Zadanie 1:

$$\sin[(k+1)\varphi]\cos(k\varphi) = \frac{1}{2i}\left(e^{i(k+1)\varphi} - e^{-i(k+1)\varphi}\right)\frac{1}{2}\left(e^{ik\varphi} + e^{-ik\varphi}\right) =$$

$$\frac{1}{4i}\left(e^{i(2k+1)\varphi} - e^{i\varphi} + e^{i\varphi} - e^{-i(2k+1)\varphi}\right) = \frac{1}{4i}\left(\left[e^{i(2k+1)\varphi} - e^{-i(2k+1)\varphi}\right] + \left[e^{i\varphi} - e^{-i\varphi}\right]\right) =$$

$$\frac{1}{2}\left[\sin(2k+1)\varphi + \sin\varphi\right]$$

$$\sum_{k=0}^{n} \sin\varphi = (n+1)\sin\varphi, \qquad \sum_{k=0}^{n} \sin(2k+1)\varphi = \operatorname{Im}\left(\sum_{k=0}^{n}\left(e^{i\varphi}\right)^{2k+1}\right) = \operatorname{Im}\left(e^{i\varphi}\sum_{k=0}^{n}\left(e^{2i\varphi}\right)^{k}\right) =$$

$$\operatorname{Im}\left(e^{i\varphi}\,\frac{1-e^{2i\varphi(n+1)}}{1-e^{2i\varphi}}\right) = \operatorname{Im}\left(\frac{e^{i\varphi(n+1)}\left(e^{-i\varphi(n+1)} - e^{i\varphi(n+1)}\right)}{e^{-i\varphi} - e^{i\varphi}}\right) = \operatorname{Im}\left(e^{i\varphi(n+1)}\,\frac{\sin(n+1)\varphi}{\sin\varphi}\right) =$$

$$\frac{\sin^2(n+1)\varphi}{\sin\varphi}$$

$$\sum_{k=0}^{n} \sin(k+1)\varphi\cos k\varphi = \frac{n+1}{2}\sin\varphi + \frac{\sin^2(n+1)\varphi}{2\sin\varphi}$$

Zadanie 2:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 3 & 6 & 9 & 12 & 15 & 18 & 1 & 4 & 7 & 10 & 13 & 16 & 19 & 2 & 5 & 8 & 11 & 14 & 17 \end{pmatrix}$$

cykle $(1\ 3\ 9\ 7)(2\ 6\ 18\ 14)(4\ 12\ 16\ 8)(5\ 15)(11\ 13\ 19\ 17)$

permutacja rzędu 4

$\operatorname{sgn}\sigma = (-1)(-1)(-1)(-1)(-1) = -1$

Permutacja jest rzędu 4, wobec tego $2015 = 2016+1 = 4\cdot 250+1$ $\sigma^{2015} = \sigma$.

Zadanie 3

$$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c \\ b+2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3a+4b \\ 3c+4d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a+2c&=0 \\ b+2d&=0 \\ 3a+4b&=0 \\ 3c+4d&=0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & \boxed{2} \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \boxed{2} & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & -2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 4 & 0 & 0 \\ -3/2 & -2 & 0 & 0 \end{bmatrix} \sim$$

proporcjonalne

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 4 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ -3 & 0 & 0 & 8 \\ 3 & 4 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} a + 2c &= 0 \\ -3a + 8d &= 0 \\ 3a + 4b &= 0 \end{aligned} \qquad \begin{aligned} c &= -\tfrac{1}{2} a \\ d &= \tfrac{3}{8} a \\ b &= -\tfrac{3}{4} a \end{aligned}$$

$$a = 8\alpha : \quad \begin{bmatrix} 8\alpha & -6\alpha \\ -4\alpha & 3\alpha \end{bmatrix} = \alpha \begin{bmatrix} 8 & -6 \\ -4 & 3 \end{bmatrix}$$

$$V_1 \cap V_2 = \left\langle \begin{bmatrix} 8 & -6 \\ -4 & 3 \end{bmatrix} \right\rangle \qquad \dim V_1 \cap V_2 = 1$$

$$V_1 : \quad \begin{aligned} a + 2c &= 0 \\ b + 2d &= 0 \end{aligned} \qquad \begin{aligned} a &= -2c \\ b &= -2d \end{aligned} \qquad \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix} = c \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$$

$$V_2 : \quad \begin{aligned} 3a + 4b &= 0 \\ 3c + 4d &= 0 \end{aligned} \qquad \begin{aligned} a &= -\tfrac{4}{3} b \\ c &= -\tfrac{4}{3} d \end{aligned} \qquad \begin{aligned} b &= 3\beta \\ d &= 3\delta \end{aligned} \qquad \begin{bmatrix} -4\beta & 3\beta \\ -4\delta & 3\delta \end{bmatrix} = \beta \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ -4 & 3 \end{bmatrix}$$

