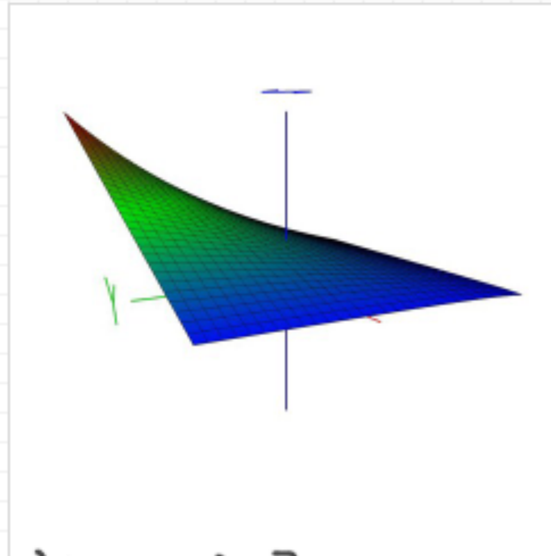
(1)

Znaleźć i zbadać punkty krytyczne funkcji $u(x,y,z)=x+y+z$ na powierzchni danej warunkiem

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1 \quad x>0,\ y>0,\ z>0$$



$$\frac{a}{x}+\frac{b}{y}-1=\frac{c}{z}$$

$$z=\frac{cxy}{ay+bx-xy}$$

$$u'=[1\ 1\ 1]$$

$$G'=\left[-\frac{a}{x^2},-\frac{b}{y^2},-\frac{c}{z^2}\right]$$

$$0=[1\ 1\ 1]-\lambda\left[-\frac{a}{x^2},-\frac{b}{y^2},-\frac{c}{z^2}\right]=\left[1+\frac{\lambda a}{x^2},\ 1+\frac{\lambda b}{y^2},\ 1+\frac{\lambda c}{z^2}\right]$$

$$0=1+\frac{\lambda a}{x^2} \quad 0=1+\frac{\lambda b}{y^2} \quad 0=1+\frac{\lambda c}{z^2}$$

$$x^2=-\lambda a \quad y^2=-\lambda b \quad z^2=-\lambda c$$

$$\lambda<0 \quad \beta=-\lambda$$

$$x^2=\beta a \quad y^2=\beta b \quad z^2=\beta c$$

$$x=\sqrt{\beta a} \quad y=\sqrt{\beta b} \quad z=\sqrt{\beta c}$$

$$\frac{a}{\sqrt{\beta a}}+\frac{b}{\sqrt{\beta b}}+\frac{c}{\sqrt{\beta c}}=1$$

$$\frac{1}{\sqrt{\beta}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)=1 \quad \beta=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2$$

$$x=\sqrt{a}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$y=\sqrt{b}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$z=\sqrt{c}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$G' = \left[-\frac{a}{x^2}, -\frac{b}{y^2}, -\frac{c}{z^2}\right]$$

$\sqrt{a}\sqrt{\beta} = x = \sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c})$

$\sqrt{b}\sqrt{\beta} = y = \sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c})$

$\sqrt{c}\sqrt{\beta} = z = \sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c})$

G' w punkcie krytycznym:

$$G' = \left[-\frac{a}{a\beta}, -\frac{b}{b\beta}, -\frac{c}{c\beta}\right] = -\frac{1}{\beta}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

Wektory styczne:

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\delta x \\ \delta y \\ \delta z\end{bmatrix} = 0 \quad \delta x + \delta y + \delta z = 0 \quad \left\langle \overset{\varphi}{\begin{bmatrix}1 \\ 0 \\ -1\end{bmatrix}}, \overset{\psi}{\begin{bmatrix}1 \\ -1 \\ 0\end{bmatrix}} \right\rangle$$

Druga pochodna w punkcie krytycznym:

$$u'' - \lambda G'', \quad u'' = 0 \qquad G'' = \begin{bmatrix}\frac{2a}{x^3} & & \\ & \frac{2b}{y^3} & \\ & & \frac{2c}{z^3}\end{bmatrix}$$

$$\underset{\Downarrow}{0} + \cancel{\beta}\begin{bmatrix}\frac{2\cancel{a}}{\cancel{a}\sqrt{a}\,\cancel{\beta}\sqrt{\beta}} & & \\ & \frac{2\cancel{b}}{\cancel{b}\sqrt{b}\,\cancel{\beta}\sqrt{\beta}} & \\ & & \frac{2\cancel{c}}{\cancel{c}\sqrt{c}\,\cancel{\beta}\sqrt{\beta}}\end{bmatrix} = \frac{2}{\sqrt{\beta}}\begin{bmatrix}\frac{1}{\sqrt{a}} & & \\ & \frac{1}{\sqrt{b}} & \\ & & \frac{1}{\sqrt{c}}\end{bmatrix}$$

