Sub Riemannian Geometry

M ; $g : D \times D \to \mathbb{R}$ scalar product $D \subset TM$ subbundle (lin. constraint on velocities)

Sub Riemannian Geodesic Problem

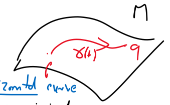
Data: $p, q \in M$, $[0,T]$

Find $\gamma : [0,T] \to M$ s.t.

1) $\gamma(0) = p$, $\gamma(T) = q$

3) $\dot\gamma(t) \in D_{\gamma(t)}$ ← horizontal curve

2) $\frac{1}{2}\int_0^T g(\dot\gamma(t), \dot\gamma(t))\,dt \to$ minimal



Questions?

1. Is there any horizontal curve joing p and q? → Chow-Rashevsky thm. If D is bracket generating then YES
2. If 1) then is there hor. curve that minizes the energy? → local. Hopf-Rinow thm. YES at least locally
3. If 2) how to find such a curve? (what are the equations?) ↳ let's do some calculations

Local description (rooted in Control Theory)

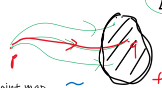
$D \overset{loc}{=} \operatorname{span}\{X_1, \dots X_k\}$ X_i are lin. independent

$g(X_i, X_j) = \delta_{ij}$ - orthonormal v.f.s

$\gamma(t)$ - horizontal

$\dot\gamma(t) = \sum u_i(t) X_i(\gamma(t))$ $u(t) = (u_1(t), \dots, u_k(t)) \to$ a control

$E_{nrg}(\gamma) = \frac{1}{2}\int_0^T \sum u_i(t)^2\,dt$ $L^2([0,T]; \mathbb{R}^k)$



Assume that $u(t)$ is optimal, i.e. the end. trajectory is a SR geodesic

The End-point map

$L^2([0,T]; \mathbb{R}^k) \xrightarrow{\tilde E} M \times \mathbb{R}$

$u(t) \longmapsto (\gamma(T); E_{nrg}(\gamma))$

s.that $\begin{cases} \dot\gamma(t) = \sum u_i(t) X_i(\gamma(t)) \\ \gamma(0) = p \end{cases}$ $\frac{1}{2}\int_0^T u_i(t)^2\,dt$

Examples

Heisenberg system $\mathbb{R}^3 \ni (x, y, z)$

$X_1 = \partial_x + y\partial_z$, $X_2 = \partial_y - x\partial_z$ no abnormal geodesics

$[X_1, X_2] = -2\partial_z$

$\operatorname{span}\{X_1, X_2, [X_1, X_2]\} = T\mathbb{R}^3$

Martinet plane

$X_1 = \partial_x$, $X_2 = \partial_y + x^2\partial_z$ $\boxed{\text{at } x = 0}$

admits abnormal geodesics $[X_1, X_2] = 2x\partial_z$; $[X_1, [X_1, X_2]] = 2\partial_z$

Also bracket-generating

$\varphi = a(t)dx + b(t)dy + c(t)dz$

$u_1(t), u_2(t)$ — controls

$\begin{cases} \dot x = u_1(t) \\ \dot y = u_2(t) \\ \dot z = x^2(t)\,u_2(t) \end{cases}$ $\begin{rcases} \langle \varphi, D \rangle = 0 \\ \Rightarrow a(t) \equiv 0 \\ b(t) + x^2 c(t) = 0 \end{rcases}$

$\begin{cases} \dot a(t) = 2u_2 x\,c(t) \\ \dot b(t) = 0 \\ \dot c(t) = 0 \end{cases}$ → $c(t) = 0$ ↯   → $x = 0$, $c(t) = c$, $u_1 \equiv 0$, $y = u_2(t)$, $z = const$

$\boxed{t \mapsto (0, t, z_0)}$

abnormal trajectory

Main idea - study the 1st derivative of the end-point map

$d\tilde E(u)[\Delta u] = \partial_s|_{s=0} \tilde E(u + s\Delta u) = \partial_s|_0 \gamma_s(T)$

where $\begin{cases} \dot\gamma_s(t) = \sum (u^i(t) + s\Delta u^i(t)) X_i(\gamma_s(t)) \\ \gamma_s(0) = p \end{cases}$ $/\partial_s|_{s=0}$

