the set $e_L^{-1}(F) \cap e_R^{-1}(F)$ is a groupoid with the set of units F.

Groupoids – structure

Orbits On the set of units E define the relation:

$$e_1 \sim e_2 \iff \exists\gamma : e_L(\gamma) = e_1\,,\ e_R(\gamma) = e_2$$

This is an equivalence relation, its classes are called orbits of Γ.

$$[e] = e_R(e_L^{-1}(e)) = e_L(e_R^{-1}(e))$$

For an orbit $O \subset E$, a set $\Gamma_O := e_L^{-1}(O) = e_R^{-1}(O) \subset \Gamma$ is a groupoid – transitive component of Γ.
Any groupoid is a disjoint union of transitive components.
Isotropy groups
For $e \in E$ a set $e_L^{-1}(e) \cap e_R^{-1}(e)$ is a group – isotropy group of e.
Points in the same orbit have isomorphic isotropy groups.

Example of transitive groupoid:

X - a set, G - a group

$$\Gamma := X \times G \times X\,,\ \ E := \{(x,e,x) : x \in X\} \simeq X$$

$$e_R(x,g,y) := (y,e,y)\,,\ \ e_L(x,g,y) := (x,e,x)$$

$$inverse:\ \ s(x,g,y) := (y,g^{-1},x)$$

$$multiplication:\ \ (x,g,y)(y,h,z) := (x,gh,z)$$

In fact, this is the most general example:
Let Γ be a transitive groupoid. Choose $e_0 \in E$ and a section $p : E \longrightarrow e_L^{-1}(e_0)$ of right projection (restricted to $e_L^{-1}(e_0)$) such that $p(e_0) = e_0$; let G be the isotropy group of e_0. The mapping:

$$E \times G \times E \ni (e_1, g, e_2) \mapsto s(p(e_1))gp(e_2) \in \Gamma$$

is an isomorphism.

Bisections

Definition

A set $B \subset \Gamma$ is a bisection iff it is a section of left and right projection over E.

Subsets of a groupoid can be "multiplied": for $A, B \subset \Gamma$ we define

$$AB := \{m(a,b) : a \in A, b \in B, (a,b) \in D(m)\}.$$

This operation turns the set of bisections into a group: neutral element is the set of identities and $B^{-1} = s(B)$. This multiplication of subsets can be used to characterize bisections:

Bisections

Lemma

Let $\Gamma \rightrightarrows E$ be a groupoid and $A \subset \Gamma$.

- *A is a section of e_R over $e_R(A)$ iff $As(A) \subset E$;*
- *A is a section of e_L over $e_L(A)$ iff $s(A)A \subset E$;*
- *A is a bisection iff $s(A)A = As(A) = E$.*

Bisections act on a groupoid by $\Gamma \ni \gamma \mapsto B\gamma := \gamma'\gamma$, where γ' is a unique element in B with $e_R(\gamma') = e_L(\gamma)$ (i.e. $\{B\gamma\} = B\{\gamma\}$ using multiplication of subsets). This action preserves right fibers i.e. $e_R(B\gamma) = e_R(\gamma)$ and maps left fibers into left fibers.

Definition

A morphism of groupoids $\Gamma \rightrightarrows E$, *and* $\Gamma' \rightrightarrows E'$ *is a* *relation* $h : \Gamma \longrightarrow\!\!\triangleright\, \Gamma'$ *that satisfies:*

