A sketch of category theoretic interface for experiment-theory relationship

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Abstract

What is the meaning of the terms *experiment* and *experimental verification*? Can they be formalised mathematically? Consideration of an experiment as something exhaustively specified by a single *set* $\mathcal{X}$ of (qualitative) elementary events and a single *set* Θ of (quantitative) parameters does not give any account for many aspects of experiments that are implicitly assumed in their construction and application, and which are required for their intersubjective reproducibility (validation). In particular, such definition does not provide any description of the necessary and sufficient conditions that allow to consider various different experimental situations as instances of the same 'experiment of a given type'. But without such description the notion of an experiment is a useless abstraction.

As a replacement for this perspective, we propose a category theoretic framework for definition of an 'experiment of a given type', as well as categorical description of relationship between predictive theory and experiment. The latter can be understood either in terms of restrictions on theoretical model construction that are introduced by the experiments of a given type, or in terms of restrictions on the experiments of a given type that are imposed by a given theoretical model.

1 Mathematical motivation

Aiming at a unified treatment of information theory in all regimes, including the non-commutative infinite-dimensional case, we have abandoned the reliance on the notion of *probability* and the spaces of probability measures on measurable spaces $(\mathcal{X}, \mho(\mathcal{X}))$ in favour of the notion of *information* and the spaces $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_*^+$ of quantum information states on W^*-algebras $\mathcal{N}$. As a consequence of this change, the Fisher–Kolmogorov [6, 9] definition of a probabilistic model $\mathcal{M}(\mathcal{X}, \mho(\mathcal{X}), \mu)$ as a subspace of the space $\mathcal{P}(\mathcal{X}, \mho(\mathcal{X}), \mu)$ of all probability measures on a countably additive algebra $\mho(\mathcal{X})$ of subsets of the "sample" set $\mathcal{X}$ dominated by a measure μ on $\mho(\mathcal{X})$, as well as related definition of a parametric probabilistic model as a subset $\Theta \times \mathcal{M}(\mathcal{X}, \mho(\mathcal{X}), \mu) \subseteq \mathbb{R}^n \times \mathcal{P}(\mathcal{X}, \mho(\mathcal{X}), \mu)$ are inapplicable. The reasons are purely mathematical: 1) in the commutative case there are many inequivalent spaces $(\mathcal{X}, \mho(\mathcal{X}))$ that represent the same mcb-algebra $\mho$ and there are many inequivalent countably additive boolean algebras $\mho(\mathcal{X})$ of subsets of $\mathcal{X}$ that can be associated with a given space $\mathcal{X}$; 2) in the non-commutative case there is simply no $\mathcal{X}$ that could be associated either with some $\omega \in \mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_*^+$ or with $\mathcal{M}(\mathcal{N})$ itself; 3) the infinite-dimensional models cannot be parametrised by $\Theta \subseteq \mathbb{R}^n$.

There is however an important aspect of the *use* of sample space $\mathcal{X}$ and the parameter space Θ in commutative statistical theory (as well as in applications of Hilbert space based approach to quantum theory) that should be incorporated into new framework: it is the relationship between $\mathcal{X}$, Θ and $\mathcal{M}$ that specifies the allowed range of variability of quantitative models $\Theta \times \mathcal{M}$ imposed by the range of variability of qualitative objects $\mathcal{X}$. For example, this can be specified as invariance of Θ and $\mathcal{M}$ under particular group of transformations of $\mathcal{X}$. In general, the relationships between $\mathcal{X}$, Θ, and $\mathcal{M}(\mathcal{X}, \mho(\mathcal{X}), \mu) \subseteq L_1(\mathcal{X}, \mho(\mathcal{X}), \mu)$ serve as a main tool for specification and control of construction of families of models $\mathcal{M}$ that are *well-behaved* under some classes of transformations of the sample space and parameter space.

Actually, in the large part of applications of probability theory to statistical inference, the sample space $\mathcal{X}$ and the parameter space Θ are used to *define* the particular *experimental situation* under consideration, which is understood as an individual instance of an 'experiment of a given type'. The behaviour of relationship between $\mathcal{X}$, Θ and $\mathcal{M}$ under the changes of $\mathcal{X}$ and Θ represents the particular *type* of experimental situation under consideration, as well as intended character of relationship between experiment of a given type and an associated theoretical model. So, if we replace $\mathcal{X}$, Θ, and $\mathcal{M}(\mathcal{X},\mho(\mathcal{X}),\mu)$ by some new mathematical structure, then we have to specify how this structure *defines* the 'experiment of a given type' and how it defines the predictive verifiability of the relationship between experiment and a theory. In particular, the passage from finite-dimensional commutative normalised setting to infinite-dimensional non-commutative finite-positive setting requires us to provide such replacement for these relationships, which would preserve good behaviour of models arising in the new construction and would also include the standard commutative approach as a special case.