Ogólny wektor styczny do powierzchni:

$$\varphi\begin{bmatrix}1 \\ 0 \\ -1\end{bmatrix} + \psi\begin{bmatrix}1 \\ -1 \\ 0\end{bmatrix} = \begin{bmatrix}\varphi+\psi \\ -\psi \\ -\varphi\end{bmatrix}$$

Druga pochodna na wektorach stycznych:

$$\begin{bmatrix}\varphi+\psi, & -\psi, & -\varphi\end{bmatrix}\frac{2}{\sqrt{\beta}}\begin{bmatrix}\frac{1}{\sqrt{a}} & & \\ & \frac{1}{\sqrt{b}} & \\ & & \frac{1}{\sqrt{c}}\end{bmatrix}\begin{bmatrix}\varphi+\psi \\ -\psi \\ -\varphi\end{bmatrix} = \frac{2}{\sqrt{\beta}}\left((\varphi+\psi)^2\frac{1}{\sqrt{a}} + \varphi^2\frac{1}{\sqrt{c}} + \psi^2\frac{1}{\sqrt{b}}\right) =$$

$$\frac{2}{\sqrt{\beta}}\left[(\varphi^2+2\varphi\psi+\psi^2)\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\psi^2 + \frac{1}{\sqrt{c}}\varphi^2\right]$$

$$\frac{2}{\sqrt{b}}\left[(\varphi^2+2\varphi\psi+\psi^2)\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\psi^2+\frac{1}{\sqrt{c}}\varphi^2\right]$$

dodatnie

$$\left[\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}\right)\varphi^2+\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)\psi^2+\frac{2}{\sqrt{a}}\psi\varphi\right]$$

$$\begin{bmatrix} \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}} & \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{a}} & \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}} \end{bmatrix}$$

$$D_1=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}>0$$

$$D_2=\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{c}}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right)-\left(\frac{1}{\sqrt{a}}\right)^2=\cancel{\left(\frac{1}{\sqrt{a}}\right)^2}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{a}}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)-\cancel{\left(\frac{1}{\sqrt{a}}\right)^2}>0$$

Druga pochodna jest d.o. $\longrightarrow$ MINIMUM

2

Znaleźć punkty krytyczne funkcji $f(x,y,z)=(x-3y)z$ na zbiorze $S=\{(x,y,z):\ 3x^2+5y^2+30z^2=32\}$. Zbadać drugą różniczkę w wybranym punkcie krytycznym.

3

Na elipsie $x^2+4y^2=4$ dane są dwa punkty $a=(-\sqrt{3},\frac{1}{2})$, $b=(1,\frac{\sqrt{3}}{2})$. Znaleźć na tej elipsie trzeci punkt c taki, że trójkąt abc ma możliwie największe pole.

4

Znaleźć i zbadać punkty krytyczne funkcji $f(x,y,z)=x^2+y^2+z^2$ na przecięciu płaszczyzny $x+y+z=0$ i walca $x^2+y^2=2$

(5)

Rozwiązać równanie $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = (y-x)z$ w $D=\{(x,y,z): x>0, y>0, z>0\}$

wprowadzając nowe współrzędne

$u = x^2+y^2$

$v = \log y/x$ i traktując w jako funkcję u i v.

$w = x+y-\log z$

$$w(u(x,y), v(x,y)) = x+y-\log z(x,y) \qquad \Big/\frac{\partial}{\partial x} \qquad \Big/\frac{\partial}{\partial y}$$

$$\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = 1 - \frac{1}{z}\frac{\partial z}{\partial x}$$

$$\frac{\partial w}{\partial u}2x - \frac{1}{x}\frac{\partial w}{\partial v} = 1 - \frac{1}{z}\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -z\left(2x\frac{\partial w}{\partial u} - \frac{1}{x}\frac{\partial w}{\partial v} - 1\right)$$

$$\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = 1 - \frac{1}{z}\frac{\partial z}{\partial y}$$

$$\frac{\partial w}{\partial u}2y + \frac{1}{y}\frac{\partial v}{\partial v} = 1 - \frac{1}{z}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{z}\left(2y\frac{\partial w}{\partial u} + \frac{1}{y}\frac{\partial w}{\partial u} - 1\right)$$