$$hm = m'(h \times h)\,,\; s'h = hs\,,\;\; he = e'$$

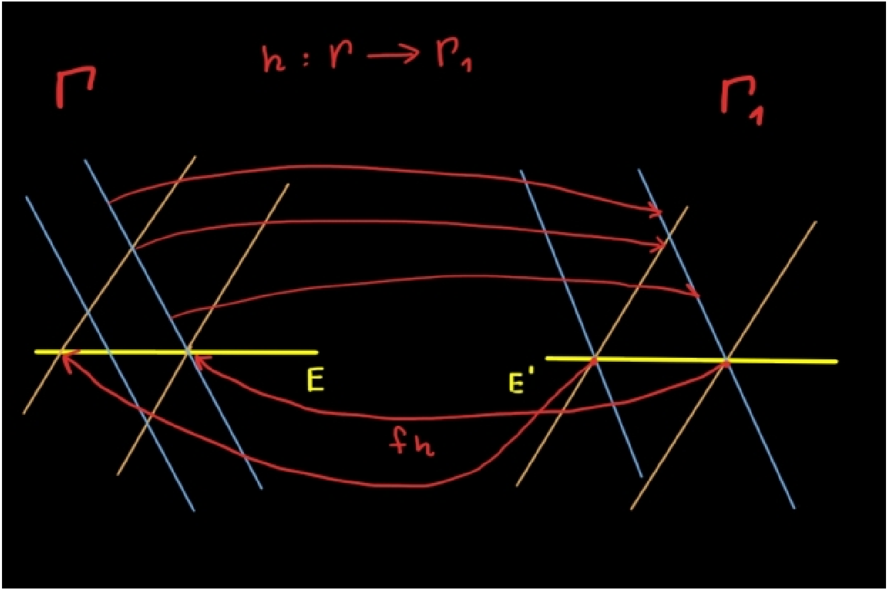
It follows that a morphism $h : \Gamma \longrightarrow\!\!\triangleright\, \Gamma'$ defines:
a mapping (base mapping) $\rho_h : E' \to E$
and for every $e' \in E'$ mappings

$$h_R(e') : e_R^{-1}(f_h(e')) \to e_R'^{-1}(e')$$

$$h_L(e') : e_L^{-1}(f_h(e')) \to e_L'^{-1}(e')$$

In particular $D(h)$ is a union of transitive components and $Im(h)$ is a (wide) subgroupoid of Γ'.

Morphisms



Examples of morphisms

- Groups If Γ and Γ' are groups, then any morphism is a group homomorphism.
- Sets If Γ is a "set-groupoid" (i.e. $\Gamma = E$), then any morphism $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ is $h = f^T$ for some mapping $f : E' \to \Gamma$. In particular $\Gamma := \{1\}$ is the initial object.
- The (left) regular representation The relation $l : \Gamma \longrightarrow\!\!\triangleright \Gamma \times \Gamma$ given by

$$(\gamma_1, \gamma_2; \gamma_3) \in l \iff (\gamma_1; \gamma_3, \gamma_2) \in m$$

  is a morphism from Γ to the pair groupoid $\Gamma \times \Gamma$

$$l = \{(\gamma_1\gamma_2, \gamma_2; \gamma_1) : \gamma_1, \gamma_2 \in \Gamma,\ e_R(\gamma_1) = e_L(\gamma_2)\}$$

- For any groupoid Γ the mapping

$$\Gamma \ni \gamma \mapsto (e_L(\gamma), e_R(\gamma)) \in E \times E$$

  is a morphism (to the pair groupoid).

Examples of morphisms (cont)

- Transitive components. If $\Gamma' \subset \Gamma$ is a union of transitive components and $i : \Gamma' \to \Gamma$ is the inclusion map, then $i^T : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ is a morphism.
- Restriction of morphism to its domain. If $h : \Gamma_1 \longrightarrow\!\!\triangleright \Gamma_2$ is a morphism with a domain $D(h)$, then the relation $h|_{D(h)} : D(h) \longrightarrow\!\!\triangleright \Gamma_2$ is a morphism.
- Wide subgroupoids. If $\Gamma_1 \subset \Gamma$ is a wide subgroupoid (i.e. $E \subset \Gamma_1$), the inclusion $i : \Gamma_1 \to \Gamma$ is a morphism.
- Isotropy group bundle. This is a special case of the previous example. Let Γ be a groupoid, $\Gamma' := \bigcup_{e \in E} e_L^{-1}(e) \cap e_R^{-1}(e)$ its isotropy group bundle and $i : \Gamma' \to \Gamma$ the inclusion. Then $i : \Gamma' \longrightarrow\!\!\triangleright \Gamma$ is a morphism.

