WZÓR STIRLINGA

WYKŁAD 11

1962 Stirling –

– 1770 Edinburgh

TWIERDZENIE: Dla $\varepsilon>0$ i $-\frac{\pi}{2}+\varepsilon<\arg z<\frac{\pi}{2}-\varepsilon$ zachodzi wzór

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$$\lim_{z\to\infty}\frac{\Gamma(z)}{z^{z-1/2}e^{-z}\sqrt{2\pi}}=1$$

UWAGI: Wzór Stirlinga wyprowadzać można na tysiąc sposobów. My zajmiemy się dwoma. Pierwszy z nich jest dość rachunkowy. Po drodze pojawiają się ciekawe wzory. Drugi wynika z ogólniejszej metody znajdowania asymptotycznego zachowania pewnych funkcji zwanej metodą punktu siodłowego.

DOWÓD $\varphi(z)=\frac{\Gamma(z)}{z^{z-1/2}e^{-z}\sqrt{2\pi}}$ Zamiast badać granicę φ zajmiemy się logarytmem φ

$$\log\varphi=\log\Gamma(z)-\left[\log\left(z^{z-1/2}e^{-z}\sqrt{2\pi}\right)\right]=\log\Gamma(z)-\left[\left(z-\tfrac{1}{2}\right)\log z-z+\tfrac{1}{2}\log(2\pi)\right]$$

Dowodzić więc mamy, że

$$\lim_{z\to\infty}\left[\log\Gamma(z)-\left(z-\tfrac{1}{2}\right)\log z+z-\tfrac{1}{2}\log(2\pi)\right]=0$$

czyli $\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log(2\pi)$ + coś dążące do zera. Co to jest to coś to się okaże później.

Korzystamy ze wzoru Weierstrassa: $\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)$

$$\log\Gamma(z)=-\log z-\gamma z+\sum_{n=1}^{\infty}\left[\frac{z}{n}-\log\left(1+\frac{z}{n}\right)\right]$$

$$\partial_z\log\Gamma(z)=-\frac{1}{z}-\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n}\frac{1}{1+\frac{z}{n}}\right]=-\frac{1}{z}-\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+z}\right]$$

$$\partial_z^2\log\Gamma(z)=\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{1}{(n+z)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}$$

szereg zbieżny jednostajnie

przypominamy twierdzenie o różniczkowaniu szeregów funkcyjnych wyraz po wyrazie.

$$\left|\frac{1}{n+z}\right|^2=\left|\frac{1}{(n+z)(n+\bar z)}\right|=\left|\frac{1}{n^2+n2\operatorname{Re}z+|z|^2}\right|\leq\left|\frac{1}{n^2+2n\operatorname{Re}z}\right|\underset{\operatorname{Re}z>0}{\leq}\frac{1}{n^2}$$

Trick rachunkowy polega na zastąpieniu $\frac{1}{(n+z)^2}$ pewną całką z parametrami z, n **2**

$$\int_0^\infty t e^{-t(z+n)}\,dt = t e^{-t(z+n)}\Big|_0^\infty + \int_0^\infty \frac{1}{z+n} e^{-t(z+n)}\,dt = -\frac{1}{(z+n)^2} e^{-t(z+n)}\Big|_0^\infty = \frac{1}{(z+n)^2}$$

$$t \qquad e^{-t(z+n)}$$
$$1 \qquad -(z+n)\,e^{-t(z+n)}$$

$$\partial_z^2 \log z = \sum_{n=0}^\infty \int_0^\infty t e^{-t(z+n)}\,dt = \int_0^\infty t \sum_{n=0}^\infty e^{-t(z+n)}\,dt = \int_0^\infty t e^{-tz}\frac{1}{1-e^{-t}}\,dt = \int_0^\infty \frac{t e^{-tz}}{1-e^{-t}}\,dt$$

całka zbieżna jednostajnie ze względu na n

$$|t e^{-tz} e^{-tn}| \underset{t\geq 0}{=} t e^{-t\operatorname{Re} z} e^{-tn} \leq t e^{-t\operatorname{Re} z}$$

$$\partial_z^2 \log\Gamma(z) = \int_0^\infty \frac{t e^{-tz}}{1-e^{-t}}\,dt$$ Korzystając z tego wzoru „schodzimy" z rzędem pochodnej

