AARHUS UNIVERSITET

TENSOR PRODUCTS OF C*-ALGEBRAS

PART I.

FINITE TENSOR PRODUCTS

by

A. Guichardet

April 1969

Lecture Notes Series
No. 12.
TENSOR PRODUCTS OF C*-ALGEBRAS

PART I.

FINITE TENSOR PRODUCTS

by

A. Guichardet

April 1969

Lecture Notes Series

No. 12.
Introduction

Given two $C^*$-algebras $A_1$ and $A_2$, we can form their algebraic tensor product $A_1 \otimes A_2$ and look for reasonable norms on it such that the completion is a $C^*$-algebra; more precisely we shall study the $C^*$-crossnorms and define in a natural manner two such norms: the largest one and the smallest one; their properties are rather similar to those of the norms $\wedge$ and $\bar{\wedge}$ which arise in the case of Banach spaces. Then we shall be concerned with the properties of the completions of $A_1 \otimes A_2$ with respect to these norms, in particular their types (liminar, postliminar, antiliminar, with continuous trace), representations, states and traces. The results are still valid for the tensor product of an arbitrary finite number of $C^*$-algebras, but we consider the case of two $C^*$-algebras for the sake of simplicity of notations.

Throughout these lectures we shall use the notations of Dixmier's book on $C^*$-algebras; moreover $S(A)$ will denote the set of all states of a $C^*$-algebra $A$, endowed with the weak topology; it is compact if $A$ has a unit element, but not necessarily locally compact in the general case; $C_1(A)$ will denote the set of all normed characters (or central states) of $A$, also endowed with the weak topology; it is not locally compact in general, even when $A$ has a unit element (see [7], § 3, prop.6), but is Polish if $A$ is separable (see [2], 7.4.2). For each Banach $*$-algebra $A$ with approximate identity we denote by $C^*(A)$ the
enveloping $C^*$-algebra of $A$.

A $C^*$-algebra is said to be simple if it has no proper closed two-sided ideal; a state $f$ on a $C^*$-algebra is factorial if the associated representation $\pi_f$ is factorial. All our vector spaces and algebras are complex; for any two vector spaces or algebras $E_1$, $E_2$, we denote by $E_1 \otimes E_2$ their algebraic tensor product; every element $x$ of this space can be written, but not in a unique manner, as $x = \sum_{n=1}^{\infty} x_{1,n} \otimes x_{2,n}$. 
§ 1. Preliminaries.

n.1.1. Tensor products of Banach spaces.

If $E_1$ and $E_2$ are Banach spaces, we say that a seminorm $p$ on $E_1 \otimes E_2$ is a cross seminorm (resp. a subcross seminorm) if

$$p(x_1 \otimes x_2) = \langle x_1, x_2 \rangle \quad \forall x_i \in E_i.$$  

The cross norm.

There exists a largest subcross seminorm on $E_1 \otimes E_2$ (which is in fact a crossnorm):

$$\| x \|_\wedge = \inf \sum_{n=1}^N \| x_{1,n} \| \cdot \| x_{2,n} \|$$

where the inf is taken for all families $(x_{1,n}, x_{2,n})$ satisfying $x = \sum_{n=1}^N x_{1,n} \otimes x_{2,n}$. The completion of $E_1 \otimes E_2$ for this crossnorm will be denoted by $E_1 \hat{\otimes} E_2$; it possesses the following universal property: for each continuous bilinear mapping $u$ of $E_1 \times E_2$ into a Banach space $F$ there exists a unique continuous linear mapping $v : E_1 \hat{\otimes} E_2 \to F$ such that $v(x_1 \otimes x_2) = u(x_1, x_2)$.

Functorial properties. For $i = 1, 2$ let $u_i$ be a continuous linear mapping of $E_i$ into a Banach space $F_i$; the linear mapping $u_1 \hat{\otimes} u_2 : E_1 \otimes E_2 \to F_1 \otimes F_2$ can be extended in a continuous linear mapping $u_1 \hat{\otimes} u_2 : E_1 \hat{\otimes} E_2 \to F_1 \hat{\otimes} F_2$; evidently

$$(u_1 \hat{\otimes} u_2)(x_1 \otimes x_2) = u_1(x_1) \otimes u_2(x_2).$$

If $u_1$ is surjective and $F_1$ has the quotient norm of $E_1$, then $u_1 \hat{\otimes} u_2$ is surjective, $F_1 \hat{\otimes} F_2$ has the quotient norm of $E_1 \hat{\otimes} E_2$ and moreover
\[ \text{Ker } u_1 \hat{\otimes} u_2 = \text{Ker } u_1 \hat{\otimes} E_2 + E_1 \hat{\otimes} \text{Ker } u_2. \]

**Example 1.** If \( X \) is a measurable space with a measure \( \mu \), and if \( E \) is a Banach space, \( L^1(X, \mu) \hat{\otimes} E \) is canonically isomorphic to \( L^1(X, \mu, E) \), the space of all \( \mu \)-integrable mappings of \( X \) into \( E \); this isomorphism carries each element of the form \( f \hat{\otimes} \xi \) into the mapping \( x \mapsto f(x) \). *In particular *

\[ L^1(X, \mu_1) \hat{\otimes} L^1(X, \mu_2) \sim L^1(X \times X, \mu_1 \otimes \mu_2). \]

The \( \hat{\otimes} \) norm.

If \( f_1 \) is a linear functional on \( E_1 \) we can consider the linear functional \( f_1 \hat{\otimes} f_2 \) on \( E_1 \hat{\otimes} E_2 \) characterized by the property

\[ (f_1 \hat{\otimes} f_2)(x_1 \hat{\otimes} x_2) = f_1(x_1) \cdot f_2(x_2), \]

the \( \hat{\otimes} \) norm is defined by

\[ \| x \|_{\hat{\otimes}} = \sup \| (f_1 \hat{\otimes} f_2)(x) \| \]

where the sup is taken for all \( f_1 \) which are continuous and of norm \( < 1 \); this is the smallest crossnorm which is reasonable in a certain sense (see [6]); the completion of \( E_1 \hat{\otimes} E_2 \) for this norm will be denoted by \( E_1 \hat{\otimes} E_2 \).

**Functorial properties.** If we have continuous linear mappings
\[ u_i : E_i \rightarrow F_i \]
there is a continuous linear mapping \( u_1 \hat{\otimes} u_2 : E_1 \hat{\otimes} E_2 \rightarrow F_1 \hat{\otimes} F_2 \); if \( u_1 \) is isometric, \( u_1 \hat{\otimes} u_2 \) is also isometric.

**Example 2.** If \( X \) is a locally compact topological space we denote by \( C_c(X) \) the space of all complex continuous functions \( f \) on \( X \) which vanish at infinity; that means that for each number
a > 0 the set of \( x \in X \) such that \( |f(x)| \geq a \) is compact; this is a Banach space for the sup-norm; if \( X \) is compact we write \( \mathcal{C}(X) \) instead of \( \mathcal{C}_c(X) \). Then if \( E \) is a Banach space, \( \mathcal{C}_c(X) \hat{\otimes} E \) is canonically isomorphic to \( \mathcal{C}_c(X, E) \); in particular

\[
\mathcal{C}_c(X_1) \hat{\otimes} \mathcal{C}_c(X_2) \sim \mathcal{C}_c(X_1 \times X_2).
\]

Definition 1. If \( \| \cdot \|_\ast \) is any subcross norm on \( E_1 \otimes E_2 \) we denote by \( E_1 \hat{\otimes} E_2 \) the completion of \( E_1 \otimes E_2 \) for this norm.

Bibliography. [6],[18].

n.1.2. \textit{Tensor products of Banach }\ast\textit{-algebras.}

(For the definition of Banach \( \ast \)-algebras, see [2], 1.2.1; see also [4] and [5].)

Let us consider two Banach algebras \( A_1 \) and \( A_2 \); \( A_1 \otimes A_2 \) is an algebra with a multiplication verifying

\[
(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 \;
\]

the \( \ast \) norm is not always a norm of algebra, i.e. does not always verify \( \| a \cdot b \|_\ast \leq \| a \|_\ast \cdot \| b \|_\ast \) (see [5]); but it is easy to see that the \( \ast \) norm does; if \( A_1 \) and \( A_2 \) are Banach \( \ast \)-algebras \( A_1 \hat{\otimes} A_2 \) is a normed \( \ast \)-algebra with an involution satisfying

\[
(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*;
\]

then \( A_1 \hat{\otimes} A_2 \) is a Banach \( \ast \)-algebra; it possesses the following universal property: let us say that two morphisms \( u_1 \) and \( u_2 \) of \( A_1 \) and \( A_2 \) into a Banach \( \ast \)-algebra \( B \) commute if

\[
u_1(a_1) \cdot u_2(a_2) = u_2(a_2) \cdot u_1(a_1) \quad \forall a_1 \in A_1;
\]
then for each pair \((u_1, u_2)\) of commuting continuous morphisms there exists a unique continuous morphism \(v: A_1 \circledast A_2 \to B\) such that
\[
v(a_1 \circledast a_2) = u_1(a_1).u_2(a_2) \quad \forall a_1 \in A_1.
\]
Conversely if \(A_1\) and \(A_2\) admit unit elements, each continuous morphism \(v\) can be obtained in this manner.

**Example 3.** If \(G_1\) and \(G_2\) are locally compact groups, the Banach \({\ast}\)-algebra \(L^1(G_1 \times G_2)\) is canonically isomorphic to the tensor product \(L^1(G_1) \circledast L^1(G_2)\).

**n.1.3. Tensor products of Hilbert spaces and von Neumann algebras.**

(See [1])

The Hilbert tensor product of two Hilbert spaces \(H_1, H_2\) will be denoted by \(H_1 \circledast H_2\); we recall that it is the completion of \(H_1 \circledast H_2\) for a scalar product which verifies
\[
(x_1 \circledast x_2, y_1 \circledast y_2) = (x_1, y_1)(x_2, y_2).
\]
If \(\mathcal{A}_1\) is a von Neumann algebra in \(H_1\) the von Neumann algebra in \(H_1 \circledast H_2\) generated by all operators \(a_1 \circledast a_2\) with \(a_1 \in \mathcal{A}_1\) will be denoted by \(\mathcal{A}_1 \circledast \mathcal{A}_2\); it is a factor iff \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are factors; it is equal to \(\mathcal{L}(H_1 \circledast H_2)\) iff
\[
\mathcal{A}_1 = \mathcal{L}(H_1).
\]

**Proposition 0.** Let \(\mathcal{A}\) be a factor in a Hilbert space \(H\) and \(\mathcal{A}'\) its commutant; the morphism \(u: \mathcal{A} \circledast \mathcal{A}' \to \mathcal{L}(H)\) defined by \(u(a \circledast a') = a a'\) is injective; in particular if \(a\) and \(a'\) are not zero, \(a a'\) is also not zero.

For the proof see [1], p. 31, exercise 6.
Distributivity with respect to Hilbert integrals.

For \( i = 1,2 \) let \( X_i \) be a measurable space with a measure \( \mu_i, s_i \mapsto H_i, s_i \) a \( \mu_i \)-measurable field of Hilbert spaces; it is easy to construct on the field \( (s_1, s_2) \mapsto H_1, s_1 \otimes H_2, s_2 \) a structure of \( \mu_1 \otimes \mu_2 \)-measurable field of Hilbert spaces and an isomorphism

\[
U : \int_{H_1, s_1} \mu_1(s_1) \otimes \int_{H_2, s_2} \mu_2(s_2)
\]

\[
\leq \int_{H_1, s_1} \otimes H_2, s_2 \cdot d(\mu_1 \otimes \mu_2)(s_1, s_2)
\]

with the following property: for each vector \( x_i = \int x_i, s_i \cdot d \mu_i(s_i) \) one has

\[
U(x_1 \otimes x_2) = \int x_1, s_1 \otimes x_2, s_2 \cdot d(\mu_1 \otimes \mu_2)(s_1, s_2).
\]
§ 2. Representations of the algebraic tensor product of two $C^*$-algebras.

We consider two $C^*$-algebras $A_1$ and $A_2$.

n.2.1. Tensor products of representations.

Let $\pi_1$ be a representation of $A_1$ in a Hilbert space $H_1$; we can form the representation of $A_1 \otimes A_2$ in $H_1 \otimes H_2$ defined by

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2);$$

the von Neumann algebra generated by this representation is clearly $\pi_1(A_1)'' \otimes \pi_2(A_2)'';$ it follows that $\pi_1 \otimes \pi_2$ is factorial (resp. irreducible) iff $\pi_1$ and $\pi_2$ have the same property. The equivalence (resp. quasiequivalence) class of $\pi_1 \otimes \pi_2$ depends only on the analogous classes for $\pi_1$ and $\pi_2$.

If we have a measurable field $\mathcal{M}$ of representations $\pi_i, s_i$, we can write, with the notations of n.1.3

$$\int \omega \pi_{1,s_1} d\mu_{1}(s_1) \otimes \int \omega \pi_{2,s_2} d\mu_{2}(s_2) \sim$$

$$\int \omega \pi_{1,s_2} \otimes \pi_{2,s_2} \cdot d(\mu_{1} \otimes \mu_{2})(s_1,s_2);$$

in particular for discrete sums

$$(\bigoplus \pi_{1,s}) \otimes (\bigoplus \pi_{2,t}) \sim \bigoplus_{s,t} (\pi_{1,s} \otimes \pi_{2,t}).$$
1.2.2. **Restrictions of a representation of** \( A_1 \otimes A_2 \).

**Definition 2.** We shall say that a representation \( \pi \) of \( A_1 \otimes A_2 \) is a **subcross representation** if

\[
\| \pi(a_1 \otimes a_2) \| \leq \| a_1 \| \cdot \| a_2 \| \quad \forall a_1 \in A_1 .
\]

If \( A_1 \) and \( A_2 \) have unit elements \( e_1 \) and \( e_2 \), every representation of \( A_1 \otimes A_2 \) is subcross because we have

\[
\| \pi(a_1 \otimes e_2) \| = \| \pi(a_1 \otimes e_2 \cdot e_1 \otimes a_2) \|
\]

\[
\leq \| \pi(a_1 \otimes e_2) \| \cdot \| \pi(e_1 \otimes a_2) \|
\]

\[
\leq \| a_1 \| \cdot \| a_2 \|
\]

since \( a_1 \mapsto \pi(a_1 \otimes e_2) \) and \( a_2 \mapsto \pi(e_1 \otimes a_2) \) are representations of \( A_1 \) and \( A_2 \) respectively.