Examples of morphisms (cont)

- Cartesian product A cartesian product of groupoids is defined in a natural way (coordinatewise). The relations

$$i_1 = \{(\gamma_1, e_2; \gamma_1) : \gamma_1 \in \Gamma_1, e_2 \in E_2\},$$

$$i_2 = \{(e_1, \gamma_2; \gamma_2) : e_1 \in E_1, \gamma_2 \in \Gamma_2\}$$

are morphisms

$$i_1 : \Gamma_1 \longrightarrow\!\triangleright \Gamma_1 \times \Gamma_2 \ , \ i_2 : \Gamma_2 \longrightarrow\!\triangleright \Gamma_1 \times \Gamma_2$$

But projections $\pi_1(\pi_2) : \Gamma_1 \times \Gamma_2 \to \Gamma_1(\Gamma_2)$ are not morphisms. So cartesian product of groupoids is not a product in categorical sense (it is rather like a tensor product).

Examples of morphisms (cont)

- Group actions If a group G acts on a set X, then the relation

$$\{(gx, x; g) : g \in G, x \in X\}$$

is a morphism from G to $X \times X$. Any morphism $G \longrightarrow\!\triangleright X \times X$ is of this kind.

- Morphism into groups If G is a group and $h : \Gamma \longrightarrow\!\triangleright G$ is a morphism, then $f_h(E_2) =: e_0$ and the orbit of e_0 is $\{e_0\}$, i.e. $e_L^{-1}(e_0) = e_R^{-1}(e_0) =: \Gamma_0$, and h is a group homomorphism $\Gamma_0 \to G$.

  In particular if X has more then 1 element the set of morphisms from $X \times X$ to G is empty.

- Morphism from groups. If G is a group and Γ is a groupoid, then morphisms $h : G \longrightarrow\!\triangleright \Gamma$ are just group homomorphisms from G to a group of bisections of Γ.

Examples of morphisms (cont)

- "Inner automorphisms" If $B \subset \Gamma$ is a bisection, the mapping

$$Ad_B : \Gamma \ni g \mapsto BgB^{-1} \in \Gamma$$

is a morphism and $Ad_B Ad_C = Ad_{BC}$.

- If $h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ is a morphism, $B, C \subset \Gamma$ are bisections, then $h(B)$ is a bisection,

$$h(B)h(C) = h(BC)\,,\; h(s(B)) = s'(h(B)) \;\; and$$

$$h\,Ad_B = Ad_{h(B)}h$$

- If M, N are manifolds and $f : N \to M$ is a smooth map, then $T^*f : T^*M \longrightarrow\!\!\triangleright T^*N$ is a morphism (cotangent lift).

Properties of morphisms

Proposition

Let $h : \Gamma \longrightarrow\!\!\triangleright\, \Delta$ *be a morphism of groupoids and* $G, G_1 \subset \Gamma$ *subgroupoids.*

1. $h(G) \subset \Delta$ *is a subgroupoid;*
2. *If* $G \cap G_1 = \emptyset$ *then* $h(G) \cap h(G_1) = \emptyset$*;*
3. $h|_G : G \longrightarrow\!\!\triangleright\, h(G)$ *is a morphism.*
4. *If* h *is surjective and* G *is a transitive component then* $h(G)$ *is a union of transitive components.*

A morphism is determined by its value on any fiber in every transitive component (contained in its domain).

Lemma

Let $\Gamma \rightrightarrows E$ *and* $\Delta \rightrightarrows F$ *be groupoids and* $h, k : \Gamma \longrightarrow\!\!\triangleright\, \Delta$ *morphisms. Assume* Γ *is transitive and for some* $e \in E$*:* $h|_{e_R^{-1}(e)} = k|_{e_R^{-1}(e)}$*. Then* $h = k$.

Actions and morphisms

A groupoid $\Gamma \rightrightarrows E$ can act on a set X equipped with a mapping to E.

Definition

Let $\Gamma \rightrightarrows E$ be a groupoid, X a set and $\rho : X \to E$ a mapping. Define the set

$$\Gamma_{e_R}\times_{\rho} X := \{(\gamma, x) \in \Gamma \times X : e_R(\gamma) = \rho(x)\}.$$

An action of Γ on X is a mapping: $\Gamma_{e_R}\times_{\rho} X \ni (\gamma, x) \mapsto \gamma x \in X$ that satisfies:

$$\rho(x)x = x \quad \gamma_1(\gamma_2 x) = (\gamma_1\gamma_2)x$$

i.e. if one side is defined, the other is also and then they are equal.

