Zadanie 5.

$$\int \frac{1+\sin x}{1+\cos x}\,dx =$$

$t = \operatorname{tg}\frac{x}{2}$ $\quad dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\,dx = (1+t^2)\frac{1}{2}dx$ $\quad dx = \frac{2}{1+t^2}dt$

$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2t}{1+t^2}$ $\quad \cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

$$= \int \frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{t^2+2t+1}{1+t^2+1-t^2}\,\frac{2}{1+t^2}\,dt = \int \frac{(t^2+2t+1)}{1+t^2}\,dt$$

$$= \int \frac{(1+t^2)+2t}{1+t^2}\,dt = \int\left(1+2\frac{t}{1+t^2}\right)dt = t + \log(1+t^2) + C$$

$$= \boxed{\operatorname{tg}\frac{x}{2} + \log\left(1+\operatorname{tg}^2\frac{x}{2}\right)+C}$$

$$\int \frac{\sqrt{x^2-1}}{x^3} = \int \frac{\operatorname{sh}^2 t}{\operatorname{ch}^3 t}\,dt = \int \frac{8u^2}{(1+u^2)^3}\,du = \left| \begin{array}{ll} f(u)=u & g'(u)=\frac{4u}{(1-u^2)^3} \\ f'(u)=1 & g(u)=-\frac{1}{(1-u^2)^2} \end{array} \right|$$

$x = \operatorname{ch} t$ $\quad dx = \operatorname{sh} t\,dt$

$u = \operatorname{th}\frac{t}{2}$ $\quad dt = \frac{2du}{1-u^2}$

$\operatorname{sh} t = \frac{2u}{1-u^2}$ $\quad \operatorname{ch} t = \frac{1+u^2}{1-u^2}$

$$-\frac{2u}{(1+u^2)^2} + 2\int \frac{du}{(1+u^2)^2} = -\frac{2u}{(1+u^2)^2} + 2\int \frac{1+u^2-u^2}{(1+u^2)^2}\,du = -\frac{2u}{(1+u^2)^2} + 2\operatorname{arctg} u +$$

$$-\int u\frac{2u}{(1+u^2)^2} = \left| \begin{array}{ll} u & \frac{2u}{(1+u^2)^2} \\ 1 & -\frac{1}{1+u^2} \end{array} \right| = -\frac{2u}{(1+u^2)^2} + 2\operatorname{arctg} u - \left(-\frac{u}{1+u^2} + \int \frac{du}{u^2+1}\right) =$$

$$= -\frac{2u}{(1+u^2)^2} + 2\operatorname{arctg} u + \frac{u}{1+u^2} - \operatorname{arctg} u + C$$

$$= \underbrace{-\frac{2u}{(1+u^2)^2} + \frac{u}{1+u^2}} + \operatorname{arctg} u + c$$

$$\frac{u}{1+u^2}\left(1-\frac{2}{1+u^2}\right) = \frac{u}{1+u^2}\,\frac{1+u^2-2}{1+u^2} = \frac{u\,(u^2-1)}{1+u^2}$$

$$x = \operatorname{ch} t = \frac{1+u^2}{1-u^2} \quad x(1-u^2) = 1+u^2 \quad x - xu^2 = 1+u^2 \quad x-1 = u^2(1+x) \quad u^2 = \frac{x-1}{x+1}$$

$$1+u^2 = 1+\frac{x-1}{x+1} = \frac{x+1+x-1}{x+1} = \frac{2x}{x+1}$$

$$u^2-1 = \frac{x-1}{x+1} - 1 = \frac{x-1-x-1}{x+1} = \frac{-2}{x+1}$$

$$u^2 = \frac{x-1}{x+1} = \frac{(x-1)^2}{x^2-1} \Rightarrow u = \frac{x-1}{\sqrt{x^2-1}}$$

$$\frac{u\,(u^2-1)}{(1+u^2)^2} = \frac{x-1}{\sqrt{x^2-1}}\,\frac{(-\cancel{2})\,(x+1)^{\cancel{2}}}{\cancel{(x+1)}\,(\underset{2}{\cancel{4}}x^2)} = -\frac{\overbrace{(x-1)(x+1)}^{x^2-1}}{\sqrt{x^2-1}\;2x^2} = -\frac{\sqrt{x^2-1}}{2x^2}$$

$$\boxed{= -\frac{\sqrt{x^2-1}}{2x^2} + \operatorname{arctg}\frac{x-1}{\sqrt{x^2-1}} + C}$$