**Proposition 1.** To each subcross representation \( \pi \) of \( A_1 \otimes A_2 \) in a Hilbert space \( H \) one can associate canonically representations \( \pi_1 \) and \( \pi_2 \) of \( A_1 \) and \( A_2 \) in \( H \) such that

\[
\pi(a_1 \otimes a_2) = \pi_1(a_1) \cdot \pi_2(a_2) = \pi_2(a_2) \cdot \pi_1(a_1) \quad (1)
\]

for each \( a_i \) in \( A_i \); moreover one has

\[
\pi_1(a_1) = \text{strong limit of } \pi(a_1 \otimes v_t)
\]

\[
\pi_2(a_2) = \text{strong limit of } \pi(u_s \otimes a_2)
\]

where \((u_s)\) and \((v_t)\) are arbitrary approximate identities of \( A_1 \) and \( A_2 \). Finally if \( \pi \) is faithful or non degenerate or factorial, \( \pi_1 \) and \( \pi_2 \) have the same property.

**Proof.** We choose an approximate identity \((u_s)\) of \( A_1 \) and prove that for each \( a_2 \) in \( A_2 \), \( \pi(u_s \otimes a_2) \) has a strong limit; set \( H_1 = \pi(A_1 \otimes A_2) \cdot H \); we must prove that \( \pi(u_s \otimes a_2) \cdot x \) has a
limit for each \( x \in \overline{H}_1 \); since the family \( \pi(u_s \cdot a_2) \) is bounded (because \( \pi \) is subcross), we can take \( x \) in \( H_1 \); by linearity we can suppose \( x \) has the following form

\[
x = \pi(b_1 \cdot b_2).y \quad \text{where} \quad b_1 \in A_1, \quad y \in H;
\]

then

\[
\pi(u_s \cdot a_2).x = \pi(u_s b_1 \cdot b_2) \cdot y;
\]

this converges to \( \pi(b_1 \cdot a_2 b_2).y \) since

\[
\| \pi(u_s b_1 \cdot a_2 b_2).y - \pi(b_1 \cdot a_2 b_2).y \| = \| \pi((u_s b_1 - b_1) \cdot a_2 b_2).y \| \\
\leq \| u_s b_1 - b_1 \| . \| a_2 b_2 \| . \| y \|
\]

we have thus proved that \( \pi(u_s \cdot a_2) \) has a strong limit which is independent of the approximate identity \( (u_s) \); denote it by \( \pi_2(a_2) \); as easily verified \( \pi_2 \) is a representation; define \( \pi_1 \) in an analogous manner; to prove (1):

\[
\pi_1(a_1) \cdot \pi_2(a_2) = \lim \pi(a_1 \cdot v_t) \cdot \lim \pi(u_s \cdot a_2) \\
= \lim \pi(a_1 u_s \cdot v_t a_2) \\
= \pi(a_1 \cdot a_2)
\]

since

\[
\| \pi(a_1 u_s \cdot v_t a_2) - \pi(a_1 \cdot a_2) \| \\
\leq \| \pi((a_1 u_s - a_1) \cdot v_t a_2) \| + \| \pi(a_1 \cdot (v_t a_2 - a_2)) \| \\
\leq \| a_1 u_s - a_1 \| \cdot \| v_t a_2 \| + \| a_1 \| \cdot \| v_t a_2 - a_2 \|
\]

The last assertion is trivial.

Remark 1. From the above proof we also deduce the following: if \( A_i \) is a Banach \( * \)-algebra with approximate identity, to each
subcross representation $\pi$ of $A_1 \otimes A_2$, one can associate representations $\pi_1$ and $\pi_2$ of $A_1$ and $A_2$ verifying (1).

**Definition 3.** The representations $\pi_1$ and $\pi_2$ associated with $\pi$ will be called the restrictions of $\pi$ to $A_1$ and $A_2$; if $A_1$ and $A_2$ have unit elements, they are nothing but the usual restrictions of $\pi$.

**Proposition 2.** If $\pi_1$ (or $\pi_2$) is a type I factor representation, $\pi$ is equivalent to a tensor product of representations.

In fact we can write

$$H = H_1 \otimes H_2$$

$$\pi_1(a_1) = \rho_1(a_1) \otimes I$$

$$\pi_2(a_2) = I \otimes \rho_2(a_2)$$

where $\rho_1$ is some representation of $A_1$ in $H_1$; then

$$\pi(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2).$$

**Proposition 3.** If $\pi_1$ is a representation of $A_1$ with $\pi_1$ non-degenerate, the restriction of $\pi_1 \otimes \pi_2$ to $A_2$ is a multiple of $\pi_2$.

In fact this restriction is given by

$$\rho_2^*(a_2) = \lim (\pi_1 \otimes \pi_2)(u_s \otimes a_2)$$

$$= \lim \pi_1(u_s) \cdot \pi_2(a_2)$$

$$= I \otimes \pi_2(a_2).$$

**Lemma 1.** If $A_1$ has no unit element, each $C^*$ subcross norm $\rho$ on $A_1 \otimes A_2$ can be extended to a $C^*$ subcross norm $\tilde{\rho}$ on $\tilde{A}_1 \otimes \tilde{A}_2$.

The same holds for $A_2$.

**Proof.** Choose a non-degenerate isometric representation $\pi$ of $A_1 \otimes A_2$ in a space $H$; its restrictions $\pi_1$ and $\pi_2$ are faithful
and non degenerate; \( \tilde{\pi}_1 \) extends to a representation \( \tilde{\pi}_1 \) of \( \tilde{A}_1 \), which is faithful because \( \tilde{\pi}_1(A_1) \) does not contain the scalars; the bilinear mapping

\[
\tilde{A}_1 \times A_2 \longrightarrow \mathcal{L}(H)
((a_1, h_1), a_2) \longmapsto \tilde{\pi}_1(a_1, h_1). \pi_2(a_2)
\]
gives rise to a linear mapping

\[
\rho : \quad \tilde{A}_1 \otimes A_2 \longrightarrow \mathcal{L}(H)
\sum_{n=1}^{N} (a_1, n, h_1, n) \otimes a_2, n \longmapsto \sum_{n=1}^{N} \tilde{\pi}_1(a_1, n, h_1, n). \pi_2(a_2, n);
\]

\( \rho \) is easily verified to be a representation. We now prove that \( \rho \) is faithful: take an element \( b \) in \( \text{Ker } \rho \); for each element \( a \) in \( A_1 \otimes A_2 \) we have

\[
b \ a \in \text{Ker } \rho \ \wedge (A_1 \otimes A_2) = \text{Ker } \pi = \{0\};
\]
then for each \( x \) in \( H \otimes H \):

\[
0 = (\tilde{\pi}_1 \otimes \pi_2)(ba).x = (\tilde{\pi}_1 \otimes \pi_2)(b).(\pi_1 \otimes \pi_2)(a).x;
\]

since \( \tilde{\pi}_1 \) and \( \pi_2 \) are non degenerate, the same holds for \( \pi_1 \otimes \pi_2 \), the elements \( (\pi_1 \otimes \pi_2)(a).x \) are dense in \( H \otimes H \), and we see that \( (\tilde{\pi}_1 \otimes \pi_2)(b) = 0 \); since \( \tilde{\pi}_1 \otimes \pi_2 \) is faithful, \( b = 0 \) and \( \rho \) is faithful.

Now setting \( \tilde{\rho}(a) = \| \rho (a) \| \) for each \( a \) in \( \tilde{A}_1 \otimes A_2 \) we get a \( C^* \) norm which clearly extends \( \rho \) and is subcross since

\[
\tilde{\rho}((a_1, h_1) \otimes a_2) = \| \rho ((a_1, h_1) \otimes a_2) \|
\leq \| \tilde{\pi}_1(a_1, h_1) \| \cdot \| \pi_2(a_2) \|
\leq \| (a_1, h_1) \| \cdot \| a_2 \|.
\]

**Bibliography** [8], [20].
§ 3. The $\mathcal{V}$ crossnorm.

3.1. Definition of the $\mathcal{V}$ crossnorm.

We consider two $C^*$-algebras $A_1$ and $A_2$; denote by $\| \cdot \|_\mathcal{V}$ the LUB of all $C^*$-subcross seminorms on $A_1 \hat{\otimes} A_2$; this is clearly a $C^*$-subcross seminorm majorized by $\| \cdot \|_\mathcal{V}$; this is in fact a crossnorm because if $\pi_1$ is a faithful representation of $A_1$,

$\pi_1 \otimes \pi_2$ is a faithful representation of $A_1 \hat{\otimes} A_2$ and

$$\mathcal{V}(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) \| = \| a_1 \| \cdot \| a_2 \| .$$

If $A_1$ and $A_2$ have units each $C^*$-seminorm on $A_1 \hat{\otimes} A_2$ is majorized by $\| \cdot \|$, since each representation is subcross. The elementary properties of the $\mathcal{V}$ norm are summarized in the following:

**Theorem 1.** The LUB of all $C^*$-subcross seminorms on $A_1 \hat{\otimes} A_2$ is a $C^*$-crossnorm $\| \cdot \|_\mathcal{V}$; the representations of the completion $A_1 \hat{\otimes} A_2$ are in bijective correspondence with the subcross representations of $A_1 \hat{\otimes} A_2$, and in particular with all representations of $A_1 \hat{\otimes} A_2$ if $A_1$ and $A_2$ have units. The $C^*$-algebra $A_1 \hat{\otimes} A_2$ has the following universal property: if we have two commuting morphisms $u_i$ of $A_i$ into some $C^*$-algebra $B$, there exists a unique morphism $u : A_1 \hat{\otimes} A_2 \longrightarrow B$ such that

$$u(a_1 \otimes a_2) = u_1(a_1).u_2(a_2).$$

Note that the representations of $A_1 \hat{\otimes} A_2$ are also the same as those of $A_1 \hat{\otimes} A_2$, so that $A_1 \hat{\otimes} A_2$ is the enveloping $C^*$-algebra of $A_1 \hat{\otimes} A_2$. More generally we have the following:

**Proposition 4.** If $A_1$ and $A_2$ are Banach $*$-algebras with approximate identities, $C^*(A_1 \hat{\otimes} A_2)$ is canonically isomorphic to $C^*(A_1) \hat{\otimes} C^*(A_2)$. 
Proof. Let \( u_1 \) and \( u \) the canonical morphisms \( A_1 \to C^*(A_1) \) and \( A_1 \hat{\otimes} A_2 \to C^*(A_1 \hat{\otimes} A_2) \); take some faithful representation \( \pi \) of \( C^*(A_1) \hat{\otimes} C^*(A_2) \) in a Hilbert space \( H \); let \( \pi_i \) be the restriction of \( \pi \) to \( C^*(A_i) \); \( \pi_1 \circ u_1 \) and \( \pi_2 \circ u_2 \) are commuting morphisms of \( A_1 \) and \( A_2 \) into \( \mathcal{L}(H) \); hence there exists a representation \( \hat{\rho} \) of \( A_1 \hat{\otimes} A_2 \) in \( H \) such that

\[
\hat{\rho}(a_1 \hat{\otimes} a_2) = \pi_1(u_1(a_1)) \cdot \pi_2(u_2(a_2));
\]

now there exists a representation \( \tilde{\rho} \) of \( C^*(A_1 \hat{\otimes} A_2) \) in \( H \) with

\[
\tilde{\rho}(u(a)) = \rho(a) \quad \forall a \in A_1 \hat{\otimes} A_2;
\]

in particular

\[
\tilde{\rho}(u_1(a_1) \hat{\otimes} u_2(a_2)) = \rho(a_1 \hat{\otimes} a_2) = \pi_1(u_1(a_1)) \cdot \pi_2(u_2(a_2)) = \pi(u_1(a_1) \otimes u_2(a_2));
\]

decides that \( \text{Im} \tilde{\rho} = \text{Im} \pi \); we must now show that \( \tilde{\rho} \) is faithful or, equivalently, that for each representation \( \sigma \) of \( C^*(A_1 \hat{\otimes} A_2) \) in a space \( K \) there exists a representation \( \tilde{\sigma} \) of \( \text{Im} \tilde{\rho} \) in \( K \) such that \( \tilde{\sigma} \circ \tilde{\rho} = \sigma \); set \( \tau = \tilde{\sigma} \circ \tilde{\rho} \); by remark 1, \( \tau \) admits restrictions \( \tau_1 \) and \( \tau_2 \); \( \tau_1 \) extends to a representation \( \tilde{\tau}_1 \) of \( C^*(A_1) \) in \( K \); since \( \tilde{\tau}_1 \) and \( \tilde{\tau}_2 \) commute, they define a representation \( \tilde{\tau} \) of \( C^*(A_1) \hat{\otimes} C^*(A_2) \) in \( K \) with

\[
\tilde{\tau}(u_1(a_1) \hat{\otimes} u_2(a_2)) = \tau_1(a_1), \tau_2(a_2) = \tau(a_1 \hat{\otimes} a_2) = \sigma(u_1(a_1) \hat{\otimes} u_2(a_2));
\]

set \( \tilde{\sigma} = \tilde{\tau} \circ \pi^{-1} \); we have

\[
\tilde{\sigma}(\tilde{\rho}(u(a_1 \hat{\otimes} a_2))) = \tilde{\tau}(u_1(a_1) \hat{\otimes} u_2(a_2)) = \sigma(u(a_1 \hat{\otimes} a_2))
\]

whence \( \tilde{\sigma} \circ \tilde{\rho} = \sigma \).
Corollary 1. If $G_1$ and $G_2$ are locally compact groups, $C^*(G_1 \times G_2)$ is canonically isomorphic to $C^*(G_1) \hat{\otimes} C^*(G_2)$.

Bibliography [10].

n.3.2. Tensor products of states and representations.