The similar conclusion can be drawn by analysing von Neumann's approach to quantitative specification of quantum theoretic models, which is based on spectral representation of commutative subalgebras of algebra $\mathfrak{B}(\mathcal{H})$ over some Hilbert space $\mathcal{H}$ in terms of functions of $L_\infty(\mathcal{X},\mho(\mathcal{X}),\mu)$ acting on some $L_2(\mathcal{X},\mho(\mathcal{X}),\mu)$ space. This approach assumes that experimental situations can be completely described in terms of orthogonal projections on $\mathcal{H}$, and provides no principles of specification of particular non-degenerate density matrices corresponding to a given experimental situation. The semi-spectral approach to quantum measurement and estimation theory showed that the choice of particular operators and density matrices (hence, spaces $\mathcal{N}(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H})$ and $\mathcal{M}(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H})^+_*$) over some suitable Hilbert space $\mathcal{H}$ depends crucially on the detailed qualitative and quantitative description of a particular experimental setting under consideration, and it might be ambiguous and ill-defined if such description is not provided (see [2, 7, 4, 3, 5] for details). In order to encompass the complexity of operational description of experimental situation, the description in terms of spectral measures is relaxed to description in terms of semi-spectral measures, which are constructed by appealing to some operational considerations. However, in general there is no bijection between semi-spectral measures and self-adjoint operators. This undermines justification of the spectral theorem as an underlying principle of construction of quantum theoretic models (as well as consideration of self-adjoint operators in particular Hilbert space representation as 'observables'), but does not provide new mathematical framework. Moreover, the semi-spectral approach does not provide any general principle of selecting a space of density matrices given a particular operational description of experimental situation. Hence, this approach exposes the operational features underlying the structure of quantum theory, but it lacks the necessary mathematical generality allowing to express these features independently of Hilbert spaces, spectral theory, and commutative probability theory. Our approach to quantum theory, together with the new framework for definition of experiments and their relationship with predictive theories, is aimed to provide a new solution to these problems.

2 Conceptual motivations

In order to implement the above ideas, we will reformulate the McCullagh–Brøns [13, 1] categorical framework for experimentally sound statistical models. The detailed examples from statistical theory that motivate the main definitions of this framework are given in the original work of McCullagh [13]. However, McCullagh does not provide any detailed conceptual justification of the specific choice of definitions he introduces. Moreover, one may ask to what extent these examples may serve as a sufficient motivation for a setting aimed to cover also quantum theory. By this reason, we will develop here the independent conceptual motivation, that takes its roots in the theory of design and analysis of experiments as presented by the book of Hinkelmann & Kempthorne [8] and in the earlier works of author on the intersubjective bayesian interpretation [10, 12, 11].

The scientific inquiry can be divided into two layers: experimental and theoretical. The elementary notions used to describe the experimental layer are: *observation unit*, *experimental unit*, *treatment* (called also *intervention*), and *registration scale*. An *observation* is defined as an assignment of a registration scale to the observation unit. Observations are collected in observation protocols and form a passive component of the experimental layer. An *experimental design* (called 'error-control design' in [8]) is defined as an assignment of a treatment to an experimental unit. A *fact* is defined as an assignment of a response scale to an experimental design. Facts are collected in experimental protocols and form an active component of the experimental layer. In what follows, we will simplify the discussion by assuming that the experimental units and observation units can be identified. This assumption is implicit in the McCullagh–Brøns approach.

According to the analysis of various possible meanings of the notion 'causality' when applied at the experimental layer, this notion should be understood as dependence of facts upon active choice of treatment (intervention) [8]. This allows to consider treatments as "causes" and to consider the corresponding facts as "effects". This consideration is independent of anything that could be proposed at the theoretical layer.

In our opinion, the elementary notions used to describe the theoretical layer are *theoretical models* (called also *information models*) and *hypotheses*. Hypotheses can be represented as statements about relationships between theoretical *source-and-response parameters* (called also *input-and-output* parameters), and may include also some additional *control parameters*. The *theory* is defined as an assignment of hypotheses to a theoretical model. Hinkelmann & Kempthorne consider theoretical model as a sort of 'generalisation' of observations. This allows us to consider theoretical model as a passive component of theoretical layer. But what is then the active component of this layer? It seems that the key element of this component is an inductive inference procedure, which is a process of drawing (assigning) judgements from (to) evidences. The evidence and the resulting judgements are specified inside the theoretical layer, using theoretical source-and-response parameters. In principle, evidences and judgements are just some hypotheses. However, the inductive inference procedure requires to use also some theoretical model, that is associated with hypotheses under consideration through a given theory. As a result, one can define a *predictive theory* as a theory equipped with an inductive inference procedure, and consider it as an active component of a theoretical layer. The hypotheses play the role of the 'user data interface' for this active component, which is to some extent similar to the role played by treatments at the experimental layer.