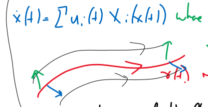
$\begin{cases} \partial_s \dot\gamma_{s=0}(t) = \sum \Delta u^i(t) X_i(\gamma(t)) + \\ \quad + \sum u^i(t) \frac{\partial X_i}{\partial q}(\gamma(t)) \cdot \partial_s\gamma(t) \\ \partial_s|_0 \gamma(0) = 0 \end{cases}$ ⊖

the linear part

$\dot v(t) = \sum u_i(t) \frac{\partial X_i}{\partial q} v(t)$ is a variation of $\dot x(t) = \sum u_i(t) X_i(x(t))$ where

$\begin{cases} \dot x(t) = \cancel{b(t)} + A(t)\dot x(t) \\ x(0) = 0 \end{cases}$

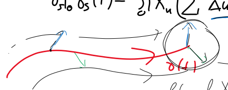
$x(t) = \int_0^t A^{t\tau} b(\tau)\,d\tau$ $A^{t\tau}$ is a fundamental solution of $\dot x(t) = A(t)x(t)$



$\gamma(t)$ ~ $u(t) \to X_u(q) = \sum u_i(t) X_i(q)$ control v.f.

→ tangent map of the flow of $X_{u(t)}$

All in all $\partial_s|_0 \gamma_s(T) = \int_0^T TX_u^{Tt}\left(\sum \Delta u^i(t) X_i(\gamma(t))\right)dt$ $D_{\gamma(t)}$



$\gamma(t)$ → integrate   flow of $X_{u(t)}(q)$

the derivative of the energy

$\partial_s|_{s=0} E_{nrg}(u + s\Delta u) = \partial_s|_{s=0} \frac{1}{2}\int_0^T \sum (u_i + s\Delta u_i)^2\,dt = \int_0^T \sum u_i(t)\,\Delta u_i(t)\,dt$

$d\tilde E(u)[\Delta u] = \left(\int_0^T TX_u^{Tt}\left[\sum \Delta u_i(t) X_i(\gamma(t))\right]dt; \int_0^T \sum u_i(t)\Delta u_i(t)\,dt\right)$

$\Delta u \in L^2([0,T], \mathbb{R}^k)$

Conclusion

The main lemma

If u(t) is optimal, then the image of $d\tilde E(u)$ cannot be the whole $T_{(\gamma(T), E_{nrg}(\gamma))} M \times \mathbb{R}$

Proof. Assume the contrary. So $\exists\ \Delta u^\alpha$ - controls, $\alpha = 1, \dots, m+1$ s.that $d\tilde E(u)[\Delta u^\alpha]$ spann the whole $T_\cdot M \times \mathbb{R}$

So consider

$\phi : \mathbb{R}^m \longrightarrow M \times \mathbb{R}$

$(t^\alpha) \longmapsto \tilde E(u + \sum t_\alpha \Delta u^\alpha)$ → C^1 map

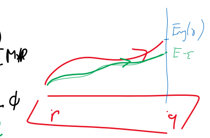
$\phi(0) = \tilde E(u) = (\gamma(T), E_{nrg}(\gamma))$

$\partial_{t_\alpha} \phi(0) = d\tilde E(u)[\Delta u^\alpha]$ so $\operatorname{Im} d\phi(0)$ is the whole $T_\cdot M \times \mathbb{R}$

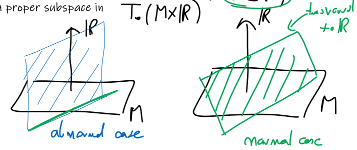
So ϕ is a local diffeom. around 0.