If $\pi_1$ is a representation of a $C^*$-algebra $A_1$, the algebraic tensor product $\pi_1 \hat{\otimes} \pi_2$ can be extended to a representation $\pi_1 \hat{\otimes} \pi_2$ of $A_1 \hat{\otimes} A_2$; this representation has the same properties as $\pi_1 \otimes \pi_2$ (see n.2.1.). On the other hand each representation $\pi$ of $A_1 \hat{\otimes} A_2$ admits restrictions $\pi_1$ and $\pi_2$ which have properties analogous to those of § 2.2. Consider now two states $f_1, f_2$ on $A_1, A_2$; $f_1 \hat{\otimes} f_2$ is continuous for the $\omega$ norm because setting $f_i = \omega_{x_i} \circ \pi_i$ we have

$$f_1 \hat{\otimes} f_2 = \omega_{x_1 \otimes x_2} \circ (\pi_1 \otimes \pi_2);$$

its extension to $A_1 \hat{\otimes} A_2$ will be denoted by $f_1 \hat{\otimes} f_2$; it is easy to see that

$$\pi_{f_1 \hat{\otimes} f_2} \simeq \pi_{f_1} \hat{\otimes} \pi_{f_2};$$

consequently $f_1 \hat{\otimes} f_2$ is pure (resp. factorial) iff $f_1$ is.

Proposition 5. The mappings $(f_1, f_2) \mapsto f_1 \hat{\otimes} f_2$ of $S(A_1) \times S(A_2)$ into $S(A_1 \hat{\otimes} A_2)$ and $(\pi_1, \pi_2) \mapsto \pi_1 \hat{\otimes} \pi_2$ of $\hat{A}_1 \times \hat{A}_2$ into $\hat{A}_1 \hat{\otimes} \hat{A}_2$ are continuous.

Proof. For the first mapping we must show that the mapping $(f_1, f_2) \mapsto (f_1 \hat{\otimes} f_2)(a)$ is continuous for each $a$ in $A_1 \hat{\otimes} A_2$; by equicontinuity we can take $a$ in $A_1 \hat{\otimes} A_2$ and the assertion becomes trivial.
The second mapping is obtained by passing to the quotients in the following commutative diagram:

\[
\begin{array}{ccc}
P(A_1) \times P(A_2) & \longrightarrow & P(A_1 \circledast A_2) \\
\downarrow T & & \downarrow \\
\hat{A}_1 \times \hat{A}_2 & \longrightarrow & A_1 \circledast A_2
\end{array}
\]

and \(T\) is open as the direct product of two open mappings ([2], 3.4.11).
§ 4. Definition and first properties of the \( * \) crossnorm.

n.4.1. **Definition of the \( * \) crossnorm.**

**Lemma 2.** Let \( A_1 \) be a concrete \( C^* \)-algebra in a Hilbert space \( H_1 \); realize \( A_1 \otimes A_2 \) in \( H_1 \otimes H_2 \) with the operator norm \( \| \cdot \| \); then for each state \( f_1 \) on \( A_1 \) and each \( a \) in \( A_1 \otimes A_2 \) we have
\[
| (f_1 \otimes f_2)(a) | \leq \| a \|
\]
for each representation \( \pi_1 \) of \( A_1 \) we have
\[
| (\pi_1 \otimes \pi_2)(a) | \| \leq \| a \|
\]

**Proof.** We have the first inequality for each pair of vector states since \( \omega_{x_1} \otimes \omega_{x_2} = \omega_{x_1 \otimes x_2} \); then for each pair \( (f_1, f_2) \) where the states \( f_1 \) and \( f_2 \) have the form
\[
f_1 = \omega_{x_1,1} + \ldots + \omega_{x_1,n_1}
\]
with \( \| x_1,1 \|^2 = 1 \)
\[
f_2 = \omega_{x_2,1} + \ldots + \omega_{x_2,n_2}
\]
with \( \| x_2,1 \|^2 = 1 \)

because
\[
| (f_1 \otimes f_2)(a) | \leq | (\omega_{x_1,1} \otimes \omega_{x_2,j})(a) |
\]
\[
= \| \omega_{x_1,1} \otimes x_2,j \| (a) \| \leq \| x_1,1 \|^2 \cdot \| x_2,j \|^2 \cdot \| a \| \leq \| a \|
\]

and finally for each pair of states \( (f_1, f_2) \) by continuity since the \( f_i \) of the previous form are dense in \( S(A_i) \) (see [2], 3.4.4).

Second assertion: \( \pi_i \) being a sum of cyclic representations we can assume \( \pi_i \) is cyclic, in a space \( K_i \), with a cyclic vector \( x_1 \) which defines a state \( f_i \) of \( A_i \); denote by \( f \) the extension of \( f_1 \otimes f_2 \) to \( A \), the uniform closure of \( A_1 \otimes A_2 \) in \( L^2(H_1 \otimes H_2) \) and consider the representation \( \pi_f \) of \( A \) in a
space $H_f$; as in [2], 2.4.2 one can construct an isomorphism of $H_f$ onto $K_1 \oplus K_2$ which carries $\pi_f(a) \cdot x_f$ into the vector $(\pi_1 \oplus \pi_2)(a) \cdot (x_1 \oplus x_2)$ for each $a$ in $A_1 \otimes A_2$, and $\pi_f(a)$ into $(\pi_1 \oplus \pi_2)(a)$; then

$$\| (\pi_1 \oplus \pi_2)(a) \| = \| \pi_f(a) \| < \| a \|.$$

**Lemma 3.** Let $A_1$ and $A_2$ be abstract C*-algebras, $\pi_1$ and $\rho_1$ representations of $A_1$ with $\text{Ker} \pi_1 \subset \text{Ker} \rho_1$; then

$$\| (\pi_1 \oplus \pi_2)(a) \| \geq \| (\rho_1 \oplus \rho_2)(a) \| \quad \forall \ a \in A_1 \otimes A_2.$$

**Proof.** Denote by $H_1$ and $K_1$ the spaces of $\pi_1$ and $\rho_1$ and set $B_1 = \pi_1(A_1) \in \mathcal{L}(H_1)$; there exists a representation $\sigma_i$ of $B_i$ in $K_i$ such that $\rho_i = \sigma_i \circ \pi_i$; we have the following mappings

$$A_1 \otimes A_2 \xrightarrow{\pi_1 \otimes \pi_2} B_1 \otimes B_2 \xrightarrow{\sigma_1 \otimes \sigma_2} \mathcal{L}(K_1 \oplus K_2)$$

and their composition is $\rho_1 \otimes \rho_2$; by the preceding lemma, for each $a$ in $A_1 \otimes A_2$:

$$\| (\rho_1 \otimes \rho_2)(a) \| = \| (\sigma_1 \otimes \sigma_2)( (\pi_1 \otimes \pi_2)(a) ) \| \leq \| (\pi_1 \otimes \pi_2)(a) \|.$$

**QED**

We are now in a position to define the $*$-crossnorm:

**Theorem 2.** Let $A_1$ and $A_2$ be two C*-algebras and $a$ an element of their algebraic tensor product; for all faithful representations $\pi_i$ of $A_i$ the number $\| (\pi_1 \otimes \pi_2)(a) \|$ has the same value, which we shall denote by $\| a \|_*$; $\| \cdot \|_*$ is a C*-crossnorm; for each representation $\rho_i$ of $A_i$, $\rho_1 \otimes \rho_2$ is continuous.
for that norm; finally \( \| a \|_* = \sup_{f_1 \in \mathcal{A}_1} \| \pi_1 \circ f_2 \| (a) \| \cdot \| \cdot \| \mathcal{F}_1 \| \mathcal{F}_2 \| \mathcal{A}_2 \| \).  

**Definition 4.** The completion of \( A_1 \circ A_2 \) for the \( * \) norm is denoted by \( A_1 \circ A_2 \); for each representation \( \pi_1 \) or state \( f_1 \) of \( A_1 \), \( \pi_1 \circ \tau_2 \) and \( f_1 \circ f_2 \) are the extensions of \( \pi_1 \circ \tau_2 \) and \( f_1 \circ f_2 \) to \( A_1 \circ A_2 \).

The identity mapping of \( A_1 \circ A_2 \) extends to a morphism \( A_1 \circ A_2 \rightarrow A_1 \circ A_2 \), so that the second algebra appears as a quotient of the first.

**Example 4.** \( \mathcal{L}^p(H_1) \circ \mathcal{L}^p(H_2) \) is nothing but \( \mathcal{L}^p(H_1 \circ H_2) \).

**Theorem 3.** If \( A_1 \) or \( A_2 \) is postliminar the \( * \) and \( \circ \) norms are identical.

In fact for each \( a \) in \( A_1 \circ A_2 \) we have \( \| a \|_\circ = \sup_{\pi \in \mathcal{A}_1 \circ \mathcal{A}_2} \| \pi(a) \| \) where \( \pi \in \mathcal{A}_1 \circ \mathcal{A}_2 \), but by proposition 2 such a \( \pi \) is equivalent to a tensor product, and we get \( \| a \|_\circ \leq \| a \|_* \).

QED

For each locally compact group \( G \) we denote by \( \mathcal{C}_r(G) \) the image of \( \mathcal{C}_r(G) \) in the left regular representation of \( G \) in the space \( L^2(G) \).

**Proposition 6.** If \( G_1 \) and \( G_2 \) are locally compact groups, \( \mathcal{C}_r(G_1 \times G_2) \) is canonically isomorphic to \( \mathcal{C}_r(G_1) \circ \mathcal{C}_r(G_2) \).

Set \( G = G_1 \times G_2 \), \( \pi_1 \) and \( \pi \) = left regular representation of \( G_1 \) and \( G \), \( U = \) canonical isomorphism of \( L^2(G_1) \circ L^2(G_2) \) onto \( L^2(G) \); for each \( f_1 \) in \( L^1(G_1) \), \( U \) carries the operator 

\( (\pi_1 \circ \pi_2)(f_1 \circ f_2) \) into \( \pi(f) \) where \( f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) \); these elements \( \pi(f) \) are contained and total in \( \mathcal{C}_r(G) \), thus
the uniform closure of \((\pi_1 \otimes \pi_2)(L^1(G_1) \otimes L^1(G_2))\) is carried by \(U\) into \(C^*_r(G) \ast\) but this uniform closure is equal to \(C^*_r(G_1) \otimes C^*_r(G_2)\).

**Corollary 2.** If \(G_1\) and \(G_2\) are amenable, the \(\ast\) and \(\nu\) norms on \(C^*_r(G_1) \otimes C^*_r(G_2)\) are equal.

By [15] a group \(G\) is amenable iff \(C^*_r(G) \sim C^*(G)\); on the other hand \(G_1 \times G_2\) is amenable if \(G_1\) and \(G_2\) are.

**Remark 2.** Consider a locally compact group \(G\) and a \(C^*\)-algebra \(A\); \(C^*(G) \otimes A\) is nothing but the crossed product \(C^*(G,A)\) defined by Zeller-Meier and Leptin among others, where the action of \(G\) in \(A\) is trivial; analogously \(C^*_r(G) \otimes A\) is equal to \(C^*_r(G,A)\); if moreover \(G\) is amenable we have \(C^*(G,A) \sim C^*_r(G,A)\), so that the \(\ast\) and \(\nu\) norms on \(C^*(G) \otimes A\) are identical.

**Bibliography** [24],[28].

**n.4.2. The fundamental property of the \(\ast\) norm.**

**Theorem 4.** The \(\ast\) norm is the smallest \(C^*\)-subcross norm on \(A_1 \otimes A_2\).

**Proof.** By lemma 1 we can suppose that \(A_1\) and \(A_2\) have units \(e_1\) and \(e_2\). Let \(p\) be a \(C^*\)-subcross norm on \(A_1 \otimes A_2\); we have to prove that for each \(\pi_i\) in \(\hat{A}_1\):

\[
\|\pi_1 \otimes \pi_2\|(a) \leq p(a) \quad \forall a \in A_1 \otimes A_2 \quad (2)
\]

let \(E\) be the set of all pairs \((\pi_1, \pi_2)\) in \(\hat{A}_1 \times \hat{A}_2\) for which (2) holds; \(E\) is closed because the mapping \((\pi_1, \pi_2) \mapsto \pi_1 \otimes \pi_2\) of \(\hat{A}_1 \times \hat{A}_2\) into \(\hat{A}_1 \otimes \hat{A}_2\) is continuous (cf. proposition 5) and the mapping \(\pi \mapsto \|\pi(a)\|\) of \(\hat{A}_1 \otimes \hat{A}_2\)
into $\mathcal{R}_+$ is lower semicontinuous ([2], 3.3.2). We will now show that $E$ is dense in $\hat{A}_1 \times \hat{A}_2$; suppose it is not dense; its complement contains a non empty elementary open set of the form $U_1 \times U_2$, where $U_1$ is the set of all $\pi_1 \circ \pi_2^{-1} a_1$, where $\pi_2$ is not identical to zero on some ideal $I_1$ of $A_1$; choose a non zero positive $a_1$ in $I_1$; then

$$(\pi_1, \pi_2) \in E \implies \pi_1(a_1) \text{ or } \pi_2(a_2) = 0 \implies (\pi_1 \circ \pi_2)(a_1 \circ a_2) = 0.$$ 

Denote by $A$ the completion of $A_1 \circ A_2$ under the norm $p$, by $B$ the commutative sub $C^*$-algebra of $A_1$ generated by $a_1$ and $e_1$, by $B \circ A_2$ the closure of $B \circ A_2$ in $A$, by $\rho$ an irreducible representation of $B \circ A_2$ in a space $K$ such that $\rho(a_1 \circ a_2) \neq 0$; since $\rho | B$ is factorial and of type I we can write

$$K = K_1 \circ K_2$$

$$\rho(b_1 \circ b_2) = \rho_1(b_1) \circ \rho_2(b_2) \quad \forall b_1 \in B, b_2 \in A_2$$

where $\rho_1$ and $\rho_2$ are irreducible representations of $B$ and $A_2$ in $K_1$ and $K_2$. On the other hand by [2], 2.10.2, there exist a Hilbert space $H$ and an irreducible representation $\pi$ of $A$ in $H$ such that $K$ is invariant under $\pi(B \circ A_2)$ and $(\pi \circ B \circ A_2)_{K} = \rho; \pi | A_2$ is factorial, and of type I since its restriction to $K$ is $\rho | A_2$, which is a multiple of $\rho_2$; thus $\pi$ is a tensor product of two irreducible representations $\pi_1, \pi_2$ of $A_1, A_2$; we have $(\pi_1, \pi_2) \in E$ and $(\pi_1 \circ \pi_2)(a_1 \circ a_2)$ is not 0 since its restriction to $K$ is $\rho(a_1 \circ a_2) \neq 0$; so we have got a contradiction.
Corollary 3. If $A_1$ and $A_2$ are simple, $A_1 \hat{\otimes} A_2$ is simple too.