Action of Γ on itself by multiplication:

$$\rho : \Gamma \ni \gamma \mapsto e_L(\gamma) \in E$$

$$\Gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2 \in \Gamma$$

Action of Γ on its set of units.

$$\rho : E \ni e \mapsto e \in E$$

$$\{(\gamma, e) \in \Gamma \times E : e_R(\gamma) = e\} \ni (\gamma, e) \mapsto e_L(\gamma) \in E$$

Action on the isotropy group bundle. $X := \Gamma'$ – the isotropy group bundle of Γ; $\rho := e_L$ and the action:

$$\{(\gamma, \gamma') : e_R(\gamma) = e_L(\gamma')\} \ni (\gamma, \gamma') \mapsto \gamma\gamma' s(\gamma) \in \Gamma'$$

Actions and morphisms (cont)

Actions from morphisms. Let $h : \Gamma_1 \longrightarrow\!\!\!\triangleright\, \Gamma_2$ be a morphism with the base map $f_h : E_2 \to E_1$. Put $\rho := f_h \cdot e_L : \Gamma_2 \to E_1$ The mapping

$$(\gamma_1, \gamma_2) \mapsto m_2(h_R(e_2)(\gamma_1), \gamma_2)\,,\ e_2 := e_L(\gamma_2)$$

is an action of Γ_1 on Γ_2.

If we use relations the definition of a groupoid action can be presented in a more group-like style:

Definition

Let $\Gamma \rightrightarrows E$ *be a groupoid and* X *a set. An action of* Γ *on* X *is a relation* $\Phi : \Gamma \times X \longrightarrow\!\triangleright X$ *that satisfies:*

$$\Phi(m \times id) = \Phi(id \times \Phi)\,,\ \ \Phi(e \times id) = id\,.$$

Actions and morphisms (cont)

Next proposition states the equivalence of both definitions.

Proposition

Let $\Phi : \Gamma \times X \longrightarrow\!\!\triangleright X$ *be an action in a sense of def. 0.8. Then*

1. *For every* $x \in X$ *there exists unique* $e \in E$ *such that* $(x; e; x) \in \Phi$*, i.e.* Φ *defines a mapping* $\rho : X \to E$*;*
2. $D(\Phi) = \Gamma_{e_R}\!\times_{\rho} X$*;*
3. $(y; \gamma, x) \in \Phi \Rightarrow \rho(y) = e_L(\gamma)$*;*
4. $(y; \gamma, x) \in \Phi \iff (x; s(\gamma), y) \in \Phi$*;*
5. Φ *is a mapping* $D(\Phi) \to X$*; this mapping is an action of* Γ *on* X *in the sense of def. 0.7;*
6. *If* Γ *acts on* X *in a sense of def. 0.7, the relation* $\Phi := \{(\gamma x; \gamma, x) : e_R(\gamma) = \rho(x)\}$ *is an action in the sense of def. 0.8.*

Actions and morphisms (cont)

If $\tilde{h}$ is an action of Γ on X then

$$Gr(h) := \{(\gamma x, x; \gamma) : e_R(\gamma) = \rho(x)\}$$

defines a morphism $h : \Gamma \longrightarrow\!\!\triangleright X \times X$.
Conversely, if $h : \Gamma \longrightarrow\!\!\triangleright X \times X$ is a morphism, then

$$\tilde{h} : \{(\gamma, x) : e_R(\gamma) = f_h(x)\} \ni (\gamma, x) \mapsto e_L(h_R(x)(\gamma)) \in X$$

defines an action of Γ on X.
So actions of groupoids on sets are just morphisms into pair groupoids

Actions and morphisms (cont)

Let $h : \Gamma_1 \longrightarrow\!\!\!\triangleright \Gamma_2$ be a morphism and $\Phi_h : \Gamma_1 \times \Gamma_2 \longrightarrow\!\!\!\triangleright \Gamma_2$ be the related action. This action commutes with multiplication in Γ_2, i.e.

$$\Phi_h(id \times m_2) = m_2(\Phi_h \times id)$$

Conversely, any action $\Phi : \Gamma_1 \times \Gamma_2 \longrightarrow\!\!\!\triangleright \Gamma_2$ that commutes with m_2 defines a morphism by:

$$h := \{(\Phi(\gamma_1, \gamma_2)s(\gamma_2); \gamma_1) : (\gamma_1, \gamma_2) \in D(\Phi)\}$$