If the elements of a theoretical model are interpreted as 'states of information', then the hypotheses assigned by a theory provide a theory-dependent classification of the 'states of information'. Usually, the inductive inference procedure is specified as a transformation between collections of states of information of a theoretical model. The fixed choice of a theory provides than the required links with hypotheses that are interpreted as evidences and judgements (inferences).

While the experimental notion of 'causality' is a relationship between treatments understood as "causes" and facts understood as "effects", the theoretical notion of 'causality' can be in principle introduced in two different ways: either as a relationship between control parameters and response parameters that form particular hypotheses, or as a relationship between hypotheses (with evidences understood as "causes" and inferences understood as "effects"). These two structures could be called, respectively, 'kinematic causality' and 'dynamic causality'. From Sections **??** and **??** we can see that in the case of our approach to quantum theory the former candidate for a theoretic notion of 'causality' takes a form of the Legendre–Fenchel transform, while the latter takes a form of the constrained maximisation of the relative entropy.

Observations, treatments, and facts belong to experimental layer only. Information models, hypotheses, and theories belong to theoretical layer only. Knowledge and experimentally verifiable theories arise as a result of association of theories with facts through some additional *interpretation* ('correspondence rules', 'association'), that consists of two independent components: passive component, providing interpretation of observations in terms of information (theoretic models), and active component, providing interpretation of treatments in terms of hypotheses (source-and-response relations). This interpretation sets a ground for analysis of the relationship of evidences and inferences to the experimental facts.

Let us recall that the main purpose of analysis of the design of experiments and its relation with theoretical models is to specify the conditions under which the selected methods of inductive inference can be regarded as 'valid in face of' a given experimental situation, where the 'validity in face of' can be also understood as 'mutual consistency' between the experimental and theoretical layers. As Hinkelmann & Kempthorne write, "We cannot talk about a theory being absolutely true. We can only talk about a theory being true in a given context of application" [8]. The goal of the above specifications of concepts describing the experimental and theoretical layers of the scientific inquiry is to enable formalisation of the above vague notions of "contextual truth", "validity in face of" or "mutual consistency" by a precise notion of an *experimentally verifiable predictive theory.*

Note that due to identification of observation units with experimental units, every fact contains 'purely active' part provided by treatment which is its "cause", and 'purely passive' part provided by an associated observation. A theory will be called *experimentally verifiable* with respect to a given collection of facts iff a hypothesis associated by an interpretation of the treatment 'underlying' any fact from this collection is equal to the hypothesis that is associated by a given theory to a result of an interpretation of the observation 'underlying' this fact. This can be illustrated more clearly by the following diagram

$$\begin{tikzcd}
 & \text{facts ("effects")} \arrow[dl] \arrow[dr] & \\
\text{experimental designs} \arrow[d] & & \text{observations} \arrow[dd, "\text{'passive' corresp. rule}"'] \\
\text{treatments ("causes")} \arrow[d, "\text{'active' corresp. rule}"] & & \\
\text{hypotheses} & & \text{theoretical model.} \arrow[ll, "\text{theory}"]
\end{tikzcd} \tag{2.1}$$

Whenever this diagram commutes the theory is *experimentally verifiable* with respect to the given collection of facts (and under particular interpretation). A predictive theory is called *experimentally verifiable* with respect to a given collection of facts iff the above condition holds for those facts that correspond to evidences or inferences of the given inductive inference procedure. That is, non-commutativity of the diagram (2.1) for those facts which are translated to hypotheses that are neither evidences nor inferences is irrelevant for experimental verifiability of a predictive theory.

In the next sections we will develop a category theoretic setting that provides a mathematical formalisation of the above concepts and their relationships. Yet, we need to justify such choice of the setting. The use of categories is motivated by McCullagh by noticing that "unless the model is embedded in a suitable structure that permits extrapolation, no useful inference is possible" [13]. While this justifies introducing the language of categories at theoretical layer, there remains a need for additional justification of the categorical description of experimental layer. For this purpose, let us quote again Hinkelmann & Kempthorne,

> "In science, a reaction to a portion of the world is an observation only if that reaction can be recorded (...). To do this requires a language and description terms. It is

> necessary that an observation can be described in terms that have some meaning to others. (...) A descriptive term does not receive validation until it is agreed on and can be confirmed by any observer who follows the prescribed protocol of observation and has been educated in the use of the descriptive terms. (...) even if the process of observation is quite unclear (as it is at the fundamental level), the world of science is permeated with interpersonally validated observation." [8]