It is sufficient to show that every irreducible representation $\pi$ of $A_1 \hat{\otimes} A_2$ is faithful; denote by $\pi_1$ and $\pi_2$ the restrictions of $\pi$; $\pi|A_1 \otimes A_2$ is composed of the two following mappings:

$$A_1 \otimes A_2 \xrightarrow{u} \pi_1(A_1)^* \otimes \pi_2(A_2)^* \xrightarrow{v} \mathcal{L}(H)$$

where $u(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ and $v(T_1 \otimes T_2) = T_1 T_2$; $u$ is faithful because $\pi_1$ and $\pi_2$ are, and $v$ is faithful by proposition 0; thus $\pi|A_1 \otimes A_2$ is faithful and $a \mapsto \|\pi(a)\|$ is a $C^*$-subcross norm on $A_1 \otimes A_2$; then $\|\pi(a)\| \geq \|a\|$ for each $a$ in $A_1 \otimes A_2$ and consequently for each $a$ in $A_1 \hat{\otimes} A_2$.

Bibliography [20].

n.4.3. The property $(T)$.

Definition 5. A $C^*$-algebra $A$ is said to have property $(T)$ if for every $C^*$-algebra $B$ the $*$ and $\nu$ norms on $A \otimes B$ coincide. Then by theorem 4 all $C^*$-subcross norms are identical. By theorem 3 every postliminar $C^*$-algebra has property $(T)$, and by remark 2 so does the $C^*$-algebra of every amenable locally compact group.

Proposition 7. If $A$ is the closure of the union of a family of sub $C^*$-algebras $A_i$ which have property $(T)$, then $A$ has also property $(T)$.

The union $\cup A_i \otimes B$ is dense in $A \otimes B$ for the topology of the $\nu$ norm; the $*$ and $\nu$ norms are continuous functions for this topology; on the other hand the restrictions of
these norms to each \( A_i \otimes B \) are \( C^* \)-subcross norms, and consequently must coincide.

**Remark 3.** It is unknown whether the property (T) passes to the quotients.

**Example 5 (of a \( C^* \)-algebra not having property (T)).**

Denote by \( G \) the free group with two generators \( u \) and \( v \), by \( \pi \) the left regular representation of \( G \) in \( H = L^2(G) \), by \( A \) the \( C^* \)-algebra generated by \( \pi(G) \), by \( U \) the automorphism \( f \mapsto \overline{f} \) of \( H \), which carries \( \pi \) into the right regular representation, and by \( \rho \) the representation of \( A \otimes A \) in \( H \) defined by \( \rho(a_1 \otimes a_2) = a_1 U a_2 U \). The \(*\) norm on \( A \otimes A \) is the operator norm in the Hilbert space \( H \otimes H \sim L^2(G \times G) \); we shall prove that \( \rho \) is not continuous for this norm.

Suppose \( \rho \) is continuous; choose an \( \varepsilon \) with \( 0 < \varepsilon < 1/12 \); let \( \delta \) be the Dirac function at the unit element on \( G \), considered as an element of \( H \), \( \pi^2 = \pi \otimes \pi \) the left regular representation of \( G \times G \); it is easy to check that

\[
(\omega_{\delta} \circ \rho)(\pi^2(u, u)) = 1 \quad \forall u \in G
\]

on the other hand the von Neumann algebra \( \mathcal{B} \) generated by \( A \otimes A \) is standard and it follows that every normal state of \( \mathcal{B} \) is vectorial; every state of \( A \otimes A \) can be extended to a state of \( \mathcal{B} \), and thus is a weak limit of vector states; in particular there a normed vector \( x \) in \( H \otimes H \) verifying

\[
|\omega_x(\pi^2(u, u)^{-1}) - (\omega_{\delta} \circ \rho)(\pi^2(u, u)^{-1})| \leq \varepsilon^2/2
\]

and the same for \( v \); this is equivalent to

\[
|\langle \pi^2(u, u)^{-1} x, x \rangle - 1 | \leq \varepsilon^2/2
\]
it follows that
\[ \| \tilde{u}^2(u, u)^{-1} x - x \| \leq \varepsilon \]
and the same for \( v \). For every subset \( E \) of \( G \times G \) let \( P_E \) be the projection in \( L^2(G \times G) \) associated with \( E \); we have
\[ P(g_1, g_2)E = \pi^2(g_1, g_2) \cdot P_E \cdot \pi^2(g_1, g_2)^{-1} \quad \forall g_1, g_2 \in G ; \]
hence
\[ (P(u, u)E x \mid x) = (P_E \cdot \pi^2(u, u)^{-1} x \mid \pi^2(u, u)^{-1} x) \]
\[ |(P_E x \mid x) - P(u, u)E x \mid x)| \leq |(P_E x \mid x) - (P_E x \mid \pi^2(u, u)^{-1} x)| \]
\[ + |(P_E x \mid \pi^2(u, u)^{-1} x) - (P_E \pi^2(u, u)^{-1} x \mid \pi^2(u, u)^{-1} x)| \]
\[ \leq 2 \varepsilon \]
\[ (P(u, u)E x \mid x) \geq (P_E x \mid x) - 2 \varepsilon \quad \forall E \quad . \tag{3} \]

Take \( E = B \times G \) where \( B \) is the set of all words of the form \( v^m u^n v^p \ldots \) with \( m \neq 0 \); the sets \( (u, u)^n E \) are mutually disjoint, so that
\[ \sum_{n=0}^{\infty} (P(u, u)^n E x \mid x) \leq 1 \quad ; \tag{4} \]
by virtue of (3)
\[ (P(u, u)^n E x \mid x) \geq (P_E x \mid x) - 2n \varepsilon \quad ; \]
it follows from (4) that
\[ 3 (P_E x \mid x) - 6 \varepsilon \leq 1 \]
or
\[ (P_{B \times G} x \mid x) \leq 1 / 3 + 2 \varepsilon \quad ; \]
in the same manner we get by taking \( E = A \times B \) , \( A = G - B \):
\[ (P_{A \times G} x \mid x) \leq 1 / 3 + 2 \varepsilon \quad ; \]
but \( P_{B \times G} + P_{\Delta \times G} = I \), so that
\[
1 = (x \mid x) \leq 2/3 + 4 \varepsilon < 1
\]
which is absurd.

Remark 4. The analogous of corollary 3 for \( A_1 \hat{\otimes} A_2 \) is not true. In fact denote by \( \mathcal{A} \) the von Neumann algebra (factor of type II\(_1\)) generated by \( A \); it is simple by [1], p. 275, cor. 3; let \( \sigma \) be the representation of \( \mathcal{A} \hat{\otimes} \mathcal{A} \) in \( H \) defined by
\[
\sigma(a_1 \otimes a_2) = a_1 a_2 ;
\]
consider the following diagramm
\[
\begin{array}{ccc}
A \hat{\otimes} A & \overset{\iota}{\longrightarrow} & \mathcal{A} \hat{\otimes} \mathcal{A} \\
\downarrow & & \downarrow \chi \\
A \hat{\otimes} A & \longrightarrow & \mathcal{A} \hat{\otimes} \mathcal{A}
\end{array}
\]
if \( \chi \) is injective, \( \sigma \circ \iota = \rho \) will pass to the quotient in a mapping \( A \hat{\otimes} A \longrightarrow \mathcal{L}(H) \); but we have just proved that this is not the case; then \( \chi \) is not injective and \( \mathcal{A} \hat{\otimes} \mathcal{A} \) is not simple.

Bibliography [20].
§ 3. Tensor products of states and of continuous linear functionals.

n. 5. 1. Tensor products of continuous linear functionals.

Proposition 8. If $f_1$ is a continuous linear functional on $A_1$, one has

$$| (f_1 \circ f_2)(a) | \leq \| f_1 \| \cdot \| f_2 \| \cdot \| a \| \quad \forall \, a \in A_1 \hat{\otimes} A_2.$$

Proof. We can suppose $\| f_1 \| = 1$; if we embed $A_1$ in its enveloping von Neumann algebra $\mathcal{A}_1^w$, $A_1 \hat{\otimes} A_2$ is embedded in the algebra $\mathcal{A}_1^w \hat{\otimes} A_2^w$; let $f_1 = u_1 \varphi_1$ be the polar decomposition of $f_1$ where $u_1$ is a partially isometric element in $A_1^w$ and $\varphi_1$ a state; for each $a = \sum a_1,n \hat{\otimes} a_2,n \in A_1 \otimes A_2$ we have

$$(f_1 \circ f_2)(a) = \sum f_1(a_1,n).f_2(a_2,n)$$

$$= \sum \varphi_1(u_1 a_1,n).\varphi_2(u_2 a_2,n)$$

$$= (\varphi_1 \circ \varphi_2)((u_1 \hat{\otimes} u_2)a)$$

whence by lemma 2

$$\| (f_1 \circ f_2)(a) \| \leq \| (u_1 \hat{\otimes} u_2)a \| \leq \| a \| \| (u_1 \hat{\otimes} u_2) \|.$$

Corollary 4. The $\ast$ norm is not smaller than the $\hat{\otimes}$ norm.

Proposition 9. If $A_1$ (or $A_2$) is commutative, the norms $\| \cdot \|$, $\| \cdot \ast \|$ and $\| \cdot \hat{\otimes} \|$ are identical.

By theorem 3 the first two are identical; writing $A_1 = \mathcal{C}_0(X)$ we have $A_1 \hat{\otimes} A_2 \sim \mathcal{C}_0(X,A_2)$; hence the norm $\hat{\otimes}$ is a $C^\ast$-crossnorm and by theorem 4 is greater than the $\ast$ norm.
Definition 6. If $\Pi$ is any $C^*$-crossnorm on $A_1 \hat{\otimes} A_2$ and if $f_1$ is a continuous linear functional on $A_1$, by proposition 8, $f_1 \hat{\otimes} f_2$ extends to a continuous linear functional on $A_1 \hat{\otimes} A_2$ (see definition 1), which we denote by $f_1 \hat{\otimes} f_2$; we have

$$\| f_1 \hat{\otimes} f_2 \| = \| f_1 \| \cdot \| f_2 \| .$$

In the same manner if $\pi_1$ is a representation of $A_1$ we get a representation $\pi_1 \hat{\otimes} \pi_2$ of $A_1 \hat{\otimes} A_2$; if $f_1$ is a state, $f_1 \hat{\otimes} f_2$ is also a state; by the proof of lemma 2, $\pi f_1 \hat{\otimes} f_2$ is equivalent to $\pi f_1 \hat{\otimes} \pi f_2$; hence $f_1 \hat{\otimes} f_2$ is factorial (resp. pure) iff $f_1$ has the same property.

Proposition 10. For each non-zero element $a$ in $A_1 \hat{\otimes} A_2$ there exist pure states $f_1$ and $f_2$ such that $(f_1 \hat{\otimes} f_2)(a) \neq 0$.

Proof. Realize $A_1$ in some Hilbert space $H_1$ and $A_1 \hat{\otimes} A_2$ in $H_1 \hat{\otimes} H_2$; there exist vectors $x$ and $y$ in $H_1 \hat{\otimes} H_2$ such that $(a \underline{x} | y) = 0$; then there exist vectors $x_1$ and $y_1$ in $H_1$ such that

$$(a \underline{x_1} \hat{\otimes} x_2 | y_1 \hat{\otimes} y_2) \neq 0$$

i.e.

$$(\omega x_1, y_1 \hat{\otimes} \omega x_2, y_2)(a) \neq 0;$$

$\omega x_1, y_1$ is a linear combination of states, hence there exist states $f_1$ and $f_2$ such that $(f_1 \otimes f_2)(a) \neq 0$; finally $f_1$ is a weak limit of linear combinations of pure states.

Proposition 11. For every $a$ in $A_1 \hat{\otimes} A_2$ we have

$$\| a \|^2 = \sup_{f_1 \hat{\otimes} f_2} (f_1 \hat{\otimes} f_2)(b^* a^* a b)$$

where $f_1$ is a state of $A_1$ and $b$ an element of $A_1 \hat{\otimes} A_2$ with $(f_1 \hat{\otimes} f_2)(b^* b) \leq 1$. 
Proof. Clearly the left handside is greater than the right one. To prove the converse inequality realize $A_1$ in a Hilbert space $H_1$ and $A = A_1 \otimes A_2$ in $H = H_1 \otimes H_2$; by decomposing $H_1$ in a direct sum of cyclic subspaces we can suppose $A_1$ admits a cyclic unit vector $x_i$; then $x = x_i \otimes x_2$ is cyclic for $A$; let $f_i = \omega x_i$; for each $\varepsilon > 0$ there exists an $y$ in $H$ with $\|y\| \leq 1$ and $\|ay\| \geq \|a\| - \varepsilon$; there exist $a, b \in A$ with $\| y - b x \| \leq \varepsilon$ and $\| b x \| \leq 1$; then

$$\| a y \| - \varepsilon \leq \| a y \|$$

$$\leq \| a(y - b x)\| + \| a b x \|$$

$$\| a b x \| \geq \| a y \| - \varepsilon - \| a \| \varepsilon$$

$$(f_1 \otimes f_2)(b^* a^* a b)) = (b^* a^* a b x \mid x) = \| a b x \|$$

$$\geq \| a y \| - \varepsilon - \| a \| \varepsilon$$

and finally

$$(f_1 \otimes f_2)(b^* b) = \| b x \|^2 \leq 1.$$  

Bibliography [24],[28].

n.5.2. Restrictions of states.