So morphisms are actions that commute with groupoid multiplication – exactly as for group homomorphisms

Action groupoids, morphisms and functors

Groupoids are special categories and "standard" definition of morphism is a functor, i.e map $f : \Gamma \to \Delta$ such that

$$f(E) \subset F \quad (F \text{ is a set uf units in } \Delta)$$

$$\gamma, \gamma' \in \Gamma^{(2)} \Rightarrow f(\gamma), f(\gamma') \in \Delta^{(2)} \text{ and then } f(\gamma\gamma') = f(\gamma)f(\gamma').$$

Let $\Phi : \Gamma \times X \longrightarrow\!\!\triangleright X$ be an action of a groupoid $\Gamma \rightrightarrows E$ on X with a base map $\rho : X \to E$. The following definitions make sense:

$$\begin{aligned} E_\Phi &:= \{(\rho(x), x) : x \in X\}\,, \\ s_\Phi &: D(\Phi) \ni (\gamma, x) \mapsto (s(\gamma), \Phi(\gamma, x)) \in D(\Phi)\,, \\ m_\Phi &: D(\Phi) \times D(\Phi) \longrightarrow\!\!\triangleright D(\Phi)\,, \\ Gr(m_\Phi) &:= \{(\gamma_1\gamma_2, x; \gamma_1, \Phi(\gamma_2, x), \gamma_2, x) : (\gamma_1, \gamma_2) \in D(m)\,, (\gamma_2, x) \in D(\Phi)\} \end{aligned}$$

$(D(\Phi), m_\Phi, s_\Phi, E_\Phi)$ is a groupoid; it is called the action groupoid for the action Φ and is denoted by $\Gamma \times_\Phi X$.

Action groupoids, morphisms and functors

Let $\Gamma \rightrightarrows E$ and $\Delta \rightrightarrows F$ be groupoids and $h : \Gamma \longrightarrow \Delta$ a morphism. Composition of h with the mapping (morphism)

$$\Delta \ni \delta \mapsto (e_L(\delta), e_R(\delta)) \in F^2$$

gives a morphism $\Gamma \longrightarrow F^2$, i.e. the action $\phi_h : \Gamma \times F \longrightarrow F$. Its domain is $D(\phi_h) := \{(\gamma, f) : e_R(\gamma) = \rho_h(f)\}$ and the action is

$$(\gamma, f) \mapsto e_L(h_f^R(\gamma)).$$

The morphism h defines also a mapping

$$D(\phi_h) \ni (\gamma, f) \mapsto h_f^R(\gamma) \in \Delta,$$

this mapping is a functor from the action groupoid $\Gamma \times_{\phi_h} F$ to Δ.

Action groupoids, morphisms and functors

Conversely:

an action ϕ of Γ on F and a functor $K : \Gamma \times_\phi F \to \Delta$ satysfying

$$K(e, f) = f \text{ for } (e, f) \in D(\phi) \cap (E \times F)$$

defines a morphism $h : \Gamma \longrightarrow\!\!\!\triangleright \Delta$ by

$$Gr(h) := \{(K(\gamma, f), \gamma) : (\gamma, f) \in \Gamma \times_\phi F\}$$

Morphisms as functors between action categories

Definition

Let Γ be a groupoid; a Γ-set is a pair (X, Φ), where X is a set and Φ an action of Γ on X. Let (X, Φ) and (Y, Ψ) be Γ-sets. A map $f : X \to Y$ is equivariant iff $f\Phi = \Psi(id \times f)$.

Γ-sets with equvariant maps as morphisms form a category.
If we think of actions as of morphisms to pair groupoids, an equivariant map $f : X \to Y$ is characterized by

$$(f \times id)h_1 = (id \times f^T)h_2$$

for $h_1 : \Gamma \longrightarrow\!\!\!\!\triangleright X^2$ and $h_2 : \Gamma \longrightarrow\!\!\!\!\triangleright Y^2$.

Morphisms as functors between action categories.