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} \boxed{-2} & 0 & -4 & 0 \\ 0 & -2 & 3 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \end{bmatrix} \overset{k}{\sim} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \boxed{-2} & 3 & 0 \\ -1 & 0 & -2 & -4 \\ 0 & 1 & 0 & 3 \end{bmatrix} \overset{k}{\sim} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 0 & -2 & -4 \\ 0 & 1 & 3/2 & 3 \end{bmatrix} \overset{k}{\sim} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & -4 \\ 0 & 1 & 3 \end{bmatrix}$$

proporcjonalne

Dowolny element $V_1 + V_2$ jest kombinacją liniową $\alpha \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ -4 & 3 \end{bmatrix}$

tzn.

$$\begin{aligned} a &= 2\alpha \to \alpha = a/2 \\ b &= -2\beta \to \beta = -b/2 \\ c &= -\alpha - 4\gamma \to -4\gamma = c + a/2 \\ d &= \beta - 3\gamma \to -3\gamma = d + b/2 \end{aligned} \quad \to \quad \frac{1}{4} c + \frac{1}{8} a = \frac{1}{3} d + \frac{1}{6} b \quad / \cdot 24$$

$$6c + 3a = 8d + 4b$$

Równanie opisujące sumę $V_1 + V_2$

$$\boxed{+3a - 4b + 6c - 8d = 0}$$

Zadanie 4

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\end{bmatrix} \quad T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

Wektory $\begin{bmatrix}1\\1\end{bmatrix}$ i $\begin{bmatrix}2\\1\end{bmatrix}$ są liniowo niezależne, tworzą więc bazę $\mathbb{R}^2$. Oznaczamy tę bazę literką g, tzn $g_1=\begin{bmatrix}1\\1\end{bmatrix}$ $g_2=\begin{bmatrix}2\\1\end{bmatrix}$. Bazę standardową w $\mathbb{R}^2$ oznaczymy symbolem st. W tych oznaczeniach mamy:

$$[T]_g^{st}=\begin{bmatrix}1&1\\1&0\end{bmatrix}$$ Potrzebujemy $[T]_e^f$. Zgodnie z zasadami

$$[T]_e^f=[id]_{st}^f\,[T]_g^{st}\,[id]_e^g$$ szukamy brakujących macierzy: $f=\left(\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right)$

Wobec tego $[id]_f^{st}=\begin{bmatrix}1&1\\-1&3\end{bmatrix}$ $\quad [id]_{st}^f=\frac{1}{3+1}\begin{bmatrix}3&-1\\1&1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}3&-1\\1&1\end{bmatrix}$

$$[id]_{st}^g=? \quad [id]_g^{st}=\begin{bmatrix}1&2\\1&1\end{bmatrix} \Rightarrow [id]_{st}^g=\frac{1}{1-2}\begin{bmatrix}1&-2\\-1&1\end{bmatrix}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$$

$$[id]_e^{st}=\begin{bmatrix}-1&1\\2&2\end{bmatrix}$$

$$[id]_e^g=[id]_{st}^g\cdot[id]_e^{st}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}\begin{bmatrix}-1&1\\2&2\end{bmatrix}=\begin{bmatrix}5&3\\-3&-1\end{bmatrix}$$

$$[T]_e^f=\frac{1}{4}\underbrace{\begin{bmatrix}3&-1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}}\begin{bmatrix}5&3\\-3&-1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}2&3\\2&1\end{bmatrix}\begin{bmatrix}5&3\\-3&-1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1&3\\7&3\end{bmatrix}$$

Zadanie 5:

$$\begin{bmatrix}p-5&2&1\\2&p-2&2\\1&2&p-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix} \quad D=\det\begin{bmatrix}p-5&2&1\\2&p-2&2\\1&2&p-5\end{bmatrix}=(p-5)^2(p-2)+4+4-(p-2)-2\cdot4(p-5)=$$
$$=\ldots=p(p-6)^2$$

$$D_x=\det\begin{bmatrix}1&2&1\\2&p-2&2\\1&2&p-5\end{bmatrix}=\ldots=(p-6)^2 \quad D_y=\det\begin{bmatrix}p-5&1&1\\2&2&2\\1&1&p-5\end{bmatrix}=\ldots=2(p-6)^2$$

$$D_2 = \begin{bmatrix} p-5 & 2 & 1 \\ 2 & p-2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dots = (p-6)^2$$

Dla $p \neq 0$ i $p \neq 0$ $\quad x = \frac{1}{p} \quad y = \frac{2}{p} \quad z = \frac{1}{p}$

$p = 0$:

$$\left[\begin{array}{ccc|c} -5 & 2 & 1 & 1 \\ 2 & -2 & 2 & 2 \\ 1 & 2 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & 0 & 3 & 3 \\ 2 & -2 & 2 & 2 \\ 3 & 0 & -3 & 3 \end{array}\right]$$

sprzeczne

Dla $p=0$ układ sprzeczny

$p = 6$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \end{array}\right] \quad x + 2y + z = 1$$

$x = 1 - 2y - z$

y, z dowolne.