Consider some $C^*$-crossnorm $\| \|$ on $A_1 \otimes A_2$, some state $\varphi$ on $A_1 \otimes A_2$, $f = \omega x \otimes \pi$, and the restriction $\pi'_{1}$ of $\pi$ to $A_i$; set $f_i = \omega x \otimes \pi'_{1}$; $\pi'_{f_i}$ is equivalent to the subrepresentation of $\pi_1$ in the subspace $\overline{\pi'(A_1) \cdot x}$; the central support of this subspace in $\pi'(A_1)$' is $I$ because it contains all vectors $\pi_j(a_j), \pi_i(a_i) \cdot x$ where $j \neq i$; hence $\pi'_{f_i}$ is quasiequivalent to $\pi_1$. We call $f_i$ the restriction of $f$ to $A_i$;
we have
\[ f_1(a_1) = \lim f(a_1 \otimes v_t) \]
\[ f_2(a_2) = \lim f(u_s \otimes a_2) \]
where \((u_s)\) and \((v_t)\) are approximate identities (arbitrary) of \(A_1\) and \(A_2\); if \(f\) is factorial, \(f_1\) and \(f_2\) are also factorial; finally the restrictions of a tensor product \(f_1 \otimes f_2\) are \(f_1\) and \(f_2\).

**Proposition 12.** Let \(f\) be a pure state of \(A_1 \otimes A_2\), \(f_1\) and \(f_2\) its restrictions; if \(f_1\) (or \(f_2\)) is pure we have \(f = f_1 \otimes f_2\).

**Proof.** The projection \(E\) onto the subspace \(\pi_1(A_1)\) is minimal in \(\pi_1(A_1)\)', hence for each \(T\) in \(\pi_1(A_1)\)', \(TE\) is a scalar \(h(T)\); if \(T = \pi_2(a_2)\) we have
\[ h(\pi_2(a_2)) = (\pi_2(a_2)^E, x | x) \]
\[ = (\pi_2(a_2), x | x) = f_2(a_2) \]
then
\[ f(a_1 \otimes a_2) = (\pi_2(a_2), \pi_1(a_1), x | x) \]
\[ = (\pi_2(a_2)^E, \pi_1(a_1)^E, x | x) \]
\[ = f_2(a_2). (\pi_1(a_1), x | x) \]
\[ = f_2(a_2). f_1(a_1). \]

**Proposition 13.** In order that every pure state of \(A_1 \otimes A_2\) be a tensor product, it is necessary and sufficient that \(A_1\) or \(A_2\) is commutative.

Sufficiency: suppose \(A_1\) is commutative and take a pure state \(f\) of \(A_1 \otimes A_2\); the first restriction \(\pi_1\) of \(\pi_f\) is a multiple of some character \(\chi\), so that the first restriction \(f_1\)
of \( f \) is equal to \( \pi \); by proposition 12 we have \( f = f_1 \otimes f_2 \).

Necessity: suppose \( A_1 \) is not commutative; then it admits an irreducible representation \( \pi_i \) in a space \( H_i \) of dimension \( \geq 2 \); set \( \pi = \pi_1 \otimes \pi_2 \), take a vector \( x \) in \( H_1 \otimes H_2 \) which is not decomposable, and set \( f = \omega x \circ \pi \); we shall prove that the pure state \( f \) is not a tensor product; suppose the contrary: \( f = f_1 \otimes f_2 \); then \( \pi \) is equivalent to \( \pi_{f_1} \otimes \pi_{f_2} \), hence \( \pi_1 \) is equivalent to \( \pi_{f_1} \); there exists \( x_1 \in H_1 \) such that \( f_1 = \omega x_1 \circ \pi_1 \); then

\[
f = (\omega x_1 \circ \pi_1) \otimes (\omega x_2 \circ \pi_2) = \omega x_1 \otimes x_2 \circ \pi;\]

this implies that \( x \) is proportional to \( x_1 \otimes x_2 \), which is a contradiction.
§ 6. **Functorial properties of** $A_1 \hat{\otimes} A_2$ **and** $A_1 \hat{\otimes} A_2$.

Let us consider morphisms $u_1 : A_1 \rightarrow B_1$ where $A_1$ and $B_1$ are $C^\ast$-algebras; the function on $A_1 \hat{\otimes} A_2 : a \mapsto \|u_1 \circ u_2(a)\|$ is a $C^\ast$-subcross seminorm, consequently less than $\|a\|_\nu$; thus $u_1 \circ u_2$ can be extended to a morphism

$$u_1 \circ u_2 : A_1 \hat{\otimes} A_2 \rightarrow B_1 \hat{\otimes} B_2.$$ 

On the other hand realizing $B_1$ and $B_2$ in some Hilbert spaces we get a morphism

$$u_1 \circ u_2 : A_1 \hat{\otimes} A_2 \rightarrow B_1 \hat{\otimes} B_2;$$

clearly if $u_1$ and $u_2$ are onto, $u_1 \circ u_2$ and $u_1 \circ u_2$ are also onto; if $u_1$ and $u_2$ are injective, the same holds for $u_1 \circ u_2$.

**Remark 5.** It is not known whether $u_1$ and $u_2$ being injective implies $u_1 \circ u_2$ is injective.

**Proposition 14.** Suppose $\text{Im } u_1$ is a closed twosided ideal of $B_1$; then we have

$$\overline{\ker u_1 \circ u_2} = \overline{\ker u_1 \circ A_2} + A_1 \hat{\otimes} \overline{\ker u_2}$$

where the bar means the closure in $A_1 \hat{\otimes} A_2$.

**Proof.** Set $A = A_1 \hat{\otimes} A_2$, $u = u_1 \circ u_2$, $I_1 = \ker u_1$, $J_1 = \text{Im } u_1$, $I = I_1 \circ A_2 + A_1 \circ I_2$; $I$ is a closed twosided ideal of $A$; the canonical decomposition of $u_1$

$$A_1 \xrightarrow{u_1'} J_1 \xrightarrow{u_1''} B_1$$

gives rise to the following decomposition:

$$A \xrightarrow{u'} J_1 \hat{\otimes} J_2 \xrightarrow{u''} B_1 \hat{\otimes} B_2$$
where \( u' = u'_1 \circ u'_2 \) and \( u'' = u''_1 \circ u''_2 \).

We first prove that \( u'' \) is injective; it suffices to show that every non-degenerate subcross representation \( \pi \) of \( J_1 \circ J_2 \) can be extended to a subcross representation \( \varphi \) of \( B_1 \circ B_2 \); denote by \( \pi_1 \) and \( \pi_2 \) the (non-degenerate) restrictions of \( \pi \); they can be extended to representations \( \varphi_1 \) and \( \varphi_2 \) of \( B_1 \) and \( B_2 \) with the same weak closures (cf. [2], 2.10.4); \( \varphi_1 \) and \( \varphi_2 \) are commuting and define a representation \( \varphi \) of \( B_1 \circ B_2 \) which has the desired properties.

We have now to prove our proposition in the case where \( u_1 \) and \( u_2 \) are surjective; denote by \( w \) the canonical morphism of \( A \) onto \( A/I \); clearly \( \ker u > I \) and it is sufficient to prove that

\[
\| u(a) \| \leq \| w(a) \| \quad \forall a \in A_1 \circ A_2 ;
\]

\( w(a_1 \circ a_2) \) depends only on \( u_1(a_1) \) and \( u_2(a_2) \), let

\[
w(a_1 \circ a_2) = v(u_1(a_1), u_2(a_2))
\]

where \( v \) is a bilinear mapping \( B_1 \times B_2 \to A/I \); \( v \) defines a linear mapping (which is a morphism) \( v' : B_1 \circ B_2 \to A/I \) with

\[
v'(u(a)) = w(a) \quad \forall a \in A_1 \circ A_2 ;
\]

take \( b_1 \) in \( B_1 \) and \( a_1 \) in \( A_1 \) with \( u_1(a_1) = b_1 \); we have

\[
\| v'(b_1 \circ b_2) \| = \| v'(u(a_1 \circ a_2)) \| = \| w(a_1 \circ a_2) \| \leq \| w_1 \| \cdot \| w_2 \| ;
\]

since the norm of \( B_1 \) is the quotient norm of the norm of \( A_1 \) we get

\[
\| v'(b_1 \circ b_2) \| \leq \| b_1 \| \cdot \| b_2 \| ;
\]

so that the function on \( B_1 \circ B_2 : b \mapsto \| v'(b) \| \) is a \( C^* \)-
subcross seminorm; then for every $a \in A_1 \otimes A_2$ we have

$$\|w(a)\| = \|v'(u(a))\| \leq \|u(a)\|_v.$$  

**Corollary 5.** If $J_1$ is a closed two-sided ideal of $A_1$, $J_1 \otimes J_2$ can be identified with a closed two-sided ideal of $A_1 \otimes A_2$.

**Corollary 6.** Consider morphisms $u_i : A_i \rightarrow B_i$ and suppose $A_1$ (or $A_2$) is postliminar; then

$$\text{Ker } u_1 \otimes u_2 = \text{Ker } u_1 \otimes u_2 = \text{Ker } u_1 \otimes A_2 + A_1 \otimes \text{Ker } u_2.$$  

**Proof.** The right hand side is a closed two-sided ideal by [2], 1.8.4; set $I_1 = \text{Ker } u_1$, $C_i = \text{Im } u_i$; $I_1$ and $C_1$ are postliminar; the canonical decomposition of $u_i$ gives rise to the following commutative diagramm:

$$
\begin{array}{ccc}
\{ A_1 \otimes A_2 \} & \xrightarrow{u'} & \{ C_1 \otimes C_2 \} \\
= A_1 \otimes A_2 & \xrightarrow{u''} & B_1 \otimes B_2
\end{array}
$$

$u''$ is injective since $u'$ is injective, and we have

$$\text{Ker } u_1 \otimes u_2 = \text{Ker } u_1 \otimes u_2 = \text{Ker } u' = I_1 \otimes A_2 + A_1 \otimes I_2 = I_1 \otimes A_2 + A_1 \otimes I_2.$$  

**Bibliography** [10].
§ 7. Study of the representations of $A_1 \otimes A_2$.

n.7.1. The mappings $\Pi$, $\Pi_c$ and $\tilde{\Pi}$.

As before $A = A_1 \hat{\otimes} A_2$ denotes the completion of $A_1 \otimes A_2$ for some $C^*$-crossnorm $\|\|_\ast$; $A$ is a quotient of $A_1 \hat{\otimes} A_2$, hence Prim $A$ is a closed subset of Prim $(A_1 \hat{\otimes} A_2)$, and a closed subset of $A_1 \hat{\otimes} A_2$, and $\hat{A}$ a Borel subset of $A_1 \hat{\otimes} A_2$.

The mapping $(\pi_1, \pi_2) \mapsto \pi_1 \hat{\otimes} \pi_2$ (see definition 6) gives rise to two mappings:

$$\Pi_c : \hat{A}_1 \times \hat{A}_2 \to \hat{A}$$
$$\Pi : \hat{A}_1 \times \hat{A}_2 \to \hat{A}$$

analogously associating to every representation $\pi$ of $\hat{A}$ its restrictions we get a mapping

$$\tilde{\Pi} : \hat{A} \to \hat{A}_1 \otimes \hat{A}_2 ;$$

$\tilde{\Pi} \circ \Pi$ is the identity, so that $\Pi$ is injective and $\tilde{\Pi}$ surjective.

Theorem 5. The mapping $\Pi_c$ is bicontinuous; its image is dense if $\ast = \hat{\ast}$.

Proof. It is continuous by proposition 5; the last assertion follows from the last assertion of theorem 2; let us now prove that the mapping $T : \pi_1 \hat{\otimes} \pi_2 \to \pi_1$ is continuous; let $U$ be an open subset of $A_1$, the set of all $\pi_1$ which are not identically zero on some subset $I_1$ of $A_1$; $T^{-1}(U)$ is the set of all $\pi_1 \hat{\otimes} \pi_2$ which are not identically zero on $I_1 \otimes A_2$, hence it is open.

Theorem 6. If $A_1$ (or $A_2$) is postliminar, $\Pi_c$ and $\Pi$ are bijective; moreover if $A_1$ and $A_2$ are separable the following con-
ditions are equivalent:

(i) $A_1$ or $A_2$ is postliminar
(ii) the mapping $\gamma: \hat{A}_1 \times \hat{A}_2 \rightarrow \hat{A}_1 \otimes \hat{A}_2$ is bijective
(iii) the mapping $\gamma: \hat{A}_1 \times \hat{A}_2 \rightarrow \hat{A}_1 \otimes \hat{A}_2$ is bijective.

Proof. The first assertion has been proved in proposition 2; clearly (iii) implies (ii); to prove that (ii) implies (i) suppose $A_1$ and $A_2$ are separable and non postliminar; by [3], th.1 there exists a representation $\Pi_1$ of $A_1$ such that $\Pi_1(A_1)$ is a type II_1 hyperfinite factor; by [9], lemme 2.1, there exist a Hilbert space $K$, a factor $\mathcal{B}$ in $K$ and two isomorphisms

$$F_1 : \Pi_1(A_1)'' \rightarrow \mathcal{B}$$
$$F_2 : \Pi_2(A_2)'' \rightarrow \mathcal{B}'$$

$F_1 \circ \Pi_1$ and $F_2 \circ \Pi_2$ are commuting representations and define a representation of $A_1 \otimes A_2$ in $K$, which is irreducible and not equivalent to any tensor product since its restrictions are not of type I.

Remark 6. It is not known whether one can replace $\otimes$ by any $\otimes$ in (ii) and (iii).

Bibliography [8],[14].

n.7.2. Borel properties of $\gamma$ and $\Pi$.