$$\partial_z \log\Gamma(z) = \partial_z \log\Gamma(1) + \int_1^z\left(\int_0^\infty \frac{t e^{-ty}}{1-e^{-t}}\,dt\right)dy$$

$$\boxed{-\frac{1}{z} - \gamma + \sum_{n=1}^\infty\left[\frac{1}{n} - \frac{1}{n+z}\right]}$$

$$-1-\gamma+\sum_{n=1}^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right) = -\gamma$$

$$t e^{-ty}\left(\frac{1}{1-e^{-t}} - \frac{1}{t} + \frac{1}{t}\right) = e^{-tz}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) + e^{-tz}$$

$$\int_0^\infty \dots = \int_0^\infty t e^{-ty}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt + \frac{1}{y}$$

$$-\gamma + \int_1^z\left(\int_0^\infty t e^{-tz}\left(\frac{1}{1-e^{t}} - \frac{1}{t}\right)dt\right)dy + \log z$$

$$= \int_0^\infty\left[t\left(\frac{1}{1-e^{t}} - \frac{1}{t}\right)\int_1^z e^{-ty}\,dy\right]dt = \int_0^\infty\left(\frac{1}{1-e^{t}} - \frac{1}{t}\right)e^{-tz}\,dt - \int_0^\infty\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}\,dt$$

zamiana kolejności całkowania jest dozwolona gdyż w ∞ mamy

$$\left|t\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-tz}\right| = |t(\dots)|\,e^{-\operatorname{Re} z\, t} \leq |t(\dots)|\,e^{-t}$$ a w zerze

$$\lim_{t\to 0} \underbrace{t\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)}_{(t-1+e^{-t})/t(1-e^{-t}) = \left(t-1+1-t+\frac{t^2}{2}-\dots\right)/t(1-1+t\dots) \xrightarrow{t\to 0} \frac{1}{2}} e^{-t} = 0$$

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Ostatecznie

$$\partial_z \log\Gamma(z) = -\gamma + \log z - \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-tz}\,dt + \underbrace{\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}\,dt}$$

Okazuje się, że ostatnia całka upraszcza się z γ.

Istotnie:

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n}\frac{1}{k}} - \underbrace{\log(n+1)}$$

Wzór znany z zadań z całek z parametrem:

$$\log z = \int_0^\infty \frac{e^{-t} - e^{-zt}}{t}\,dt$$

(Wzory Froullaniego)

$$\frac{1}{k} = \int_0^\infty e^{-kt}\,dt$$

$$\sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\int_0^\infty e^{-kt}\,dt = \int_0^\infty \left(\sum_{k=1}^{n} e^{-kt}\right)dt = \int_0^\infty \frac{1-e^{-nt}}{1-e^{-t}}\,e^{-t}\,dt$$

Mamy:

$$\gamma = \lim_{n\to\infty} \dots = \lim_{n\to\infty} \int_0^\infty \left[\frac{1-e^{-nt}}{1-e^{-t}}e^{-t} - \frac{e^{-t} - e^{-(n+1)t}}{t}\right]dt = \lim_{n\to\infty}\left\{\int_0^\infty \left(\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t}\right)dt + \int_0^\infty e^{-(n+1)t}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)dt\right\} = \int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt$$

ma skończoną gr. w zerze

$$\partial_z \log\Gamma(z) = \log z - \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-tz}\,dt$$

Całkujemy dalej:

$$\log\Gamma(z) = \log\Gamma\left(\tfrac{1}{2}\right) - \int_{\frac{1}{2}}^{z}\left(\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-tu}\,dt\right)du$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\log\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\log\pi$$

$$\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)e^{-tu} + \frac{1}{2}e^{-tu}$$

$$\int_0^\infty (\dots) + \frac{1}{2u}$$

$$\log\Gamma(z) = \frac{1}{2}\log\pi + \int_{1/2}^{z}\log y\,dy - \frac{1}{2}\int_{1/2}^{z}\frac{1}{u}\,du - \int_{1/2}^{z}\Big[\int_0^\infty (\ \cdots\ )e^{-tu}\,dt\Big]du =$$

$$= \frac{1}{2}\log\pi + (u\log u - u)\Big|_{1/2}^{z} - \frac{1}{2}\log z - \frac{1}{2}\log\frac{1}{2} - \cdots =$$