A morphism $h : \Gamma \longrightarrow\!\!\triangleright \Delta$ defines a functor H_h from Δ-sets to Γ-sets by composition: having an action of Δ on X i.e. morphism $k : \Delta \longrightarrow\!\!\triangleright X^2$ and a morphism $h : \Gamma \longrightarrow\!\!\triangleright \Delta$, we have an action of Γ on X by $kh : \Gamma \longrightarrow\!\!\triangleright X^2$.
This functor doesn't change sets and equivariant maps, in other words, if For_Γ, For_Δ are forgetful functors to the category of sets (i.e $For_\Gamma(X, \Phi) = X$ and $For_\Gamma(f) = f$, where f is an equivariant map between Γ-sets X and Y) it satisfies $For_\Delta H_h = For_\Gamma$.
Conversely any such functor defines a morphism of groupoids:

Proposition

Let H be a functor from Γ-sets to Δ-sets satisfying $For_\Delta H = For_\Gamma$. There exists unique morphism $h : \Delta \longrightarrow\!\!\triangleright \Gamma$, such that H is the composition with h.

Manifolds: smooth, Hausdorff, paracompact, second countable.
Submanifold=embedded submanifold
$r : X \longrightarrow\!\!\triangleright Y$ is a differential relation if $Gr(r)$ is a submanifold in $Y \times X$.
Tangent lift If $r : X \longrightarrow\!\!\triangleright Y$ then $Tr : TX \longrightarrow\!\!\triangleright TY$
$Gr(Tr) := TGr(r) \subset TY \times TX$
Cotangent lift $T^*(r) : T^*X \longrightarrow\!\!\triangleright T^*Y$:

$$(\beta, \alpha) \in T^*(r) \iff \forall (v, w) \in Gr(Tr) \, : \, \beta(v) = \alpha(w)$$

Differential groupoids (cont)

Transversality Let $r : X \longrightarrow\!\!\triangleright Y$ and $s : Y \longrightarrow\!\!\triangleright Z$.
Relations r, s have *simple composition* if

$$\forall (z, y) \in Gr(sr)\, \exists!\, y \in Y : (z, y) \in s\,,\ (y, x) \in r$$

Relations s, r have transverse ($s \pitchfork r$) composition iff

- Ts and Tr have simple composition;
- T^*s and T^*r have simple composition;
- sr is a differential relation.

A relation $r : X \longrightarrow\!\!\triangleright Y$ is a differential reduction iff $r = fi^T$ for $i : C \to X$ – inclusion map of a submanifold C and $f : X \to Y$ – surjective submersion. (i.e. r is a surjective submersion from a submanifold in X).

Differential groupoids

- Γ a manifold;
- m, e, s differential relations;
- m differential reduction;
- $m \pitchfork (m \times id)$, $m \pitchfork (id \times m)$, $m \pitchfork (id \times e)$, $m \pitchfork (e \times id)$;

Then e_L, e_R are surjective submersion.

Morphisms

$h : \Gamma \longrightarrow\!\!\triangleright \Gamma'$ differential relation; $m' \pitchfork (h \times h)$ and $h \pitchfork e$.Then $f_h : E' \to E$ is smooth.

Symplectic groupoids

Let $\Gamma \rightrightarrows E$ be a differential groupoid. Γ is symplectic groupoid if Γ is symplectic and $m : \Gamma \times \Gamma \longrightarrow\!\!\!\!\triangleright\, \Gamma$ is a symplectic relation.
Then E is a Poisson manifold in a canonical way:
There exists unique Poisson bracket on E such that $e_R : \Gamma \rightarrow E$ is a Poisson map.
If Γ, Γ' are symplectic groupoids, then morphisms are morphisms of diff groupoids which are symplectic relations. Base maps of morphisms of symplectic groupoids are (complete) Poisson maps.

Tangent and cotangent lifts If $\Gamma \rightrightarrows E$ is a differential groupoid then $T\Gamma \rightrightarrows TE$ is a differential groupoid with the structure (Tm, Te, Ts) and $T^*\Gamma \rightrightarrows (TE)^0$ is a differential groupoid with the structure $(T^*m, T^*e, -T^*s)$.

- If $X \rightrightarrows X$ is a manifold then its cotangent lift is $T^*X \rightrightarrows X$ (bundle of groups).
- If G is a group then $T^*G \rightrightarrows \mathfrak{g}^*$ is a transformation groupoid $G \times \mathfrak{g}^*$ with the coadjoint action.

T and T^* are functors on the category of differential groupoids (in fact T^* is a functor to the category of symplectic groupoids).