In this and the following numbers we suppose $A_1$ and $A_2$ separable; for each $n = 1, 2, \ldots$ we take a Hilbert space $H_n$ of dimension $n$ and identify $H_n \otimes H_m$ with $H_{nm}$ by means of a fixed isomorphism; the spaces $\text{Fac}_n(A_1)$, $\text{Fac}_n(A)$, $\text{Fac}(A_1)$, $\text{Fac}(A)$ are endowed with their usual Borel structures; $\Theta$ and $\Theta$ are the canonical mappings $\text{Fac}(A_1) \rightarrow \hat{A}_1$ and
Fac \( (A) \rightarrow \hat{A} \); \( \hat{R}_1 \) and \( \hat{R} \) are the quasi-equivalence relations in Fac \( (A_1) \) and Fac \( (A) \); (Fac \( (A_1) \times \text{Fac} \( (A_2) \))/\( (\hat{R}_1 \times \hat{R}_2) \) has the quotient Borel structure of the product Borel structure and \( \hat{A}_1 \times \hat{A}_2 \) has the product Borel structure; the canonical bijection

\[
(\text{Fac} \( (A_1) \times \text{Fac} \( (A_2) \))/\( (\hat{R}_1 \times \hat{R}_2) \) \rightarrow \hat{A}_1 \times \hat{A}_2 (5)
\]

is easily seen to be Borel.

**Lemma 4.** The mapping \( (\pi_1, \pi_2) \rightarrow \pi_1 \circ \pi_2 \) of \( \text{Fac} \( (A_1) \times \text{Fac} \( (A_2) \) \rightarrow \text{Fac} \( (A) \) \) is Borel.

It suffices to show that for each \( n \) and \( m \) the mapping \( \text{Fac}_n(A_1) \times \text{Fac}_m(A_2) \rightarrow \text{Fac}_{nm}(A) \) is Borel; or that are Borel the mappings

\[
(\pi_1, \pi_2) \rightarrow (\pi_1 \circ \pi_2)(a).x
\]

where \( a \in A, x \in H_n \otimes H_m \); or the mappings

\[
(\pi_1, \pi_2) \rightarrow (\pi_1 \circ \pi_2)(a_1 \circ a_2).(x_1 \circ x_2)
\]

\[= \pi_1(a_1).x_1 \circ \pi_2(a_2).x_2
\]

but these mappings are continuous.

**Remark 7.** We do not know whether \( \Pi \) is Borel, because we do not know whether the mapping (5) is biborel.

**Lemma 5.** The restriction mapping \( \pi \rightarrow (\pi_1, \pi_2) \) of Fac \( (A) \) into \( \text{Fac} \( (A_1) \times \text{Fac} \( (A_2) \) \) is Borel.

It is sufficient to show that for each \( n \) the mapping \( \text{Fac}_n(A) \rightarrow \text{Fac}_n(A_1) \times \text{Fac}_n(A_2) \) is Borel; or that the mappings \( \pi \rightarrow \pi_1(a_1).x_1 \) are Borel for \( a_1 \in A_1, x_1 \in H_n \);
but such a mapping is the pointwise limit of the mappings
\[ \pi \rightarrow \pi(a_t \otimes v_t)x \] and we can choose a countable approximate identity \((v_t)\).

**Proposition 15.** The mapping \( \tilde{\pi} \) is Borel.

In fact the composed mapping
\[ \text{Fac}(A) \rightarrow \text{Fac}(A_1) \times \text{Fac}(A_2) \rightarrow (\text{Fac}(A_1) \times \text{Fac}(A_2))' \rightarrow \hat{A}_1 \times \hat{A}_2 \]
is Borel, and \( \tilde{\pi} \) is obtained from it by passing to the quotient:

**Proposition 16.** The image of \( \pi \) is a Borel subset of \( \hat{A} \).

It suffices to prove that the set \( E = \Theta^{-1}(\text{Im} \, \pi) \) is Borel in \( \text{Fac}(A) \); for each \( \pi \) in \( \text{Fac}(A) \) set \( f(\pi) = \pi_1 \otimes \pi_2 \)
where \( \pi_1 \) and \( \pi_2 \) are the restrictions of \( \pi \); \( f \) is a Borel mapping; we have
\[ E = \{ \pi \mid f(\pi) \text{ is quasi-equivalent to } \gamma \} \]
\[ = \{ \pi \mid (\pi, f(\pi)) \in \text{graph of } \hat{A} \}; \]
but this graph is Borel by [2], 7.2.3, and the mapping \( \pi \rightarrow (\pi, f(\pi)) \) is Borel.

**Bibliography** [8],[14].

n. 7.3. **Product of measures on \( \hat{A}_1 \) and \( \hat{A}_2 \).**

Given two standard Borel measures \( \nu_1 \) and \( \nu_2 \) on \( \hat{A}_1 \) and \( \hat{A}_2 \) we shall construct a "product" measure on \( \hat{A} \); the construction is made somewhat difficult by the fact quoted in remark 7.
Proposition 17. There exists a standard Borel subset $E_1$ of $A_1$, carrying $\mu_1$ and such that $\Pi \mid E_1 \times E_2$ is a Borel isomorphism of $E_1 \times E_2$ onto a standard Borel subset of $\bar{A}$; if one takes $E_1$ with these properties and sets $\mu = \Pi (\mu_1 \otimes \mu_2)$, the quasi-equivalence class $\int_{\pi}^\# \cdot d\mu (\pi)$ is the tensor product of the quasi-equivalence classes $\int_{\pi_1}^\# \cdot d\mu_1 (\pi_1)$ and $\int_{\pi_2}^\# \cdot d\mu_2 (\pi_2)$; moreover $\mu$ is central iff $\mu_1$ and $\mu_2$ are central.

Proof. By [12], th. 6.3, there exist a standard Borel subset $E_1$ carrying $\mu_1$, and a Borel mapping $R_1 : E_1 \rightarrow \text{Fac} (A_1)$ such that $\otimes_{i} \cdot R_1 = \text{identity}$; consider the (Borel) composed mapping

$$E_1 \times E_2 \xrightarrow{a} \text{Fac} (A_1) \times \text{Fac} (A_2) \xrightarrow{b} \text{Fac} (A) \xrightarrow{c} \bar{A}$$

where $a = R_1 \times R_2$, $b = \text{tensor product}$, and $c = \text{restriction}$ of $\otimes$ to $\text{Im} (b \circ a)$; $c \circ b \circ a$ is the restriction of $\Pi$ to $E_1 \times E_2$, hence $b \circ a$ is injective and its image meets each quasi-equivalence class in one point at most; by [2], B 21, this image is Borel standard and $b \circ a$ is a Borel isomorphism; by [2], 7.2.3, $c$ is a Borel isomorphism and its image, which is nothing but $\Pi (E_1 \times E_2)$, is Borel standard; we have thus proved the first assertion.

The measure $\mu$ defined in the statement is carried by $E = \Pi (E_1 \times E_2)$; let $R$ be the inverse mapping of $c$; the quasi-equivalence classes $\int_{\pi}^\# \cdot d\mu (\pi)$ and $\int_{\pi_1}^\# \cdot d\mu_1 (\pi_1)$ contain respectively the representations $\rho = \int_{E}^\# \cdot R (\pi) \cdot d\mu (\pi)$ and $\rho_1 = \int_{E_1}^\# \cdot R_1 (\pi_1) \cdot d\mu_1 (\pi_1)$, but by n.2.1 we have

$$\rho_1 \otimes \ell_2 \simeq \int_{E_1 \times E_2}^\# \cdot R_1 (\pi_1) \otimes R_2 (\pi_2) \cdot d(\mu_1 \otimes \mu_2) (\pi_1, \pi_2);$$ (6)
transporting by means of \( \Pi \) we get
\[
f_1 \otimes f_2 \sim \int_{E} R(\pi) \cdot d\nu(\pi) = f \tag{7}
\]
which proves the second assertion.

As for the last assertion denote by \( \alpha(\pi), \alpha(\pi_1), \alpha_1 \) and \( \beta_1 \) the von Neumann algebras generated by \( R(\pi), R_1(\pi_1), \rho \) and \( \rho_1 \); to say that \( \rho \) is central amounts to say that \( \alpha = \int \alpha(\pi) \cdot d\nu(\pi) \); but according to (6) and (7), \( \alpha \) can be identified with \( \alpha_1 \otimes \alpha_2 \) and \( \int \alpha(\pi) \cdot d\nu(\pi) \) with the tensor product \( \int \alpha(\pi_1) \cdot d\nu_1(\pi_1) \otimes \int \alpha(\pi_2) \cdot d\nu_1(\pi_2) \); since we have always \( \alpha \subseteq \int \alpha(\pi) \cdot d\nu(\pi) \) and \( \alpha_1 \subseteq \int \alpha(\pi_1) \cdot d\nu_1(\pi_1) \), we shall have \( \alpha = \int \alpha(\pi) \cdot d\nu(\pi) \) iff \( \alpha_1 = \int \alpha(\pi_1) \cdot d\nu_1(\pi_1) \), the last assertion being a consequence of the

**Lemma 6.** If \( \alpha_1 \subseteq \beta_1 \) are von Neumann algebras in a Hilbert space \( H \), \( \alpha_1 \otimes \alpha_2 = \beta_1 \otimes \beta_2 \) implies \( \alpha_1 = \beta_1 \).

**Proof:** suppose \( \alpha_1 \neq \beta_1 \); there exists \( T \in \beta_1 \), \( T \notin \alpha_1 \); then \( T \otimes I \notin \alpha_1 \otimes \mathcal{L}(H_2) \) as is easily deduced from the matrix representation of the elements of \( \alpha_1 \otimes \mathcal{L}(H_2) \).

**Bibliography** [8].

n.7.4. **Some properties of** \( \pi_1 \otimes \pi_2 \) **and** \( A_1 \otimes A_2 \).

**Lemma 7.** Consider \( C^* \) - algebras \( A_1 \) and \( B_1 \) in \( \mathcal{H} \) Hilbert space \( H_1 \) and \( A_1 \otimes A_2, B_1 \otimes B_2 \) as \( C^* \) - algebras in \( H_1 \otimes H_2 \); then
\[
B_1 \otimes B_2 \subset A_1 \otimes A_2 \implies B_1 \subset A_1.
\]

Suppose \( B_1 \notin A_1 \); there exists \( a \in B_1 \), \( a \notin A_1 \); there exists a continuous linear functional \( f_1 \) on \( \mathcal{L}(H_1) \) which is zero on \( A_1 \) but not on \( a \); take a continuous linear functional \( f_2 \) on \( \mathcal{L}(H_2) \) which is non-zero on \( B_2 \); by proposition 8 we have a
continuous linear functional $f_1 \otimes f_2$ on $\mathcal{L}(H_1) \otimes \mathcal{L}(H_2)$; it is zero on $A_1 \hat{\otimes} A_2$ but not on $B_1 \hat{\otimes} B_2$ – which is a contradiction.

**Definition 7.** An irreducible representation $\pi$ of a C*-algebra $A$ in a space $H$ is **traceable** (resp. **compact**) if $\pi(A)$ contains (resp. is equal to) $\mathcal{L} \bar{\varphi}(H)$.

**Proposition 18.** Let $\pi_1$ be an irreducible representation of $A_1$; $\pi_1 \hat{\otimes} \pi_2$ is traceable (resp. compact) if and only if $\pi_1$ and $\pi_2$ have the same property.

It is known that $\pi_1(a_1) \otimes \pi_2(a_2)$ is compact iff $\pi_1(a_1)$ and $\pi_2(a_2)$ are; hence $\pi_1 \hat{\otimes} \pi_2$ is compact iff $\pi_1$ and $\pi_2$ are, and $\pi_1 \hat{\otimes} \pi_2$ is traceable if $\pi_1$ and $\pi_2$ are; the converse is a consequence of lemma 7 and example 4.

**Theorem 7.** The C*-algebra $A_1 \hat{\otimes} A_2$ is liminar (resp. postliminar) if and only if $A_1$ and $A_2$ have the same property.

**Proof.** Suppose $A_1$ is liminar; every $\pi \in A_1 \hat{\otimes} A_2$ is of the form $\pi_1 \hat{\otimes} \pi_2$, hence compact since $\pi_1$ and $\pi_2$ are compact; consequently $A_1 \hat{\otimes} A_2$ is liminar. Conversely if $A_1 \hat{\otimes} A_2$ is liminar, for each $\pi_i \in \hat{A}_i$, $\pi_i$ is compact since $\pi_1 \hat{\otimes} \pi_2$ is and $A_1$ is liminar.

Suppose now $A_1$ is postliminar; each factor representation $\pi$ of $A_1 \hat{\otimes} A_2$ is of the form $\pi_1 \hat{\otimes} \pi_2$, hence is of type I; by [17], $A_1 \hat{\otimes} A_2$ is postliminar. Conversely if $A_1 \hat{\otimes} A_2$ is postliminar, for each $\pi_i \in \hat{A}_i$, $\pi_1 \hat{\otimes} \pi_2$ is of type I; it follows ([16], ch.3, § 4) that $\pi_i$ is of type I; hence $A_i$ is postliminar.
Proposition 19. Each compact irreducible representation of \( A_1 \hat{\otimes} A_2 \) is the tensor product of two compact irreducible representations.

It suffices to prove that if \( \alpha \) is a continuous factor in some Hilbert space \( H \), and \( S \) and \( T \) are non zero elements in \( \alpha \) and \( \alpha' \) respectively, then \( ST \) is not compact; we can suppose \( S \) and \( T \) are positive because \( ST \) compact implies \( ST \; T^* S^* = S \; S^* T T^* \) compact; then there exist spectral projections \( P_E \) and \( P_F \) of \( S \) and \( T \) such that
\[
S_E \; \geq \; h > 0 \quad \text{and} \quad T_F \; \geq \; k > 0 ;
\]
for \( x \in E \cap F \) we have \( Tx \in E \), hence
\[
\| STx \| \; \geq \; h \| Tx \| \; \geq \; hk \| x \| ;
\]
\( E \cap F \) is infinite dimensional since there exist projections \( Q_1, Q_2, \ldots \) in \( \alpha' \), non zero, mutually orthogonal, whose sum is \( P_F \), and we have \( P_E Q_n \neq 0 \; \forall \; n \); this proves that \( ST \) is not compact.

Bibliography [8],[22],[28],[29].

Remark 8. It is not known whether every traceable irreducible representation of \( A_1 \hat{\otimes} A_2 \) is a tensor product.
§ 8. Further results on the type of $A_1 \tilde{\otimes} A_2$.

n.8.1. Antiliminar algebras.