Zadanie 4

$\sum_{n=1}^{\infty} \frac{1}{n\log^2 n}$ Stosujemy lemat o zagęszczaniu, który każe sprawdzić zbieżności szeregu $\sum 2^k a_{2^k}$

$$\cancel{2^k}\frac{1}{\cancel{2^k}[\log 2^k]^2} = \frac{1}{(k\log 2)^2} = \frac{1}{\log^2 2\, k} = b_k$$

szereg $\sum b_k$ jest zbieżny, zatem na mocy kryterium zagęszczeniowego $\sum \frac{1}{n\log^2 n}$ też jest zbieżny

$\sum \left(1-\frac{1}{\sqrt{n}}\right)^n$ a_n zachowuje się najprawdopodobniej jak $e^{-\sqrt{n}}$. Sprawdźmy, czy szereg $e^{-\sqrt{n}}$ jest zbieżny?

Wiadomo, że

$\lim_{n\to\infty} n^2 e^{-\sqrt{n}} = 0$ tzn, dla wystarczająco dużych n $n^2 e^{-\sqrt{n}} < 1$

zatem $e^{-\sqrt{n}} < \frac{1}{n^2}$ szereg $\sum e^{-\sqrt{n}}$ jest więc zbieżny na mocy I-go kryterium porównawczego. Pozostaje zbadać granicę

$$x_n = e^{\sqrt{n}}\left(1-\frac{1}{\sqrt{n}}\right)^n \quad \log x_n = \sqrt{n} + n\log\left(1-\frac{1}{\sqrt{n}}\right) \simeq \sqrt{n} + n\left(-\frac{1}{\sqrt{n}} + \frac{2}{n} - \dots\right) =$$

$$\sqrt{n} - \sqrt{n} + \frac{2}{\sqrt{n}} - \dots \xrightarrow[n\to\infty]{} 0$$

$\log x_n \to 0$ $\boxed{x_n \to 1.}$ Wyjściowy szereg jest więc zbieżny na mocy drugiego kryterium porównawczego z szeregiem $\sum e^{-\sqrt{n}}$

Zadanie 3

$$I_n = \int_0^1 x^{2n-1} e^{x^2} = \frac{1}{2} e^{x^2}\left(x^{2n-2}\right)\Big|_0^1 - \frac{1}{2}(2n-2)\underbrace{\int_0^1 x^{2n-3} e^{x^2}\,dx}_{I_{n-1}} =$$

$$f'(x) = x e^{x^2} \qquad g(x) = x^{2n-2}$$

$$f(x) = \frac{1}{2} e^{x^2} \qquad g'(x) = (2n-2)x^{2n-3}$$

$$= \frac{1}{2} e \cdot 1 - \frac{1}{2} e^0 \cdot 0 - I_{n-1} = \frac{1}{2} e - (n-1) I_{n-1}$$

$$I_1 = \int_0^1 x e^{+x^2} = \frac{1}{2} e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1)$$

$$I_n = \frac{1}{2}e - (n-1) I_{n-1} = \frac{1}{2} e - (n-1)\left[\frac{1}{2} e - (n-2) I_{n-2}\right] = \frac{1}{2} e - (n-1)\frac{1}{2} e +$$

$$(n-1)(n-2)\left[\frac{1}{2} e - (n-3) I_{n-3}\right] = \frac{1}{2} e - \frac{(n-1)!}{(n-2)!}\frac{1}{2} e + \frac{(n-1)!}{(n-3)!}\frac{1}{2} e - \dots$$

$$\pm (n-1)(n-2)\cdots 2\left[\frac{1}{2} e - 1 \cdot I_1\right] = \frac{1}{2} e\left[1 - \frac{(n-1)!}{(n-2)!} + \frac{(n-1)!}{(n-3)!} - \dots \pm \frac{(n-1)!}{1!}\right] \mp$$

$$\mp (n-1)!\, I_1 =$$

$$= \frac{1}{2} e \sum_{k=1}^{n-1} (-1)^{k-1} \frac{(n-1)!}{(n-k)!} + (-1)^{n-1}(n-1)!\left[\frac{1}{2} e - \frac{1}{2}\right] =$$

$$= \frac{1}{2} e \sum_{k=1}^{n} (-1)^{k-1} \frac{(n-1)!}{(n-k)!} + (-1)^n (n-1)!\, \frac{1}{2}$$

$$I_n = \frac{(n-1)!\, e}{2} \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{(n-k)!} + \frac{(-1)^n}{2}(n-1)!$$