Thus, there is a need for formalisation of the language used for intersubjective description of experiments. The use of category theory as a setting for this language allows to solve a specific problem: defining the conditions for intersubjective reproducibility of experiments ("interpersonal validation of observations"). The *categories of* experimental units, treatments and registration scales (as well as derived categories of experimental designs, observations and facts) allow to precisely specify the range of allowed variability of individual experimental situations that are considered as various instances of the same experiment of a given type. This variability is provided by the morphisms of the respective categories. Moreover, the categorical formulation allows to express the 'conceptual' diagram (2.1) as the mathematical diagram (4.1) that asserts certain property of functors between experimental and theoretical categories. As a result, the category theoretic setting for intersubjectivity of experiments and for predictive theories allows also to define the notion of an intersubjective experimental verifiability of a predictive theory. This *does not* solve the meta-physical problem "how *the* intersubjective agreement is possible?", but it allows to solve the practical (physical, predictive) problem "how to express necessary and sufficient conditions required for *an* intersubjective agreement?". In this sense, 'intersubjectivity' of scientific inquiry amounts to controlled variation of particular context of experimental and theoretical layers of scientific inquiry over the various individual instances of experimental situation, which preserves the particular form of relationship between these two layers.

The above conceptual setting provides a ground for the formalisation of the relationship between experiment and theory carried in the next sections. Apart from a mathematical goal of developing a framework which is independent of the notions of sample space, measure space, and relative frequency, its aim is to resolve at least some of the interpretational problems of quantum theory. In particular, the problem of various incompatible meanings assigned to the notion of quantum "measurement", as well as the problem of the "intersubjective" character of quantum theoretic inferences. Our approach replaces the notion of "measurement" (that suggests a sort of passive learning about preexisting properties) by an analysis of the experimental and theoretical structures associated with the notions of "intervention" and "observation". It also allows to maintain intersubjectivity without appealing to relative frequencies of outcomes, while carrying definite links between theory and experiment (which is not the case in the personalistic approach to quantum theory, see [14]).

3 Experimental universes and theories

Consider three categories: **Subj** of ***subjects of experimental inquiry*** ("things subjected to treatment and observation", "objects under experimental consideration", "experimental units", "statistical units"), **Config** of ***experimental treatments*** ("interventions", "(actively chosen) configurations", "settings", "experimental inputs", "experimental causes"), and **Scale** of ***response scales*** ("spaces of possible outcomes", "registration scales", "spaces of registration states of measuring device"). We consider these categories as 'qualitative', which means that they need not be equipped with any evaluation to some number field. Their role is to provide mathematical formalism allowing to express the operational definition (description, concept) of the 'experiment of a given type'. These categories are required to be equipped with the functors mapping to some common 'reference' category $\mathcal{E}_{\mathrm{ex}}$,

$$U_{\mathbf{Subj}} : \mathbf{Subj} \to \mathcal{E}_{\mathrm{ex}}, \quad U_{\mathbf{Config}} : \mathbf{Config} \to \mathcal{E}_{\mathrm{ex}}, \quad U_{\mathbf{Scale}} : \mathbf{Scale} \to \mathcal{E}_{\mathrm{ex}}.$$

For example, if $\mathcal{E}_{\text{ex}} = \mathbf{Set}$, then the above functors can be given by forgetful functors, allowing to consider subjects of inquiry, treatments, and response scales as sets. The role of category $\mathcal{E}_{\text{ex}}$ is to provide an underlying mathematical universe which fixes an environment allowing to construct various relations between objects of **Subj**, **Config**, and **Scale** 'as if' they were some "elementary objects" of $\mathcal{E}_{\text{ex}}$. By this reason, $\mathcal{E}_{\text{ex}}$ might be given also by some suitable topos.

Consideration of these categories instead of single objects is aimed to encode the criteria of *intersubjective reproducibility* of an experiment of a given *type* under the possible changes of construction of individual instances of this experiment (its subject, scale, and treatment).

Given $O \in \text{Ob}(\mathbf{Subj})$, $C \in \text{Ob}(\mathbf{Config})$, $S \in \text{Ob}(\mathbf{Scale})$, the morphism

$$x : U_{\mathbf{Subj}}(O) \to U_{\mathbf{Config}}(C) \tag{3.1}$$

will be called ***experimental design***, while the morphism

$$q : U_{\mathbf{Subj}}(O) \to U_{\mathbf{Scale}}(S) \tag{3.2}$$

will be called an ***observation*** ("experimental response", "elementary event", "experimental sample", "elementary observation").