$$= \frac{1}{2}\log\pi + \underbrace{z\log z} - z - \frac{1}{2}\cancel{\log\frac{1}{2}} + \frac{1}{2} - \underbrace{\frac{1}{2}\log z} + \frac{1}{2}\cancel{\log\frac{1}{2}} + \cdots$$

$$\left(z - \frac{1}{2}\log z\right) - z + \frac{1}{2} + \frac{1}{2}\log\pi - \int_0^\infty\Big[\Big(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\int_{1/2}^{z} e^{-tu}\,du\Big]dt$$

$$-\frac{1}{t}e^{-zt} + \frac{1}{t}e^{-t/2}$$

$$\int_0^\infty\Big(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\frac{1}{t}e^{-t/2} = \frac{1}{2} + \frac{1}{2}\log\frac{1}{2}$$

Wzór Pringsheime

$$= \left(z - \frac{1}{2}\log z\right) - z + \cancel{\frac{1}{2}} + \frac{1}{2}\log\pi - \cancel{\frac{1}{2}} - \frac{1}{2}\log\frac{1}{2} + I(z) =$$

$$\frac{1}{2}\log 2\pi$$

$$= \left(z - \frac{1}{2}\log z\right) - z + \frac{1}{2}\log 2\pi + \int_0^\infty\Big(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\frac{e^{-tz}}{t}\,dt$$

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Wyprowadzenie wzoru Pringsheime

$$\int_0^\infty\Big(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\Big)\frac{e^{-t/2}}{t}\,dt = \int_0^\infty\Big(\frac{1 + e^{-t/2} - e^{-t/2}}{\underbrace{1-e^{-t}}} - \frac{1}{t} - \underbrace{\frac{1}{2}}\Big)\frac{e^{-t/2}}{t}\,dt =$$

$$\int_0^\infty\Big(\underbrace{\frac{\cancel{2}^{\,1} + 2e^{-t/2} - \cancel{1} + e^{-t}}{2(1-e^{-t})}} - \frac{e^{-t/2}}{1-e^{-t}} - \frac{1}{t}\Big)\frac{e^{-t/2}}{t}\,dt = \int_0^\infty\Big[\frac{(1+e^{-t/2})^2}{2(1-e^{-t/2})(1+e^{-t/2})}e^{-t/2} - \frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t/2}}{t}\Big]\frac{dt}{t}$$

$$= \int_0^\infty \frac{(1+e^{-t/2})^2}{2(1-e^{-t/2})(1+e^{-t/2})} e^{-t/2} - \frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t/2}}{t}\Big]\frac{dt}{t}$$

$$= \int_0^\infty \Big[\frac{1-e^{-t/2}}{2(1+e^{-t/2})} e^{-t/2} - \frac{e^{-t/2}}{t} - \frac{e^{-t}}{1-e^{-t}}\Big] dt =$$

$$= \int_0^\infty \Big[\frac{1+e^{-t/2}}{2(1-e^{-t/2})} e^{-t/2} - \frac{2e^{-t/2}}{t}\Big]\frac{dt}{t} + \int_0^\infty \Big[-\frac{e^{-t}}{1-e^{-t}} + \frac{e^{-t/2}}{t}\Big] dt =$$

$\uparrow\ u = -t/2$

$$= \int_0^\infty \Big[\frac{1+e^{-u}}{2(1-e^{-u})} e^{-u} - \frac{e^{-u}}{u}\Big]\frac{du}{u} + \int_0^\infty \dots$$

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$$= \int_0^\infty \Big[\frac{\cancel{e^{-u}} + e^{-2u} - \cancel{2}e^{-u}}{2(1-e^{-u})} - \frac{e^{-u}}{u} + \frac{e^{-u/2}}{u}\Big]\frac{du}{u} = \int_0^\infty \Big[\frac{e^{-u}(\cancel{e^{-u}-1})}{2\cancel{(1-e^{-u})}} + \frac{e^{-u/2}-e^{-u}}{u}\Big]\frac{du}{u}$$