Zadanie 2

$$g(x) = \begin{cases} ax+b & x \le 0 \\ \left(\frac{\arcsin x}{x}\right)^{1/x^2} & x > 0 \end{cases} \qquad \lim_{x\to 0^-} g(x) = b$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \left(\frac{\arcsin x}{x}\right)^{1/x^2} = \ ?$$

liczymy $\lim_{x\to 0^+} \log g(x) = \lim_{x\to 0^+} \frac{1}{x^2}\log\left(\frac{\arcsin x}{x}\right)$ potrzebne jest rozwinięcie w szereg arcsinx

$$\arcsin x\Big|_{x=0} = 0 \qquad \arcsin' x = \frac{1}{\sqrt{1-x^2}}\Big|_{x=0} = 1 \qquad \arcsin'' x = \left(\left(1-x^2\right)^{-1/2}\right)' =$$

$$+\frac{1}{\cancel{2}}\left(1-x^2\right)^{-3/2}\cdot \cancel{2}x = \frac{x}{(1-x^2)^{3/2}}\Big|_{x=0} = 0$$

$$\arcsin^{(3)} x = \left[x\left(1-x^2\right)^{-3/2}\right]' = \left(1-x^2\right)^{-3/2} +$$

$$x\left(-\frac{3}{\cancel{2}}\right)\left(1-x^2\right)^{-5/2}\left(-\cancel{2}x\right) = \left(x^2-1\right)^{-3/2} + 3x^2\left(x^2-1\right)^{-5/2}\Big|_{x=0} = 1.$$

$$\arcsin x = 0 + x + \frac{1}{2!}\cdot 0\cdot x^2 + \frac{1}{3!}\cdot 1\cdot x^3 =$$

$$= x + \frac{1}{6}x^3$$

$$\lim \frac{1}{x^2}\log\left(\frac{x+\frac{1}{6}x^3}{x}\right) = \lim_{x\to 0^+} \frac{1}{x^2}\log\left(1+\frac{x^2}{6}\right) = \frac{1}{6}$$

Funkcja g jest ciągła jeśli $b = e^{1/6}$

$$g'(0) = \lim_{t\to 0} \frac{g(t)-g(0)}{t} = \lim_{t\to 0} \frac{g(t)-e^{1/6}}{t}$$

$$t\to 0^- \quad \frac{at+e^{1/6}-e^{1/6}}{t} = a$$

$$\lim_{t\to 0^+} \frac{g(t)-e^{1/6}}{t} = \lim_{t\to 0^+} \frac{\left(\frac{\arcsin t}{t}\right)^{1/t^2} - e^{1/6}}{t}$$

$$\frac{1}{t^2}\log\left(\frac{\arcsin t}{t}\right) = \frac{1}{t^2}\log\left(\frac{t+\frac{1}{6}t^3+\frac{3}{40}t^5}{t}\right) = \frac{1}{t^2}\left(\frac{1}{6}t^2+\frac{3}{40}t^4-\frac{1}{2}\left(\frac{1}{36}t^4\right)\right) =$$

$$\frac{1}{6}+\left(\frac{3}{40}-\frac{1}{72}\right)t^2 = \frac{1}{6}+\left(\frac{27}{360}-\frac{5}{360}\right)t^2 = \frac{1}{6}+\frac{11}{180}t^2$$

$$\exp(\cdots) = e^{1/6}\, e^{11/180\, t^2} = e^{1/6}\left(1+\frac{11}{180}t^2\right)$$

$$\frac{e^{1/6}\left(1+\frac{11}{180}t^2\right)-e^{1/6}}{t} \xrightarrow[t\to 0.]{} 0$$

$$a=0$$

Zadanie 1.

$$f(x) = e^{-\frac{1}{x^2-4}}$$

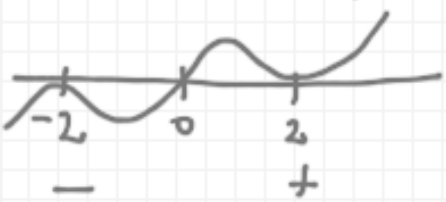
(1) Dziedzina funkcji jest $\mathbb{R} \setminus \{\pm 2\}$

$\lim\limits_{x\to-\infty} f(x) = 1$ $\quad \lim\limits_{x\to+\infty} f(x) = 1$ $\quad \lim\limits_{x\to-2^-} f(x) = 0 = \lim\limits_{x\to 2^+} f(x)$ $\quad \lim\limits_{x\to-2^+} f(x) = +\infty = \lim\limits_{x\to 2^-} f(x)$