According to the definition (3.1), the experimental design is a function between two sets. However, the *category of experimental designs* should preserve the knowledge about the categorical structure of subjects of inquiry and treatments. For this reason, it is defined as a comma category $U_{\mathbf{Subj}} \downarrow U_{\mathbf{Config}}$. For convenience of notation, we will denote it as $\mathbf{Subj} \downarrow_{\mathcal{E}_{\text{ex}}} \mathbf{Config}$ or just as $\mathbf{Subj} \downarrow \mathbf{Config}$. By definition, its objects are given by all designs (3.1), while its morphisms are given by such pairs of arrows

$$(f, g) \in \text{Mor}(\mathbf{Subj}) \times \text{Mor}(\mathbf{Config}), \quad f : O \to O', \quad g : C \to C',$$

that for any design $x' : U_{\mathbf{Subj}}(O') \to U_{\mathbf{Config}}(C')$ the diagram

$$\begin{tikzcd} U_{\mathbf{Subj}}(O) \arrow[r, "x"] \arrow[d, "U_{\mathbf{Subj}}(f)"'] & U_{\mathbf{Config}}(C) \arrow[d, "U_{\mathbf{Config}}(g)"] \\ U_{\mathbf{Subj}}(O') \arrow[r, "x'"] & U_{\mathbf{Config}}(C') \end{tikzcd} \tag{3.3}$$

commutes [15, 13, 1].

In ordinary statistical estimation, an elementary event (observation) is considered to be an element of a sample space $\mathcal{X}$. In general, the sample space can be defined, using (3.2), as a hom-set $\text{Hom}_{\mathcal{E}_{\text{ex}}}(U_{\mathbf{Subj}}(O), U_{\mathbf{Scale}}(S))$, but this object lacks important information about the variability (categorical structure) of **Subj** and **Scale**. For this reason, the ***category of observations*** is defined as a product category $\mathbf{Scale} \times \mathbf{Subj}^{op}$, with objects given by responses (S, O) and morphisms given by arrows $(S, O) \to (S', O')$ for any pair of arrows $h : O' \to O$ in **Subj** and $j : S \to S'$ in **Scale**. The objects of category $\mathbf{Scale} \times \mathbf{Subj}^{op}$ implement the notion of an 'observation protocol' from Section 2. The morphism of hom-sets

$$\text{Hom}_{\mathcal{E}_{\text{ex}}}(U_{\mathbf{Subj}}(O), U_{\mathbf{Scale}}(S)) \to \text{Hom}_{\mathcal{E}_{\text{ex}}}(U_{\mathbf{Subj}}(O'), U_{\mathbf{Scale}}(S'))$$

generalises the transformation $\mathcal{X} \to \mathcal{X}'$ of sample spaces.

The above constructions can be joined together, forming a categorical description of an experimental setup (of a given type):

$$\begin{tikzcd}
 & \mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op} \arrow[dl, "(\cdot)^{op} \circ \Pi_{(\mathbf{Subj} \downarrow \mathbf{Config})^{op}}"'] \arrow[dr, "\mathrm{id}_{\mathbf{Scale}} \times \Pi_{\mathbf{Subj}^{op}}"] & \\
(\mathbf{Subj} \downarrow \mathbf{Config}) \arrow[d, "\Pi_{\mathbf{Config}}"'] & & \mathbf{Scale} \times \mathbf{Subj}^{op} \\
\mathbf{Config} & &
\end{tikzcd} \tag{3.4}$$

The projection $\Pi_{(\mathbf{Subj} \downarrow \mathbf{Config})^{op}}$ is a canonical projection of the cartesian product of categories, while the projections $\Pi_{\mathbf{Subj}^{op}}$ and $\Pi_{\mathbf{Config}^{op}}$ are the canonical projection functors of comma category,

$$\mathbf{Subj} \xleftarrow{\Pi_{\mathbf{Subj}}} \mathbf{Subj} \downarrow \mathbf{Config} \xrightarrow{\Pi_{\mathbf{Config}}} \mathbf{Config}, \tag{3.5}$$

applied to an opposite category. We will call $\mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op}$ an ***experimental universe***. The objects of this category implement the notion of an 'experimental protocol' from Section 2. The elements of these objects will be called ***experimental facts***.