Wstawiamy do równanie

$$-yz\left(2x\frac{\partial w}{\partial u} - \frac{1}{x}\frac{\partial w}{\partial u} - 1\right) + zx\left(2y\frac{\partial w}{\partial u} + \frac{1}{u}\frac{\partial w}{\partial v}\right) = z(y-x)$$

$$\frac{\partial w}{\partial u}\left[-2x\cancel{y}z + 2\cancel{y}xz\right] + \frac{\partial w}{\partial v}\left(\frac{zy}{x} + \frac{zx}{u}\right) + \cancel{2x-2y} = z(\cancel{y-x})$$

$$z\left(\frac{y}{x}+\frac{x}{y}\right)\frac{\partial w}{\partial v} = 0 \longrightarrow \frac{\partial w}{\partial v} = 0 \qquad w = f(u) \qquad f \in C^1(\mathbb{R}_+)$$

$w = x+y-\log z$

$f(x^2+y^2) - (x+y) = -\log z \qquad z = e^x e^y e^{-f(x^2+y^2)}$

(4)

$f(x,y,z) = x^2+y^2+z^2$

$F^1(x,y,z) = x+y+z$

$F^2(x,y,z) = x^2+y^2-2$

$f' - \lambda F^{1'} - \mu F^{2'} = 0$

$[2x \;\; 2y \;\; 2z] - \lambda[1 \;\; 1 \;\; 1] - \mu[2x \;\; 2y \;\; 0] = 0$

$2x - \lambda - 2\mu x = 0 \longrightarrow 2(1-\mu)x = \lambda \longrightarrow x = \frac{\lambda}{2(1-\mu)}$

$2y - \lambda - 2\mu y = 0 \qquad 2(1-\mu)y = \lambda \longrightarrow y = \frac{\lambda}{2(1-\mu)}$

$2z - \lambda = 0 \longrightarrow z = \frac{\lambda}{2}$

jeśli $\mu \neq 1$

$2 \cdot \frac{\lambda}{(1-\mu)2} + \frac{\lambda}{2} = 0$

$\lambda\left(\frac{1}{1-\mu} + \frac{1}{2}\right) = 0$

$\lambda = 0$ lub $\mu = 3$

$\lambda = 0 \rightarrow z = 0$

$\rightarrow x+y=0$

a my mamy $x=y$

czyli $x=y=0$ nie należy do krzywej

$x = \frac{-\lambda}{4} = y$

$z = \frac{\lambda}{2}$

$x^2+y^2 = 2$

$\frac{\lambda^2}{16} + \frac{\lambda^2}{16} = 2 \qquad \lambda^2 = 16 \quad \lambda = \pm 4$

$x=-1$, $y=-1$, $z=2$, $\mu=3$, $\lambda=4$

$x=1$, $y=1$, $z=-2$, $\mu=3$, $\lambda=-4$

Jeśli $\mu = 1$ mamy $\lambda = 0$ i $z=0$. Wtedy $x+y=0$

i z równania $x^2+y^2=2$ dostajemy:

$x=1$ $y=-1$ $z=0$ $\lambda=0$ $\mu=1$, $\quad x=-1$ $y=1$ $z=0$ $\lambda=0$ $\mu=1$

Mamy cztery punkty krytyczne. Jeden z nich możemy zbadać.

$$\begin{array}{l} x=1 \\ y=1 \\ z=-2 \\ \mu=3 \\ \lambda=-4 \end{array}$$

weźmy ten

$$\begin{bmatrix} 2x & 2y & 2z \end{bmatrix} - \lambda \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} - \mu \begin{bmatrix} 2x & 2y & 0 \end{bmatrix}$$

To była pierwsza pochodna, liczymy drugą:

$$\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} - \lambda \cdot 0 - \mu \begin{bmatrix} 2 & & \\ & 2 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} 2-2\mu & & \\ & 2-2\mu & \\ & & 2 \end{bmatrix} = \begin{bmatrix} -4 & & \\ & -4 & \\ & & 2 \end{bmatrix}$$

w wybranym punkcie krytycznym

Potrzebujemy wektory styczne do krzywej:

$$\begin{bmatrix} \underset{2}{2\cdot 1} & \underset{2}{2\cdot 1} & \underset{-4}{2\cdot(-2)} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = 0 \; ; \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = 0$$

$$2\delta x + 2\delta y - 4\delta z = 0$$

$$\left.\begin{cases} \delta x + \delta y - \delta z = 0 \\ \delta x + \delta y + \delta z = 0 \end{cases} \rightarrow \begin{array}{l} \delta z = 0 \\ \delta x + \delta y = 0 \rightarrow \delta x = -\delta y \end{array} \right\} \rightarrow \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 & & \\ & -4 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} = -4-4 = -8 < 0$$

MAKSIMUM