Noncommutative frames

- emergent, $\Delta \longleftrightarrow$ gravity

- symmetries ; localizing gauge

  \+ SW (Chamseddine)

  twisted gravity

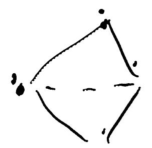
  (Wess, Aschieri, Dimitrije...

- geometry

  (differential geometry) ~ frame

  (Madore, ..)

Discretize space

1) 

2) Spectrum : x^μ (matrices)

$[x^\mu, x^\nu] \neq 0 \rightarrow$ uncert. relations

Differential geometry

- coordinate description

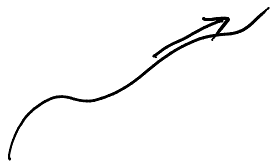
manifold $\mathcal{M}$ $\quad \mathbb{R}^n$

charts

x^μ

1) $f(x^\mu)$; $\delta(x^\mu - x^\mu_0)$

2) vector fields $X = X^\mu \partial_\mu$



$X^\mu(x)$

$$X(fg) = X(f) \cdot g + f \cdot X(g)$$

3) 1-forms χ $\quad dx^\mu$

$$dx^\mu(e_\nu) = \delta^\mu{}_\nu$$

n-dim

4) differential forms & wedge

$$dx^\mu \wedge dx^\nu = \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right)$$

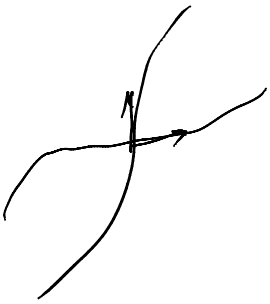
$$= P^{\mu\nu}{}_{\rho\sigma}\, dx^\rho \otimes dx^\sigma$$

5) $d:$ r-form $\rightarrow$ (r+1) form

$$df = (\partial_\mu f) \cdot dx^\mu$$

$$d(fg) = df \cdot g + f \cdot dg$$

Tetrad (moving frame)



$$g(\partial_\mu \otimes \partial_\nu) = g_{\mu\nu}$$

$$g(dx^\mu \otimes dx^\nu) = g^{\mu\nu}$$

$$e_\alpha = e^\mu{}_\alpha \partial_\mu \qquad g(e_\alpha \otimes e_\beta) = \eta_{\alpha\beta} = \text{const},$$

$$\theta^\alpha = \theta^\alpha{}_\mu dx^\mu \qquad g_{\mu\nu} = \theta^\alpha{}_\mu \theta^\beta{}_\nu \eta_{\alpha\beta}$$

$$\theta^\alpha{}_\mu e^\mu{}_\beta = \delta^\alpha{}_\beta$$

$\mu \quad \beta \quad \beta$

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma}\, \theta^\beta \theta^\gamma$$

Minkowski space

$$ds^2 = -dt^2 + (dx^i)^2 \qquad \theta^0 = dt$$

$$= -(\theta^0)^2 + (\theta^i)^2 \qquad dx^i = \theta^i$$

Schw ; $ds^2 = -\left(1-\frac{2m}{r}\right) dt^2 + \frac{1}{1-\frac{2m}{r}} dr^2 + d\Omega^2$

$$\theta^0 = \sqrt{1-\frac{2m}{r}}\, dt \ , \ \theta^1 = \frac{1}{\sqrt{1-\frac{2m}{r}}} dr$$

$$\theta^2 = r\, d\theta \quad , \ \theta^3 = r \sin\theta\, d\varphi$$

$$ds^2 = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2$$

- connection $\omega^\alpha{}_\beta = \omega^\alpha{}_{\gamma\beta}\, \theta^\gamma$

$$T^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta$$

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$$

$$\Theta = \sqrt{|g|}\; dx^1 \dots dx^n = \theta^1 \theta^2 \dots \theta^n$$

<u>Noncommutative space</u> $\mathcal{A}$

$$\mathcal{A} \not\to \mathbb{R}^n \qquad x^\mu$$

1) $$f(x^\mu) = \sum \frac{f^{(n)}(0)}{n!}\, x^n$$

2) vector fields (derivations)