An experimental universe defines the meaning of an "experimental *setup* of a given type". However, the notion of an "experiment of a given type" requires to specify also the relationship of an experimental setup with some particular theory that is subjected to verification in terms of this setup. In order to introduce a categorical description of such theory, let us first observe that the category $\mathbf{Scale} \times \mathbf{Subj}^{op}$ of elementary events represents the *passive* aspect of an experiment, while the category **Config** of treatments represents the *active* aspect of an experiment. In the role of theoretic counterparts of these categories, let us introduce the category **InfoMod** of theoretical models ("information models") and the category **Hypo** of hypotheses ("theoretical parametrisations", "information data"). The category **InfoMod** represents passive aspect of a theory, and consists of spaces of information states as objects and their transformations as morphisms. Examples of this category include cartesian closed categories and various categories of non-linear spaces (e.g., C^∞-manifolds). The main motivating example from probability theory is the category of probabilistic models $\mathcal{M}(\mathcal{X}, \mho(\mathcal{X}), \mu)$ equipped with suitable morphisms as arrows (e.g., dual Markov maps). Category **Hypo** represents active aspect of a theory, and consists of spaces of hypotheses (usually expressed in terms of some source-and-response parameters) as objects and their transformations as morphisms. Examples of this category include symmetric monoidal categories and various categories of linear spaces (e.g., dualised vector spaces). The main motivating example from probability theory is the category of parameter spaces Θ equipped with suitable morphisms as arrows (e.g., a category with a single object $*$, representing the set $\Theta \subset \mathbb{R}^n$, and morphisms given by actions of elements of a group G on Θ).

We define a ***theory*** ("classification of information by hypotheses", "parametric representation") as a covariant functor

$$\mathrm{Th} : \mathbf{InfoMod} \to \mathbf{Hypo}. \tag{3.6}$$

The natural transformation between two theories will be called their ***intertwinner***. The functor category $\mathbf{Hypo}^{\mathbf{InfoMod}}$ is a category of all theories as objects, and all intertwiners as arrows. Thus, in principle, this category can be considered as a ***theoretical universe***. But note that some additional conditions imposed on the theories Th might restrict the notion of theoretical universe to some subcategory of $\mathbf{Hypo}^{\mathbf{InfoMod}}$ that is selected by these conditions.

4 Knowledge universes and experimental verification

With these categories at hand, we can define the relationship between experiment and theory in the category theoretic terms. Consider two covariant functors: $\mathcal{R} : \mathbf{Config} \to \mathbf{Hypo}$ and $\mathcal{G} : \mathbf{Scale} \times \mathbf{Subj}^{op} \to \mathbf{InfoMod}$. The functor $\mathcal{R}$ provides a ***(active) theoretic interpretation*** (***active correspondence rule***, “association”, “generalisation”, “idealisation”) of experimental treatments (configurations) in terms of theoretic hypotheses (relations of source-and-response parameters), while the functor $\mathcal{G}$ provides a ***passive theoretic interpretation*** (***passive correspondence rule***, “association”, “generalisation”, “idealisation”) of experimental observations in terms of information models. This way the active aspect of a relationship of a theory with experiment is encoded by the interpretation functor $\mathcal{R}$, while its passive aspect is contained in the interpretation functor $\mathcal{G}$. The quadruple

$$(\mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op}, \mathbf{Hypo}^{\mathbf{InfoMod}}, \mathcal{G}, \mathcal{R})$$

will be referred to as an ***knowledge universe***. It consists of the experimental universe, theoretical universe, and functors that interpret the active and passive elements of an experiment in terms of active and passive elements of a theory, respectively. The reason for such terminology is following. By a ***state of information*** we understand any element of a theoretical model. (Examples of states of information are given by the finite positive integrals on mcb- or W^*-algebras.) As opposed to it, a ***state of knowledge*** is a state of information together with an assignment of this state to some observation. We can also define a ***state of factual knowledge***, as a state of information together with an assignment of this state to some experimental fact. This expresses the idea that *information* is just a quantitative evaluation without any fixed semantic background (such as provided by experimental/operational layer), while—as opposed to it—*knowledge* is a quantitative and theoretic evaluation that is always provided in reference to particular terms of its operational specification and communication (such as allowed subjects of experimental inquiry, allowed experimental treatments, and allowed scales of experimental response).

Following this idea, it is natural to select such theories that are not only theories of information, but also theories of knowledge, compatible with the allowed transformations of semantic terms of operational specification and communication of this information. This condition may not be satisfied by some of theories. We will say that the theory $\mathrm{Th} : \mathbf{InfoMod} \to \mathbf{Hypo}$ is ***strongly verifiable*** in a given experimental universe $\mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op}$ iff there exist covariant functors (theoretic interpretations) $\mathcal{R} : \mathbf{Config} \to \mathbf{Hypo}$ and $\mathcal{G} : \mathbf{Scale} \times \mathbf{Subj}^{op} \to \mathbf{InfoMod}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op} \arrow[dl, "(\cdot)^{op}\circ\Pi_{(\mathbf{Subj}\downarrow\mathbf{Config})^{op}}"'] \arrow[dr, "\mathrm{id}_{\mathbf{Scale}}\times\Pi_{\mathbf{Subj}^{op}}"] & \\
(\mathbf{Subj} \downarrow \mathbf{Config}) \arrow[d, "\Pi_{\mathbf{Config}}"'] & & \mathbf{Scale} \times \mathbf{Subj}^{op} \arrow[dd, "\mathcal{G}"] \\
\mathbf{Config} \arrow[d, "\mathcal{R}"'] & & \\
\mathbf{Hypo} & & \mathbf{InfoMod}, \arrow[ll, "\mathrm{Th}"']
\end{tikzcd} \tag{4.1}$$

