Zadanie: Niech $f \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$. Sprawdzić, że f spełnia równanie cząstkowe $x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = 0$ wtedy i tylko wtedy gdy istnieje $g \in C^1(]0,\infty[)$ taka, że $f(x,y) = g(x^2+y^2)$.

$\Leftarrow$

$\frac{\partial f}{\partial x} = g'(x^2+y^2)\cdot 2x \quad \frac{\partial f}{\partial y} = g'(x^2+y^2)\cdot 2y \quad x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = x g'(x^2+y^2)\cdot 2y - y g(x^2+y^2)2x =$

$= (2xy - 2xy)g'(x^2+y^2) = 0$

$\Rightarrow$ Zapiszmy równanie o którym mowa we współrzędnych biegunowych, tzn

$x = r\cos\varphi \qquad f(x(r,\varphi), y(r,\varphi)) = F(r,\varphi)$
$y = r\sin\varphi$

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi$$

$$\frac{\partial F}{\partial \varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} = -\frac{\partial f}{\partial x} r\sin\varphi + \frac{\partial f}{\partial y} r\cos\varphi$$

$$\begin{bmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\frac{1}{r}\begin{bmatrix} r\cos\varphi & -\sin\varphi \\ r\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \cos\varphi \frac{\partial F}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial F}{\partial \varphi}$$

$$\frac{\partial f}{\partial y} = \sin\varphi \frac{\partial F}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial F}{\partial \varphi}$$

$$\frac{\partial f}{\partial x} = \cos\varphi \frac{\partial F}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial F}{\partial \varphi}$$

$$\frac{\partial f}{\partial y} = \sin\varphi \frac{\partial F}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial F}{\partial \varphi}$$

$$x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = r\cos\varphi\left(\sin\varphi\frac{\partial F}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial F}{\partial \varphi}\right) - r\sin\varphi\left(\cos\varphi\frac{\partial F}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial F}{\partial \varphi}\right) =$$

$$= \cancel{r\cos\varphi\sin\varphi\frac{\partial F}{\partial r}} + \cos^2\varphi\frac{\partial F}{\partial \varphi} - \cancel{r\sin\varphi\cos\varphi\frac{\partial F}{\partial r}} + \sin^2\varphi\frac{\partial F}{\partial \varphi} = (\sin^2\varphi + \cos^2\varphi)\frac{\partial F}{\partial \varphi}$$

$$x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = \frac{\partial F}{\partial \varphi}$$

We współrzędnych biegunowych równanie przyjmuje postać

$$\frac{\partial F}{\partial \varphi} = 0 \quad \text{tzn} \quad F(r,\varphi) = h(r)$$

tzn. funkcja zależy tylko od r.

Czy zapiszemy to $h(r)$, czy $g(r^2)$ nie ma znaczenia, gdyż obie funkcje różnią się o złożenie z funkcją $\sqrt{}$, tzn $h \circ \sqrt{} = g$ (dla stosownych g i h oczywiście) a $x \mapsto \sqrt{x}$ jest klasy C^∞ na $]0,\infty[$.

$\square$

Zadanie: Wyrazić wielkość $\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \varphi}\right)$ we współrzędnych kartezjańskich.

$$u(r,\varphi) = u\left(x(r,\varphi), y(r,\varphi)\right)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = \frac{\partial u}{\partial x}(-r\sin\varphi) + \frac{\partial u}{\partial y}(r\cos\varphi)$$

$$\frac{1}{r}\frac{\partial u}{\partial \varphi} = -\sin\varphi\frac{\partial u}{\partial x} + \cos\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \varphi}\right) = -\sin\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \cos\varphi\frac{\partial}{\partial r}\frac{\partial u}{\partial y} = -\sin\varphi\left[\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right]\frac{\partial u}{\partial x} +$$

$$\cos\varphi\left[\cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}\right]\frac{\partial u}{\partial y} =$$

$$= -\sin\varphi\cos\varphi\frac{\partial^2 u}{\partial x^2} - \sin^2\varphi\frac{\partial^2 u}{\partial x\partial y} + \cos^2\varphi\frac{\partial^2 u}{\partial x\partial y} + \cos\varphi\sin\varphi\frac{\partial^2 u}{\partial y^2} =$$

$$= \sin\varphi\cos\varphi\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + \left(\cos^2\varphi - \sin^2\varphi\right)\frac{\partial^2 u}{\partial x\partial y} =$$

$$\boxed{\frac{xy}{x^2+y^2}\left(u_{xx} - u_{yy}\right) + \frac{x^2-y^2}{x^2+y^2}u_{xy}}$$

Zadanie do przemyślenia w domu:

$\Omega = \{(x,y):\ x>0, y>0\}$ W Ω wprowadzamy nowe współrzędne wzorem $u = xy$ $v = \frac{y}{x}$. Zapisać we współrzędnych u i v równanie $x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = 0$ i, jeśli się da, rozwiązać je. Naszkicować krzywe stałego u i stałego v.

$$F\big(u(x,y), v(x,y)\big) = f(x,y)$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

$$\frac{\partial F}{\partial u}y - \frac{\partial F}{\partial v}\left(\frac{y}{x^2}\right) = \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial x}y + \frac{\partial^2 F}{\partial v \partial u}\frac{\partial v}{\partial x}y - \frac{\partial^2 F}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{y}{x^2} - \frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial x}\left(\frac{y}{x^2}\right) + \frac{\partial F}{\partial v}2\frac{y}{x^3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial u^2}y^2 - \frac{y^2}{x^2}\frac{\partial^2 F}{\partial v \partial u} - \frac{\partial^2 F}{\partial u \partial v}\frac{y^2}{x^2} + \frac{\partial^2 F}{\partial v^2}\frac{y^2}{x^4} + 2\frac{\partial F}{\partial v}\frac{y}{x^3}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2\frac{\partial^2 F}{\partial u^2} - 2\frac{y^2}{x^2}\frac{\partial^2 F}{\partial v \partial u} + \frac{\partial^2 F}{\partial v^2}\frac{y^2}{x^4} + 2\frac{y}{x^2}\frac{\partial F}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial F}{\partial u} + \frac{1}{x}\frac{\partial F}{\partial v}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 F}{\partial u^2}x^2 + \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial u \partial v} + \frac{1}{x^2}\frac{\partial^2 F}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial y^2} = x^2\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial v} + \frac{1}{x^2}\frac{\partial f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 \frac{\partial^2 F}{\partial u^2} - 2\frac{y^2}{x^2}\frac{\partial^2 F}{\partial v \partial u} + \frac{\partial^2 F}{\partial v^2}\frac{y^2}{x^4} + 2\frac{y}{x^3}\frac{\partial F}{\partial v} \quad / x^2$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 \frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u \partial v} + \frac{1}{x^2}\frac{\partial^2 f}{\partial v^2} \quad / y^2$$

$$x^2\frac{\partial^2 f}{\partial x^2} = \cancel{(xy)^2\frac{\partial^2 F}{\partial u^2}} - 2y^2\frac{\partial^2 F}{\partial v \partial u} + \cancel{\frac{y^2}{x^2}\frac{\partial^2 F}{\partial v^2}} + 2\frac{y}{x}\frac{\partial F}{\partial v}$$

$$-\quad y^2\frac{\partial^2 f}{\partial y^2} = \cancel{(xy)^2\frac{\partial^2 F}{\partial u^2}} + 2y^2\frac{\partial^2 F}{\partial v \partial u} + \cancel{\frac{y^2}{x^2}\frac{\partial^2 F}{\partial v^2}}$$

$$x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = -4y^2\frac{\partial^2 F}{\partial u \partial v} + 2y/x\frac{\partial F}{\partial v}$$

$$0 = -4y^{\cancel{2}}\frac{\partial^2 F}{\partial u \partial v} + 2\cancel{y}\frac{\partial F}{\partial v}$$

$$0 = -2yx\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial F}{\partial v}$$

$$0 = -2u\frac{\partial^2 F}{\partial v \partial u} + \frac{\partial F}{\partial v} = \frac{\partial}{\partial v}\left(-2u\frac{\partial F}{\partial u} + F\right)$$

$$-2u\frac{\partial F}{\partial u} + F = g(u)$$

Zależność F od u opisana jest równaniem różniczkowym liniowym pierwszego rzędu niejednorodnym

$$-2u\frac{\partial F}{\partial u} + F = g(u)$$

Materiał dodatkowy: Zapiszmy $F(u,v) = h_v(u)$

$$-2u h_v'(u) + h_v(u) = g(u)$$

Równanie jednorodne $-2u h_v'(u) + h_v(u) = 0$ rozwiążujemy rozdzielając zmienne $h_v = 2u h_v'$

$$\frac{h_v}{2u} = \frac{dh_v}{du} \qquad \frac{du}{2u} = \frac{dh_v}{h_v} \qquad \frac{du}{u} = 2\frac{dh_v}{h_v}$$

$$\log u = 2\log h_v + C$$

$$h_v^2 = Cu$$

$$h_v = C u^{1/2} \qquad h_v(u) = C(v)\, u^{1/2}$$

Ponieważ prawa strona jest dowolna

otrzymujemy ostatecznie

$$F(u,v) = C(v)\, u^{1/2} + D(u)$$

gdzie C, D są dowolnymi funkcjami klasy C^2

$$f(x,y) = C\left(\frac{y}{x}\right)\sqrt{xy} + D(xy)$$