We shall make use of the following construction: let $f$ be a continuous linear functional on $A_1$; denote by $R_f$ the mapping $f \circ I$ of $A_1 \otimes A_2$ into $C \otimes A_2 \sim A_2$; for each $a = \sum_{n=1}^N a_1, n \otimes a_2, n$ in $A_1 \otimes A_2$ we have

$$R_f(a) = \sum_{n=1}^N < f, a_1, n > \cdot a_2, n;$$

and for each continuous linear functional $g$ on $A_2$:

$$< g, R_f(a) > = \sum < f, a_1, n > \cdot < g, a_2, n >$$

$$= < f \circ g, a >;$$

hence by proposition 8:

$$\| R_f(a) \| = \sup_{\| f \| \leq 1} | < g, R_f(a) > | \leq \| f \| \cdot \| a \|;$$

thus $R_f$ extends to a continuous linear mapping $R_f : A_1 \tilde{\otimes} A_2 \rightarrow A_2$ and we have

$$< g, R_f(a) > = < f \circ g, a > \quad (8)$$

for each $a$ in $A_1 \tilde{\otimes} A_2$ and $g$ in $A_2'$; note that $\| R_f \| \leq \| f \|$, that $R_f$ is surjective and that it is positive if $f$ is positive.

In the same manner one can define mappings $L_g : A_1 \tilde{\otimes} A_2 \rightarrow A_1$.

We set $A = A_1 \tilde{\otimes} A_2$.

Lemma 8. For every non zero $a$ in $A$ there exists a pure state $f$ on $A_1$ such that $R_f(a) \neq \emptyset$.

This is a consequence of proposition 10.
Lemma 9. If \( I \) is a twosided ideal of \( A \), \( R_f(I) \) is a twosided ideal of \( A_2 \).

Proof. We have to show that

\[ x \in I, a_2 \in A_2 \implies a_2 R_f(x) \in R_f(I); \]

take an \( \varepsilon > 0 \); there exist an \( y = \sum y_1, n \otimes y_2, n \) in \( A_1 \otimes A_2 \) with \( \|x - y\| \leq \varepsilon \), and an \( a_1 \) in \( A_1 \) with \( \|a_1\| \leq 1 \) and

\[ \|a_1 y_1, n - y_1, n\| \leq \varepsilon / \sum \|y_2, m\| \forall n; \]

we then have

\[ \| a_2 R_f(x) - a_2 R_f(y) \| \leq \| a_2 \| \cdot \| f \| \cdot \varepsilon \]

\[ \| a_2 R_f(y) - R_f((a_1 \otimes a_2)y) \| = \| R_f(\sum y_1, n \otimes a_2 y_2, n - \sum a_1 y_1, n \otimes a_2 y_2, n) \| \]

\[ = \| R_f(\sum (y_1, n - a_1 y_1, n) \otimes a_2 y_2, n) \| \]

\[ \leq \| a_2 \| \cdot \| f \| \cdot \varepsilon \]

\[ \| R_f((a_1 \otimes a_2)y) - R_f((a_1 \otimes a_2)x) \| = \| R_f((a_1 \otimes a_2)(y-x)) \| \]

\[ \leq \| a_2 \| \cdot \| f \| \cdot \varepsilon \]

whence

\[ \| a_2 R_f(x) - R_f((a_1 \otimes a_2)x) \| \leq 3 \| a_2 \| \cdot \| f \| \cdot \varepsilon; \]

since \( R_f((a_1 \otimes a_2)x) \) belongs to \( R_f(I) \) and since \( \varepsilon \) is arbitrary, the assertion follows.

Lemma 10. Denote by \( H_f \), \( \bar{\pi}_f \), \( x_f \) respectively the Hilbert space, representation and cyclic vector associated with the state \( f \) of \( A_1 \), by \( \rho \) a representation of \( A_2 \) in a space \( K \), and by \( y \) an element of \( K \); for each \( a \in A \) we have

\[ < \omega_y, \rho(R_f(a)) > = < \omega x_f, (\pi_f \otimes \rho)(a) >. \quad (9) \]
It is sufficient to prove (9) when \( a \) has the form \( a_1 \otimes a_2 \); in that case we have

\[
\langle \omega_y, \rho(R_f(a)) \rangle = \langle \omega_y, \rho(\langle f, a_1 \rangle \cdot a_2) \rangle \\
= \langle f, a_1 \rangle \cdot \langle \omega_y, \rho(a_2) \rangle \\
= \langle \omega x_f \otimes \pi_f(a_1) \rangle \cdot \langle \omega_y, \rho(a_2) \rangle \\
= \langle \omega x_f \otimes \omega_y, (\pi_f \otimes \rho)(a_1 \otimes a_2) \rangle.
\]

**Lemma 11.** If \( f \) is a pure state and \( I \) a liminar closed twosided ideal of \( A \), \( R_f(I) \) is liminar.

**Proof.** We have to show that for each irreducible representation \( \rho \) of \( A_2 \) in a space \( K \), one has

\[
\rho(R_f(I)) \subset L^\infty(K);
\]

since \( I \) is liminar we have

\[
(\pi_f \otimes \rho)(I) \subset L^\infty(H_f \otimes K);
\]

by (9)

\[
\langle g, \rho(R_f(a)) \rangle = \langle \omega x_f \otimes g, (\pi_f \otimes \rho)(a) \rangle \\
(10)
\]

for every vector state \( g \) on \( L(K) \); this is still true for every state of the form \( g = \omega y_1 + \ldots + \omega y_n \), then, by continuity, for every state of \( L(K) \) (see [2], 3.4.4); and finally for every continuous linear functional \( g \) on \( L(K) \). If \( g \) is null on \( L^\infty(K) \), \( \omega x_f \otimes g \) is null on \( L^\infty(H_f \otimes K) \), hence on the set \( (\pi_f \otimes \rho)(I) \); by (10), \( g \) is null on \( \rho(R_f(I)) \); and this proves our assertion.

**Theorem 8.** The \( C^* \)-algebra \( A_1 \otimes A_2 \) is antiliminar if and only if \( A_1 \) or \( A_2 \) is antiliminar.
The condition is necessary because if $A_1$ contains a non-zero liminar closed twosided ideal $I_1$, $I_1 \otimes I_2$ is a non-zero liminar closed twosided ideal of $A$. Conversely suppose $A_2$ is antiliminar; let $I$ be a liminar closed twosided ideal of $A$; by lemma 11, $R_f(I)$ is null for each pure state $f$ of $A_1$, and by lemma 8 this implies that $I$ is null.

Remark 9. If for some $C^*$-crossnorm $\| \cdot \|$, $A_1 \otimes A_2$ is antiliminar, $A_1$ or $A_2$ is antiliminar; in fact if $A_1$ contains a non-zero liminar ideal $I_1$, $I_1 \otimes I_2$ (the closure in $A_1 \otimes A_2$) is a non-zero closed twosided ideal in $A_1 \otimes A_2$, and is liminar since it is equivalent to $I_1 \otimes I_2$. It is not known whether $A_1$ or $A_2$ antiliminar implies $A_1 \otimes A_2$ antiliminar.

Bibliography [22].

1.8.2. Algebras with continuous trace.

For any $C^*$-algebra $A$ we denote by $p_A$ the set of all positive elements $a$ of $A$ for which the function on $\hat{A}$: $\pi \mapsto \text{Tr} \, \pi(a)$ is finite and continuous; and by $m_A$ the linear subspace spanned by $p_A$; we recall that $m_A$ is a twosided ideal and that $A$ is said to be with continuous trace if $m_A$ is dense in $A$ (see [2], 4.5.2).

We set $A = A_1 \otimes A_2$.

Lemma 12. If $f$ is a pure state of $A_1$ we have $R_f(p_A) \subset p_{A_2}$.

Proof. Choose an orthonormal basis $(x_i)$ of $H_f$ with $x_0 = x_f$, and set $f_1 = \omega x_i \otimes f_f$, $f_0 = f$; let $\rho$ be an irreducible representation of $A_2$ in a space $K$, $(y_j)$ an orthonormal basis of $K$; for every $a$ in $p_A$, $\text{Tr}((\pi_f \otimes \rho)(a))$ is finite; but
\[
\text{Tr}((\pi_f \otimes \rho)(a)) = \sum_{i,j} \langle \omega_i \otimes \omega_j, (\pi_f \otimes \rho)(a) \rangle
\]
by (9)

\[
= \sum_{i,j} \langle \omega_i, \rho(R) \rangle
\]
\[
= \sum \text{Tr} \rho(R_f(a))
\]
\[
= \text{Tr} \rho(R_f(a)) + \sum \text{Tr} \rho(R_f(a))
\]
since \(R_f(a)\) and \(R_f(a)\) are positive, we see that \(\text{Tr} \rho(R_f(a))\)
and \(\text{Tr} \rho(R_f(a))\) are finite; the function \(\rho \mapsto \text{Tr}((\pi_f \otimes \rho)(a))\)
is finite and continuous since \(a \in p_A\); for each \(i\) the function \(\rho \mapsto \text{Tr} \rho(R_f(a))\) is lower semicontinuous ([21, 3.5.9]), hence the function \(\sum_{i \neq \infty} \text{Tr} \rho(R_f(a))\) is l.s.c.; \(\text{Tr} \rho(R_f(a))\) is upper semicontinuous, and also l.s.c., hence continuous; this proves that \(R_f(a) \in p_{A_2}\).

Theorem 9. Let \(\|\|\|\) be any \(C^*\)-crossnorm on \(A_1 \otimes A_2; A_1 \otimes A_2\)
is a \(C^*\) - algebra with continuous trace if and only if \(A_1\) and \(A_2\) have the same property.

Sufficiency: since \(m_{A_1}\) is dense in \(A_1\), \(m_{A_1} \otimes m_{A_2}\) is dense in \(A_1 \otimes A_2\) and generated by elements \(a_1 \otimes a_2\) with \(a_1 \in p_{A_1}\); each \(\pi \in \hat{A_1} \otimes \hat{A_2}\) is of the form \(\pi_1 \otimes \pi_2\); for \(a_i \in p_{A_i}\) we have

\[
\text{Tr} \pi(a_1 \otimes a_2) = \text{Tr} \pi_1(a_1) \cdot \text{Tr} \pi_2(a_2)
\]
since \(\hat{A_1} \otimes \hat{A_2}\) is homeomorphic to \(\hat{A_1} \times \hat{A_2}\), we see that the function \(\pi \mapsto \text{Tr} \pi(a_1 \otimes a_2)\) is finite and continuous.

Necessity: \(A_1 \otimes A_2\) is liminar, hence identical to \(A_1 \otimes A_2\); take a pure state \(f\) of \(A_1\); by lemma 12, \(R_f(m_A) \subseteq m_{A_2}\); since
\( m_A \) is dense in \( A \) and \( R_f \) is surjective, \( m_{A_2} \) is dense in \( A_2 \).

**Remark 10.** A similar result holds for the \( C^* \)-algebras with generalized continuous trace (cf. [22], th.2).

n.8.3. **The largest postliminar ideal of** \( A_1 \hat{\otimes} A_2 \).

**Proposition 20.** Suppose \( A_1 \) (or \( A_2 \)) has property \((T)\); let \( K_1 \) be the largest postliminar ideal of \( A_1 \); then the largest postliminar ideal of \( A_1 \hat{\otimes} A_2 \) is \( K_1 \hat{\otimes} K_2 \).

**Proof.** First, \( K_1 \hat{\otimes} K_2 \) is a postliminar ideal in \( A_1 \hat{\otimes} A_2 \); now consider the following ideals

\[
K_1 \hat{\otimes} K_2 = K_1 \hat{\otimes} K_2 \subset A_1 \hat{\otimes} K_2 = A_1 \hat{\otimes} K_2 \subset A_1 \hat{\otimes} A_2 = A_1 \hat{\otimes} A_2; \\
A_1 \hat{\otimes} A_2 / A_1 \hat{\otimes} K_2 \text{ and } A_1 \hat{\otimes} K_2 / K_1 \hat{\otimes} K_2 \text{ are respectively isomorphic to } A_1 \hat{\otimes} A_2 / K_2 \text{ and } A_1 / K_1 \hat{\otimes} K_2 \text{ (cf. proposition 14), hence antiliminar by theorem 8; } A_1 \hat{\otimes} A_2 / A_1 \hat{\otimes} K_2 \text{ is isomorphic to } (A_1 \hat{\otimes} A_2 / K_1 \hat{\otimes} K_2) / (A_1 \hat{\otimes} K_2 / K_1 \hat{\otimes} K_2); \text{ this shows that } A_1 \hat{\otimes} A_2 / K_1 \hat{\otimes} K_2 \text{ is antiliminar; hence } K_1 \hat{\otimes} K_2 \text{ is the largest postliminar ideal in } A_1 \hat{\otimes} A_2.

**Remark 11.** It is not known whether the above proposition still holds without assumption.

**Bibliography** [22].

Given two von Neumann algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and two faithful normal semifinite traces $t_1$, $t_2$ on $\mathcal{A}_1$, $\mathcal{A}_2$, one can construct canonically a faithful normal semifinite trace $t_1 \hat{\otimes} t_2$ on $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ such that

$$(t_1 \hat{\otimes} t_2)(a_1 \otimes a_2) = t_1(a_1).t_2(a_2) \quad \forall a_1 \in \mathcal{A}_1^+$$

(here and in the sequel we agree that $0. \infty = \infty . 0 = 0$); in order to do this, denote by $m_1$ the definition ideal of $t_1$ and form the tensor product of the Hilbert algebras $m_1^{1/2}$, $m_2^{1/2}$; the von Neumann algebra $\mathcal{U}(m_1^{1/2} \otimes m_2^{1/2})$ is isomorphic to $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ and it suffices to transport by means of this isomorphism the natural trace of $\mathcal{U}(m_1^{1/2} \otimes m_2^{1/2})$.

We consider some $C^*$-crossnorm $\| \cdot \|$ on $A_1 \otimes A_2$.