(Inverse F. Thm) $\Rightarrow (\gamma(T), E_{nrg}(\gamma) - \varepsilon) \in \operatorname{Im}\phi$

this contradicts optimality ↯

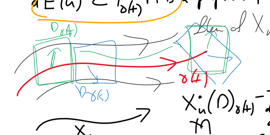


Corollary -- if u(t) is optimal then the image $d\tilde E(u)$ is a proper subspace in $T_\cdot(M \times \mathbb{R})$



Abnormal case

$d E(u) \subset T_{\gamma(t)} M$ is a proper subspace



$X_u^\cdot(D)_{\gamma(t)}$ - the smallest distribution along $\gamma(t)$ containing $D_{\gamma(t)}$ and invariant under the flow of $X_{u(t)}$

$\cap$

$T_{\gamma(t)}M$

Hamiltonian description

then $\exists\ \varphi(T) : T_{\gamma(T)}M \to \mathbb{R}$ s.that $dE(u) \subset \ker\varphi(T)$

define $\varphi(t) : T_{\gamma(t)}M \to \mathbb{R}$ by pull-backing it by the flow of $X_{u(t)}$

then $\langle \varphi(t), D_{\gamma(t)} \rangle = 0$ $\varphi(t) \neq 0$

$\dot\varphi^\alpha(t) = -\sum_{\beta, i} \varphi^\beta(t)\,u_i(t) \frac{\partial X_i^\beta}{\partial q^\alpha}$

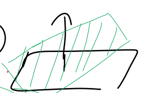
⚠ We don't know if an abnormal SR geodesic is necessarily smooth.) (open 1991)

note that if $H(\varphi, \gamma, u, t) = \langle \varphi, \sum u_i(t) X_i(\gamma) \rangle$

then $\begin{cases} \dot\gamma = \frac{\partial H}{\partial \varphi} = \sum u_i(t) X_i(\gamma) \\ \dot\varphi = -\frac{\partial H}{\partial \gamma} = -\langle \varphi(t); \sum u_i(t) \frac{\partial X_i}{\partial \gamma} \rangle \end{cases}$

Normal case

If $d\tilde E(u) \neq T_\cdot(M \times \mathbb{R})$ and is $\pitchfork$ to $\mathbb{R}$.



then $\exists\ \varphi : T_{\gamma(T)}M \to \mathbb{R}$ s.that $\operatorname{Im}_u dE(u) \subset \operatorname{graph}(\varphi(T))$

$\int_0^T TX_u^{Tt}(\sum \Delta u^i X_i)\,dt ; \int_0^T \sum u_i(t)\Delta u_i(t)\,dt$

that is $\int_0^T \sum u_i\,\Delta u^i\,dt = \int_0^T \sum \Delta u^i\,\varphi(TX_u^{Tt}(X_i))\,dt$ $\forall\ \Delta u^i \in L^2$

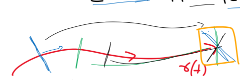
we conclude that $u_i(t) = \varphi(TX_u^{Tt}(X_i(\gamma(t))))$ a.e.

C^1 C^0 L^2 ⟶ $\gamma(t)$ C^0 C^1 C^2

$\boxed{\text{It follows that } u_i(t) \text{ are } C^\infty}$

What is the kernel of φ

If $\sum \Delta u^i u^i = 0$ then $0 = \sum \Delta u_i u_i = \varphi(TX_u^{Tt}(\sum \Delta u_i X_i(t)))$ $D \cap \dot\gamma^\perp$



It turns out that $X_u^\cdot(\dot\gamma^\perp)_{\gamma(t)} \neq T_{\gamma(T)}M$

the smallest distribution along $\gamma(t)$ containing $D_{\gamma(t)} \cap \dot\gamma^\perp$ and invariant by the flow of $X_{u(t)}$

Hamiltonian description

$\exists\ \varphi(t) : T_{\gamma(t)}M \to \mathbb{R}$

s.that

$\begin{cases} \dot\varphi^\alpha(t) = -\sum_{\beta, i} \varphi^\beta u_i(t) \frac{\partial X^\beta}{\partial q^\alpha}(\gamma(t)) = -\frac{\partial H}{\partial \gamma^\alpha} \\ \dot\gamma(t) = \sum u_i(t) X_i(\gamma(t)) = \frac{\partial H}{\partial p} \\ \langle \varphi(t), X_i \rangle = u_i(t) \end{cases}$

$H(\varphi, \gamma, u, t) = \langle \varphi, \sum u_i X_i(\gamma) \rangle - \frac{1}{2}u_i^2$

u_i is a critical point of H

Thm. Normal curves in SR Geometry are locally minimizing