$$\int_0^\infty \Big[\frac{e^{-u/2}-e^{-u}}{u} - \frac{1}{2} e^{-u}\Big] du/u = -\int_0^\infty \frac{d}{du}\Big(\frac{e^{-u/2}-e^{-u}}{u}\Big) du + \int_0^\infty \Big(\frac{e^{-u}}{u} - \frac{1}{2u}e^{-u/2} - \frac{1}{2}e^{-u}\Big) du$$

$$\frac{d}{du}\Big(\frac{e^{-u/2}-e^{-u}}{u}\Big) = \frac{(-\frac{1}{2}e^{-u/2}+e^{-u})u - e^{-u/2} + e^{-u}}{u^2} = \frac{e^{-u} - \frac{1}{2}e^{-u/2}}{u} + \frac{e^{-u} - e^{-u/2}}{u^2}$$

$$= +\lim_{u\to 0}\Big(\frac{e^{-u/2}-e^{-u}}{u}\Big) + \int_0^\infty \frac{e^{-u}-e^{-u/2}}{2u} du = \frac{1}{2} + \frac{1}{2}\log\frac{1}{2}$$

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} dx = \log\left(\frac{b}{a}\right)$$

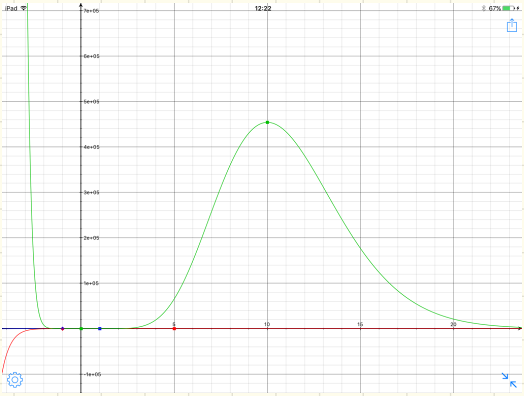
Wzory Froullaniego.

METODA PUNKTU SIODŁOWEGO

(wprowadzenie rzeczywiste)

Rozważmy $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ dla $x \in \mathbb{R}$ $x > 0$ x duże

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} = \int_0^\infty \exp\left(x \log t - t \right) dt$$



$$f(t) = x \log t - t$$

$$f'(t) = x/t - 1$$

$$f'(t) = 0 \iff t = x$$

$$f''(t) = -\frac{x}{t^2}$$

$$f''(x) = -\frac{1}{x} < 0 \quad \text{i.e. max}$$

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x) (t-x)^2 + \dots =$$

$$= x \log x - x - \frac{1}{2} \frac{(t-x)^2}{x} + \dots$$

$$\exp(f(t)) \simeq x^x e^{-x} \exp\left(-\frac{1}{2x} (t-x)^2 \right)$$

$$\int_0^\infty \exp(f(t))\, dt \simeq x^x e^{-x} \int_{x-\delta}^{x+\delta} \exp\left(-\frac{(t-x)^2}{2x} \right) dt \simeq x^x e^{-x} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-x)^2}{2x} \right) dt$$

$$= x^x e^{-x} \sqrt{2\pi x} = x^{x+1/2} e^{-x} \sqrt{2\pi}$$

$y = x+1 \qquad x = y-1$

$\Gamma(y) \simeq (y-1)^{y-1/2} e^{-y+1} \sqrt{2\pi}$ niemal wzór Stirlinga

$$\Gamma(y) \simeq (y-1)^{y-\frac{1}{2}} e^{-y+1} \sqrt{2\pi}$$ niemal wzór Stirlinga

$$(y-1)^{y-\frac{1}{2}} = \exp\left(\left(y-\tfrac{1}{2}\right)\log(y-1)\right) \simeq \exp\left(\left(y-\tfrac{1}{2}\right)\left[\log y - \tfrac{1}{y}\right]\right) \simeq$$

$$\exp\left(\left(y-\tfrac{1}{2}\right)\log y\right)\exp\left(-1+\frac{1}{2y}\right) = y^{y-\frac{1}{2}}\,\frac{1}{e}\,e^{1/2y} \approx y^{y-\frac{1}{2}}\,\frac{1}{e}$$

$$\approx y^{y-\frac{1}{2}} e^{-y} \sqrt{2\pi}$$ całkiem wzór Stirlinga

Zadanie na po świętach: uogólnić i uszczegółowić
tę metodę wraz z miarą błędu!