that is,

$$\mathrm{Th} \circ \mathcal{G} \circ \mathrm{id}_{\mathbf{Scale}} \times \Pi_{\mathbf{Subj}^{op}} = \mathcal{R} \circ \Pi_{\mathbf{Config}} \circ (\cdot)^{op} \circ \Pi_{(\mathbf{Subj}\downarrow\mathbf{Config})^{op}},$$

where equality holds for each pair of allowed experimental scale S and allowed experimental design x, and for each joint morphism of allowed scale and allowed design. Given a knowledge universe, we will call any of its verifiable theories a ***strong information representation***. We define also the *weak information representation* (and corresponding ***weak verifiability*** of a theory) as such covariant functor $\mathrm{Th} : \mathbf{InfoMod} \to \mathbf{Hypo}$ that the diagram (4.1) commutes weakly (up to a natural transformation). More precisely, Th will be called a ***weak information representation*** iff there exists a natural transformation N_1 given by

$$\begin{tikzcd} \mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op} \arrow[r, bend left, "\mathcal{R}\circ(\cdot)^{op}\circ\Pi_{\mathbf{Config}}\circ\Pi_{(\mathbf{Subj}\downarrow\mathbf{Config})^{op}}"] \arrow[r, bend right, "\mathrm{Th}\circ\mathcal{G}\circ\mathrm{id}_{\mathbf{Scale}}\times\Pi_{\mathbf{Subj}^{op}}"'] \arrow[r, phantom, "\Downarrow N_1"] & \mathbf{Hypo} \end{tikzcd} \tag{4.2}$$

By [***what?***], such N_1 is equivalent to a functor

$$\overline{N_1} : \mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op} \to (\mathrm{id}_{\mathbf{Hypo}} \downarrow \mathrm{Th}). \tag{4.3}$$

To summarise, given particular knowledge universe, the (strong or weak) commutativity of diagram (4.1) imposes a non-trivial requirement on theories $\mathrm{Th} : \mathbf{InfoMod} \to \mathbf{Hypo}$, restricting them to the class which is compatible with the allowed quantitative transformations of subjects, treatments and scales. This compatibility is understood as experimental verification of a theory. By this reason, any triple

$$(\mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op}, \mathcal{R}, \mathcal{G})$$

will be called an (experimental and interpretational) ***context of verification***.

Due to 'intersubjective' interpretation that is associated with the above categorical scheme, we can identify 'subjective/personal' perspectives on the verifiable theory Th by fixing the choice of the object in the experimental universe $\mathbf{Scale} \times (\mathbf{Subj} \downarrow \mathbf{Config})^{op}$. This object is a certain collection of facts (and it can possess its own internal mathematical structure). Recalling the discussion of experimental and theoretical notions of 'causality' in Section 2, we see that each 'stage' of a diagram (4.1) there is associated its own set of experimental causes and effects, as well as its own division of the hypothesis space into kinematic and dynamic causes and effects. We will call the choice of some 'stage' of the diagram (4.1) a ***personalisation***. From this it follows that experimental and theoretical cause-and-effect relationships may vary between two different personalisations of the same intersubjective experimentally verifiable theory.

5 Comparison with the McCullagh–Brøns approach

Our definitions of the categories and functors forming the experimental universe amount to conceptually refined restatement of the McCullagh–Brøns formalism [13, 1]. The remaining part of the above construction differs from the McCullagh–Brøns formalism in several important aspects. In particular, having in mind the requirements of quantum theory, we do not require that **InfoMod** be defined in terms of measures or probability measures. We also do not require the use of set-theoretic representations of **Hypo** and **InfoMod**. Moreover, we consider information model as an entity on its own right and of equal importance as parameter space, which leads us to consider the commutativity of the diagram (4.1) as a condition on allowed parametric representations of the model *and not* as a condition on models themselves.