$$X(fg) = X(f)\cdot g + f X(g)$$

$$hX\,? \qquad hX(fg) = h\cdot X(f)\cdot g + hf\cdot X(g)$$

$$\neq hX(f)\cdot g + f\cdot hX(g)$$

$$fh \neq hf$$

$[p, f] = X(f)$

$[p, fg] = [p, f]g + f[p, g]$ |

inner derivation

$\partial_\mu \Psi(x)$ exterior derivative

$\partial_\mu = f(x_1, x^2, \ldots)$

3) 1-forms

<u>calculus</u> $d, \ldots$

$fg \neq gf$

$f \cdot X \neq X f$

$X \wedge \omega \neq -\omega \wedge X$

NC: $\mathcal{A}$ + $[x^\mu, x^\nu] = i J^{\mu\nu}(x)$

$= i\hbar J^{\mu\nu}(x)$

$\uparrow \hbar \to 0$

d - not unique.

- respect $[\ ,\ ]$

- $d^2 = 0$

$$d(x^\mu x^\nu - x^\nu x^\mu) =$$

$$dx^\mu \cdot x^\nu + x^\mu dx^\nu - dx^\nu x^\mu - x^\nu dx^\mu$$

$$= [dx^\mu, x^\nu] + [x^\mu, dx^\nu] = i\, dJ^{\mu\nu}$$

$J^{\mu\nu} = \text{const} \qquad dJ^{\mu\nu} = 0.$

$$[x^\mu, x^\nu] = \text{const}$$

$$[dx^\mu, x^\nu] + [x^\mu, dx^\nu] = 0$$

define : $[dx^\mu, x^\nu] = 0$

$$[x^\mu, x^\nu] = i\, \overbrace{C^{\mu\nu}{}_\rho}^{\text{const}} x^\rho$$

$$[dx^\mu, x^\nu] + \underbrace{[x^\mu, dx^\nu]}_{\neq 0} = i C^{\mu\nu}{}_\rho \, dx^\rho$$

Proceed?

- from example to example
- try a general definition?

NC frames (JM)

θ^α a special role

$$[f, \theta^\alpha] = 0 . \qquad \theta^\alpha = \theta^\alpha{}_\mu dx^\mu$$

$$\theta^0 \xrightarrow{\hbar \to 0} \sqrt{1 - \frac{2m}{r}}\, dt$$

$\{\theta^\alpha\}$ frame

$$\theta^\beta(e_\alpha) = \delta^\beta_\alpha$$

$$df = (e_\alpha f) \cdot \theta^\alpha$$

$$\updownarrow$$

$$g(\theta^\alpha \otimes \theta^\beta) = \eta^{\alpha\beta} = \text{const}$$

$$e_\alpha f = [p_\alpha, f] \quad p_\alpha \ \underline{\text{momenta}}$$

$$e^\mu_\alpha = [p_\alpha, x^\mu]$$

$$dx^\mu = (e_\alpha x^\mu)\,\theta^\alpha = \boxed{e^\mu_\alpha(x)}\,\theta^\alpha$$

$$f\theta^\alpha = \theta^\alpha f$$

$$f\,dx^\mu \neq dx^\mu f$$

$$g^{\mu\nu} = g(dx^\mu \otimes dx^\nu) = g(e^\mu_\alpha \theta^\alpha \otimes e^\nu_\beta \theta^\beta$$

$$= e^\mu_\alpha\,\eta^{\alpha\beta}\,e^\nu_\beta = g^{\mu\nu}(x)$$

How this works?

1) Flat space

$$[x^\mu, x^\nu] = i J^{\mu\nu}\,\mathbb{1} = \text{const}$$

$[dx^\mu, f] = 0$

$p_\mu = (i J_{\mu\nu})^{-1} x^\nu$

$\mu, \nu, \ldots$ coor.

$\alpha, \beta \ldots$ local

$p_\alpha = \delta^\mu{}_\alpha (i J_{\mu\nu})^{-1} x^\nu$

$e^\mu_\alpha = [p_\alpha, x^\mu] = \delta^\mu_\alpha$ flat.

$\theta^\alpha = \delta^\alpha{}_\mu \cdot dx^\mu$

$d\theta^\alpha = 0 \quad (d^2 = 0)$

$\omega = 0, \ R = 0$.