Proposition 21. Let $f_i$ be a semifinite lower semicontinuous (s.f.l.s.c.) trace on $A_i$; one can construct canonically a s.f.l.s.c. trace $f$ on $A_1 \hat{\otimes} A_2$ such that the representation associated with $f$ is quasi-equivalent to the tensor product of those associated with $f_1$ and $f_2$, and that $f(a_1 \otimes a_2) = f_1(a_1).f_2(a_2)$ for every $a_i$ in $A_i^+$.

Proof. Denote by $m_i$ the definition ideal of $f_i$, by $\pi_i$ the representation associated with $f_i$, by $\mathcal{A}_i$ the von Neumann algebra generated by $\pi_i(A_i)$, by $t_i$ the faithful normal semifinite (f.n.s.f.) trace on $\mathcal{A}_i$ such that $f_i = t_i \circ \pi_i$; the pair $(\pi_1 \hat{\otimes} \pi_2$, $t_1 \hat{\otimes} t_2)$ is a traced representation (see definition 6.6.1 in [2]) for if $a_i \in m_i^{1/2}$ we have

$$(t_1 \hat{\otimes} t_2)(\pi_1 \hat{\otimes} \pi_2(a_1 \otimes a_2)) = t_1(\pi_1(a_1)).t_2(\pi_2(a_2)) < + \infty$$
and this proves that the trace class operators in \( \text{Im } (\pi_1 \otimes \pi_2) \) generate the von Neumann algebra \( \mathcal{A}_1 \otimes \mathcal{A}_2 \). It is then sufficient to set \( f = (t_1 \otimes t_2) \cdot (\pi_1 \otimes \pi_2) \).

Definition 8. The trace \( f \) constructed above is called the tensor product of \( f_1 \) and \( f_2 \) and denoted by \( f_1 \otimes f_2 \); it is a character iff \( f_1 \) and \( f_2 \) are characters. On the other hand it is immediately seen that \( f_1 \otimes f_2 \) is finite if \( f_1 \) and \( f_2 \) are finite; and conversely that if \( f_1 \otimes f_2 \) is finite and \( f_1 \) and \( f_2 \) are not identically zero, they are finite; in this case \( f_1 \otimes f_2 \) is nothing the tensor product of the central positive functionals \( f_1 \) and \( f_2 \).

Lemma 13. Let \( \mathfrak{A} \) be a factor, \( t \) a f.n.s.f. trace on \( \mathfrak{A}, \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) two factors included in \( \mathfrak{A} \), commuting and generating \( \mathfrak{A} \); suppose that \( 0 < f(a_1, a_2) < +\infty \) for at least one pair \((a_1, a_2)\) in \( \mathfrak{A}_1^* \times \mathfrak{A}_2^* \). Then there exist a f.n.s.f. trace \( t_1 \) on \( \mathfrak{A}_1 \) and an isomorphism of \( \mathfrak{A}_1 \otimes \mathfrak{A}_2 \) onto \( \mathfrak{A} \) carrying \( t_1 \otimes t_2 \) into \( t \) and \( a_1 \otimes a_2 \) into \( a_1 a_2 \) for each \( a_i \) in \( \mathfrak{A}_i \).

Proof. Denote by \( E_1 \) the (non empty) set of all \( a_1 \in \mathfrak{A}_1^+ \) such that \( 0 < t(a_1, a_2) < +\infty \) for at least one \( a_2 \) in \( \mathfrak{A}_2^+ \); define \( E_2 \) in a similar manner; let \( a_2 \) be an element of \( E_2 \); the function on \( \mathfrak{A}_1^+ : a_1 \mapsto t(a_1, a_2) \) is a trace which is normal (the verification is immediate), semifinite (since it assumes a value which is neither zero nor infinite) and faithful since \( t(a_1, a_2) = 0 \) implies \( a_1 a_2 = 0 \) which in turn implies \( a_1 = 0 \) (cf. prop. 0); choose a fixed f.n.s.f. trace \( t_1 \) on \( \mathfrak{A}_1 \) with definition ideal \( m_1 \); for each \( a_1 \) in \( \mathfrak{A}_1^+ \), we have

\[
t(a_1, a_2) = k_2(a_2) t_1(a_1)
\]  

(11)
where \( k_2(a_2) \) is a strictly positive number; in the same manner for \( a_1 \in E_1 \) and \( a_2 \in \mathcal{A}_2^+ \)
\[
t(a_1, a_2) = t_1(a_1) \cdot t_2'(a_2)
\]
where \( t_2' \) is a f.n.s.f. trace on \( \mathcal{A}_2 \) and \( k_1(a_1) \cdot 0 \); let \( m_2 \) be the definition ideal of \( t_2' \).

If \( a_1 \in m_1^+ - 0 \) and \( a_2 \in E_2 \), we have \( 0 < t_1(a_1) < +\infty \) whence by (11), \( 0 < t(a_1, a_2) < +\infty \); this proves that \( m_1^+ - 0 \subseteq E_1 \); in the same manner \( m_2^+ - 0 \subseteq E_2 \). Take \( a_1 \) in \( m_1^+ - 0 \); since \( a_1 \) lies in \( E_1 \) we have
\[
t(a_1, a_2) = k_2(a_2) \cdot t_1(a_1) = k_1(a_1) \cdot t_2'(a_2)
\]
thus \( k_1(a_1) / t_1(a_1) = k_2(a_2) / t_2'(a_2) \) is a number \( k \) independent of \( a_1 \) and \( a_2 \); if we set \( t_2 = k t_2' \) we get
\[
t(a_1, a_2) = t_1(a_1) \cdot t_2(a_2)
\]
for \( a_1 \in m_1^+ - 0 \), then, by linearity, for \( a_1 \in m_1 \); and also, by semifiniteness, for \( a_1 \in \mathcal{A}_1^+ \).

Denote by \( H, H_1, H_2 \) the Hilbert completions of the Hilbert algebras \( m_1^+, m_1^+, m_2^+ \); the bilinear mapping \( (a_1, a_2) \mapsto a_1 a_2 \) gives rise to a linear mapping \( F : m_1^+ \otimes m_2^+ \to m_1^+ \);

it is easily verified that \( F \) is an isometric \( * \)-morphism; then it can be extended to an isomorphism \( F \) of \( H_1 \otimes H_2 \) onto some closed subspace \( H' \) of \( H \); \( F(m_1^+ \otimes m_2^+) \) is invariant by the left multiplication operators \( U_b \) where \( b \) is in \( \mathcal{A}_1 \) or \( \mathcal{A}_2 \); the same holds for \( H' \), so that \( H' \) is invariant by all \( U_b \) where \( b \in \mathcal{A} \); similarly \( H' \) is invariant by all \( V_b \) where \( b \in \mathcal{A} \); hence \( H' = H \), \( F(m_1^+ \otimes m_2^+) \) is a dense sub Hilbert algebra of \( m_1^+ \), and
\[ \mathcal{U}(\mathbb{F}(m_1^\frac{1}{3} \otimes m_2^\frac{1}{3})) = \mathcal{U}(m_3^\frac{1}{3}) ; \]

finally the desired isomorphism is obtained by composing the following ones:

\[ \alpha_1 \otimes \alpha_2 \rightarrow \mathcal{U}(m_1^\frac{1}{3} \otimes m_2^\frac{1}{3}) \rightarrow \mathcal{U}(\mathbb{F}(m_1^\frac{1}{3} \otimes m_2^\frac{1}{3})) \rightarrow \]

\[ \rightarrow \mathcal{U}(m_3^\frac{1}{3}) \rightarrow \alpha . \]

**Proposition 22.** Every character \( f \) of \( A_1 \otimes A_2 \) such that \( 0 < f(a_1 \otimes a_2) < +\infty \) for at least one pair \( (a_1, a_2) \in A_1^* \times A_2^* \), is the tensor product of two characters.

**Proof.** The character \( f \) defines a representation \( \tau \) and a trace \( t \) on \( A = \tau(A_1 \otimes A_2)'' \); let \( \tau_i \) be the restrictions of \( \tau \), \( \delta \), the factor generated by \( \tau_i(A_i) \), \( t_i \) the trace constructed in lemma 13; the pair \( (\tau_i, t_i) \) is a traced representation:

in fact if \( 0 < f(a_1 \otimes a_2) < +\infty \) we have

\[ 0 < t(\tau(a_1 \otimes a_2)) = t(\tau_1(a_1), \tau_2(a_2)) \]

\[ = t_1(\tau_1(a_1)).t_2(\tau_2(a_2)) < +\infty \]

whence \( 0 < t_i(\tau_i(a_i)) < +\infty \) and our assertion follows by [2], 6.7.2. Denote by \( f_i \) the character \( t_i \circ \tau_i \), by \( \delta_i \) the representation associated with \( f_i \), by \( \delta_i \) the factor \( \tau_i(A_i)'' \) and by \( s_i \) the corresponding trace on \( \delta_i \); there exists an isomorphism \( \delta_i \rightarrow \mathcal{Q}_i \) carrying \( f_i \) in \( \tau_i \) and \( s_i \) in \( t_i \), whence an isomorphism \( \delta_1 \otimes \delta_2 \rightarrow \mathcal{Q}_1 \otimes \mathcal{Q}_2 \) carrying \( f_1 \otimes f_2 \) in \( \tau_1 \otimes \tau_2 \) and \( s_1 \otimes s_2 \) in \( t_1 \otimes t_2 \); by composing with the isomorphism \( \alpha_1 \otimes \alpha_2 \rightarrow \alpha \) of lemma 13, we see that \( f_1 \otimes f_2 = f \).

**Theorem 10.** The mapping \( (f_1, f_2) \rightarrow f_1 \otimes f_2 \) is a homeomorphism of \( C_1(A_1) \times C_1(A_2) \) onto \( C_1(A_1 \otimes A_2) \).
Proof. It is bijective by proposition 22 and continuous by proposition 5; we must now prove that the mapping \( f_1 \circ f_2 \mapsto f_1(a_1) \) is continuous for every \( a_1 \in A_1^+ \); choose an increasing approximate identity \((v_t)\) of \( A_2 \); \( f_1(a_1) \) is the limit of the filtering family \((f_1 \circ f_2)(a_1 \circ v_t)\), hence \( f_1 \circ f_2 \mapsto f_1(a_1) \) is l.s.c.; the same holds for \( f_1 \circ f_2 \mapsto f_2(a_2) \); in order to prove the continuity we can suppose that \( f_1 \circ f_2 \) is in the neighbourhood of some element \( f_1^0 \circ f_2^0 \); there exists \( a_2 \) in \( A_2 \) such that \( f_2^0(a_2) > 0 \); since \( f_1 \circ f_2 \mapsto f_2(a_2) \) is l.s.c., \( f_1 \circ f_2 \not\approx f_1^0 \circ f_2^0 \) implies \( f_2(a_2) > 0 \); then

\[
f_1(a_1) = (f_1 \circ f_2)(a_1 \circ a_2) / f_2(a_2)
\]

which proves that \( f_1 \circ f_2 \mapsto f_1(a_1) \) is u.s.c., and finally continuous.

Corollary 7. Suppose \( A_1 \) and \( A_2 \) are separable and set \( A = A_1 \otimes A_2 \); then \( \overline{\eta} \) induces a Borel isomorphism \((\widehat{A_1})_f \times (\widehat{A_2})_f \mapsto \widehat{A_f} \).

The restriction of \( \overline{\eta} \) to \( \widehat{A_f} \) is Borel by proposition 15 and injective by theorem 10; \( \widehat{A_f} \) and \((\widehat{A_1})_f \times (\widehat{A_2})_f \) are standard by [2], 7.4.3; then \( \overline{\eta} \mid \widehat{A_f} \) is a Borel isomorphism by [2], B 21.

Bibliography. [8].
Bibliography


[26] T. Turumaru. On the direct product of operator algebras. IV.

[27] T. Turumaru. Crossed products of operator algebras. Tôhoku


[31] A. Wulfsahn. The primitive spectrum of a tensor product of
<table>
<thead>
<tr>
<th>§ 1. Preliminaries.</th>
<th>pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>n.1.1. Tensor products of Banach spaces.</td>
<td>1</td>
</tr>
<tr>
<td>n.1.2. Tensor products of Banach $\ast$- algebras.</td>
<td>3</td>
</tr>
<tr>
<td>n.1.3. Tensor products of Hilbert spaces and von Neumann algebras.</td>
<td>4</td>
</tr>
</tbody>
</table>

| § 2. Representations of the algebraic tensor product of two $C^*$- algebras. |       |
| n.2.1. Tensor products of representations. | 6     |
| n.2.2. Restrictions of a representation of $A_1 \otimes A_2$. | 7     |

| § 3. The $\nu$ crossnorm.            |       |
| n.3.1. Definition of the $\nu$ crossnorm. | 11    |
| n.3.2. Tensor products of states and representations. | 13    |

| § 4. Definition and first properties of the $\ast$ crossnorm. |       |
| n.4.1. Definition of the $\ast$ crossnorm. | 15    |
| n.4.2. The fundamental property of the $\ast$ norm. | 18    |
| n.4.3. The property ($T$). | 20    |

| § 5. Tensor products of states anf of continuous linear functionals. |       |
| n.5.1. Tensor products of continuous linear functionals. | 24    |
| n.5.2. Restrictions of states. | 26    |

| § 6. Functorial properties of $A_1 \hat{\otimes} A_2$ and $A_1 \hat{\otimes} A_2$. | 29    |

| § 7. Study of the representations of $A_1 \hat{\otimes} A_2$. |       |
| n.7.1. The mappings $\Pi$, $\Pi'$, and $\overline{\Pi}$. | 32    |
| n.7.2. Borel properties of $\Pi$ and $\overline{\Pi}$. | 33    |
n.7.3. Product of measures on $\hat{A}_1$ and $\hat{A}_2$.

n.7.4. Some properties of $\hat{\pi}_1 \hat{\boxtimes} \hat{\pi}_2$ and $\hat{A}_1 \hat{\boxtimes} \hat{A}_2$.

§ 8. Further results on the type of $\hat{A}_1 \hat{\boxtimes} \hat{A}_2$.

n.8.1. Antiliminar algebras.

n.8.2. Algebras with continuous trace.

n.8.3. The largest postliminar ideal of $\hat{A}_1 \hat{\boxtimes} \hat{A}_2$.


Bibliography.