The original McCullagh–Brøns formulation *defines* a probabilistic model as a natural transformation N_2,

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Scale}\times(\mathbf{Subj}\downarrow\mathbf{Config})^{op} \arrow[dl, "\mathcal{R}\circ(\cdot)^{op}\circ\Pi_{\mathbf{Config}^{op}}\circ\Pi_{(\mathbf{Subj}\downarrow\mathbf{Config})^{op}}"'] \arrow[dr, "\tilde{\mathcal{G}}\circ\mathrm{id}_{\mathbf{Scale}}\times\Pi_{\mathbf{Subj}^{op}}"] & \\
\mathbf{Hypo} \arrow[dr, "U_{\mathbf{Hypo}}"'] \arrow[rr, phantom, "\overset{N_2}{\Longrightarrow}"] & & \mathbf{MeasSp} \arrow[dl, "\mathcal{P}"] \\
 & \mathbf{Set}. &
\end{tikzcd} \tag{5.1}$$

Here **MeasSp** is a category of measurable spaces $(\mathcal{X},\mho(\mathcal{X}))$ with some suitable (but undefined) morphisms, $U_{\mathbf{Hypo}}$ is a forgetful functor, $\tilde{\mathcal{G}}:\mathbf{Scale}\times\mathbf{Subj}^{op}\to\mathbf{MeasSp}$ is a contravariant functor that assigns a measurable space $(\mathcal{X},\mho(\mathcal{X}))$ to each "sample space" $\mathcal{X}\in\mathrm{Ob}(\mathbf{Scale}\times\mathbf{Subj}^{op})$, and $\mathcal{P}:\mathbf{MeasSp}\to\mathbf{Set}$ assigns to each measure space $(\mathcal{X},\mho(\mathcal{X}))$ the set $\mathcal{P}(\mathcal{X},\mho(\mathcal{X}))$ of *all* probability measures on $(\mathcal{X},\mho(\mathcal{X}))$. This definition has three key drawbacks: (i) it does not extend to non-commutative case; (ii) it is too restrictive in commutative case (but does not exclude possible pathological behaviour of $\mathcal{P}\circ\tilde{\mathcal{G}}$); (iii) it makes the definition of information models dependent of the particular parametrisation.

Regarding (iii), the non-parametric geometric structures can be introduced and studied on information models, making the point of view specified by (5.1) too restrictive even in commutative case. Regarding (ii), one has to note that construction of probabilistic model by an assignment $\mathcal{P}\circ\tilde{\mathcal{G}}:\mathcal{X}\to(\mathcal{X},\mho(\mathcal{X}))\to\mathcal{P}(\mathcal{X},\mho(\mathcal{X}))$ equipped with selection of $\mathcal{M}(\mathcal{X},\mho(\mathcal{X}))\subseteq\mathcal{P}(\mathcal{X},\mho(\mathcal{X}))$ through $U_{\mathbf{Hypo}}\circ\mathcal{R}$ and (weak) commutativity of (5.1) is a highly restrictive, representation-dependent procedure. Let us recall from Section **??** that for every mcb-algebra $\mho$ there exists a unique family $L_p(\mho)$ of abstract L_p spaces associated to $\mho$, and a plenty of different representations $(\mathcal{X},\mho(\mathcal{X}))$ of $\mho$. Nevertheless, for any representation $(\mathcal{X},\mho(\mathcal{X}))$ of $\mho$, the spaces $L_p(\mathcal{X},\mho(\mathcal{X}))$ and $L_p(\mho)$ are equivalent (isometrically isomorphic). In general, the statistical model is a subspace $\mathcal{M}(\mho)\subseteq L_1(\mho)^+$ of an abstract commutative L_1 space. Hence, it does not depend neither on the choice of a sample space $\mathcal{X}$ nor on the choice of a particular representation $\mathcal{X}\to\mho(\mathcal{X})$ of $\mho$. Thus, the definition of a non-parametric probabilistic or statistical model $\mathcal{M}(\mho)$ is essentially independent of the choice of $(\mathcal{X},\mho(\mathcal{X}))$. The same holds for quantum models $\mathcal{M}(\mathcal{N})\subseteq\mathcal{N}_*^+$. In this sense, the McCullagh–Brøns approach is too restrictive. However, it is also too general at the same time, because it does not exclude non-localisable measure spaces. For non-localisable measure spaces the Steinhaus–Nikodým duality $L_1(\mathcal{X},\mho(\mathcal{X}),\mu)^{\mathbf{B}}\cong L_\infty(\mathcal{X},\mho(\mathcal{X}),\mu)$ does not hold, and the standard consideration of a probabilistic model $\mathcal{M}$ as a space of Radon–Nikodým derivatives (densities) with respect to a measure μ breaks down. (This pathological situation can be removed by restricting considerations from the category **MeasSp** only to its subcategory **LocMeasSp** of localisable measure spaces.) Finally, the definition provided by (5.1) does not extend to non-commutative case, because tensor products in the category **Set** are given by cartesian products and **Set** is a cartesian closed category, while the monoidal structure of non-commutative W^*-algebras (or Hilbert spaces) is not cartesian closed.

As a result, we conclude that the commutativity of (5.1) should be interpreted only as a condition on representation of a model, and not on the model itself. The above problems motivate our proposition to replace (5.1) by (4.1). As we will see below, this replacement is required also for the compatibility with the categorical structures that arise from quantum information geometry.

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