$f(0) = \sqrt[4]{e}$

(2) Pochodna i monotoniczność

$$f'(x) = \exp\left(-\frac{1}{x^2-4}\right)\left(-\frac{2x(-1)}{(x^2-4)^2}\right) = \frac{2x}{(x^2-4)^2}\overbrace{\exp\left(-\frac{1}{x^2-4}\right)}^{>0}$$



funkcja maleje dla $x<0$ i rośnie dla $x>0$

minimum w $x=0$;

ponadto granice pochodnej są zero przy -2^- i 2^+

(3) Druga pochodna i wypukłość

$2x^4-16x^2+32-8x^4+32x^2$

$$\left\{\frac{2x}{(x^2-4)^2}\exp\left(-\frac{1}{x^2-4}\right)\right\}' = \left(\frac{2x}{(x^2-4)^2}\right)^2\exp\left(-\frac{1}{x^2-4}\right) + \exp\left(-\frac{1}{x^2-4}\right)\frac{2(x^2-4)^2-2(x^2-4)2x\cdot 2x}{(x^2-4)^4}$$

$$\exp\left(-\frac{1}{x^2-4}\right)\frac{1}{(x^2-4)^4}\left[4x^2+2x^4-16x^2+32-8x^4+32x^2\right] =$$

razy dodatnie $\cdot\left[-6x^4+20x^2+32\right] = -6(x-x_0)(x+x_0)\left(x^2-\frac{5-\sqrt{73}}{3}\right)$

$+3x^4-10x^2-16$

$\Delta = 100+4\cdot16\cdot3 = 100+16\cdot12 = 100+160+32 = 292 =$
$= 4\cdot 73$ $\quad 2\sqrt{73}$

$$x^2 = \frac{10+2\sqrt{73}}{6} = \frac{5+\sqrt{73}}{3}$$

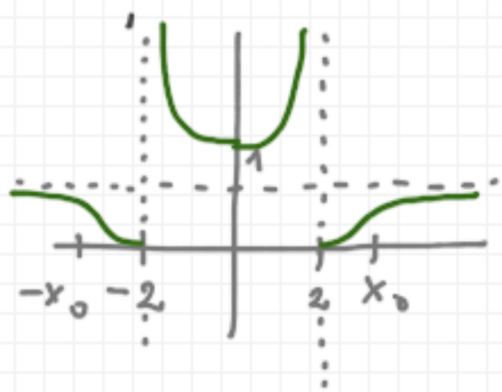
punkty przegięcia są w $x = \pm\sqrt{\frac{5+\sqrt{73}}{3}}$

$$\frac{5+\sqrt{73}}{3} < \frac{5+9}{3} = \frac{14}{3} = 4\tfrac{2}{3}$$

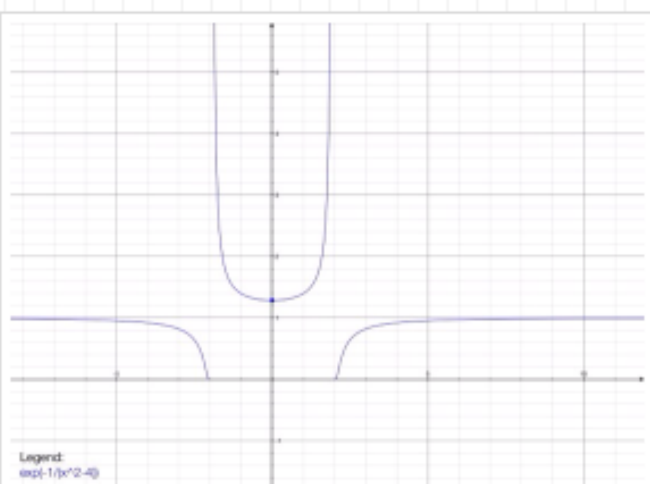
$$\frac{5+\sqrt{73}}{3} > \frac{5+8}{3} = \frac{13}{3} = 4\tfrac{1}{3} \qquad x_0 > 2$$

punkty char:	$-\infty$		$-x_0$		-2		0		2		x_0		$+\infty$
pochodna		$- - -$		$- - -$	$0 \mid$	$- - -$	0	$+ + +$	$\mid 0$	$+ + +$		$+ +$	
2 pochodna		$- - -$	0	$+ + +$	$0 \mid$	$+ + +$		$+ + +$	$\mid 0$	$+ +$	0	$- - -$	
funkcja	1	$\searrow$		$\searrow$	$0 \mid +\infty$	$\searrow$	$\sqrt[4]{e}$	$\nearrow$	$+\infty \mid 0$	$\nearrow$		$\nearrow$	1

Szkic wykresu



Wykres narysowany przez narzędzie elektroniczne



Fragment w pobliżu $x=2$ w powiększeniu