2) Fuzzy sphere

$[x^m, x^n] = \frac{i\hbar}{r} \varepsilon^{mnp} x^p$

$\cancel{dx^m}$

$d ? \longleftrightarrow e_a \longleftrightarrow p_a$,

$$p_a = \frac{1}{i\hbar} \delta_{am} x^m$$

$$e^m{}_a = [p_a, x^m] = -\frac{1}{r} \varepsilon_{mab} x^b$$

$$dx^m = \frac{1}{r} \varepsilon_{mnb} x^n \theta^b$$

$$dx^a \cdot x^a = -x^a dx^a$$

$$d(x^a x^a) = 0 .$$

$$\theta^a{}_m e^m{}_b = \delta^a_b$$

$$\theta^a{}_m = \frac{1}{r} \varepsilon^{amc} x^c + \frac{1}{r^2} i\hbar \delta^{am} - \frac{1}{\hbar} x^a x^m$$

$$g^{mn} = e^m{}_a e^{na} = \frac{1}{r^2} (r^2 \delta^{mn} - x^m x^n)$$

$$g^{ab} = \delta^{ab} \qquad a, b = 1, 2, 3$$

$\cup$

$$+\frac{1}{2}\{X^m, X^n\}$$

$$+\frac{1}{2}[X^m, X^n]$$

$$X^m, \quad X^m X^m = \mathrm{const}$$

$\rightarrow$ $\underline{2d}$

$$\theta^a \qquad 3d$$

$$\{\theta^a, \theta^b\} = 0.$$

$$\omega_{acb} = -\frac{1}{2}\frac{1}{r}\varepsilon_{abc}$$

$$T = 0 \quad \Rightarrow \quad R = \frac{3}{2r^2}$$

3) Truncated Heisenberg Space
(2d)

$$[x, y] = i\varepsilon\mu^{-2}$$

$$X = \frac{1}{\sqrt{2}\cdot\mu}\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & & \\ & & & \ddots & \\ & & & & \sqrt{n-1} \end{pmatrix}$$

$$X = \frac{1}{\sqrt{2}\cdot\mu}\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \ddots & \\ & & \ddots & \ddots & \sqrt{n-1} \\ & & & \sqrt{n-1} & \\ & & & & & \ddots \end{pmatrix}$$

$$y = \frac{i}{\mu\sqrt{2}}\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & \sqrt{2} & & \\ & -\sqrt{2} & 0 & \ddots & \\ & & \ddots & \ddots & \sqrt{n-1} \\ & & & -\sqrt{n-1} & \\ & & & & & \ddots \end{pmatrix}$$

$$[\mu x, \mu y] = i\varepsilon(1 - \mu' z)$$

$$[\mu x, \mu' z] = i\varepsilon(\mu y\, \mu' z + \mu' z\, \mu y)$$

$$[\mu y, \mu' z] = -i\varepsilon(\mu x\, \mu' z + \mu' z\, \mu x)$$

$$z = \frac{n}{\mu'}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \; 1 \end{pmatrix}$$

$\mu' \to 0 \quad (z \to 0)$ limit of Heis. alg.

$\varepsilon = 1$, $\mu' = \mu$ finite $n \times n$ reprs.

geometry?

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define p_α

$d^2 = 0$

$$[p_\alpha, p_\beta] = \frac{1}{i\varepsilon} K_{\alpha\beta} + F^\gamma{}_{\alpha\beta} p_\gamma - 2i\varepsilon Q^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta$$

$$[x^\mu, x^\nu] = i\varepsilon J^{\mu\nu}(x)$$

$$\theta^\alpha \wedge \theta^\beta = \underbrace{P^{\alpha\beta}{}_{\gamma\delta}}_{\text{"antisymmetr."}} \theta^\gamma \otimes \theta^\delta$$

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma) + i\varepsilon Q^{\alpha\beta}{}_{\gamma\delta}$$

$$\varepsilon p_1 = i\mu^2 y$$

$$\varepsilon p_2 = -i\mu^2 x$$

$$\varepsilon p_3 = i\mu\left(\mu z - \frac{1}{2}\right)$$

$$(\theta^1)^2 = 0 \qquad (\theta^2)^2 = 0 \qquad (\theta^3)^2 = 0$$

$$\theta^1\theta^2 = -\theta^2\theta^1$$

$$\{\theta^1, \theta^3\} = i\varepsilon[\theta^2, \theta^3]$$

$$\{\theta^2, \theta^3\} = i\varepsilon[\theta^3, \theta^1]$$

3 – forms $\quad \theta$

$$\theta^2 / \quad \theta^1\theta^3 + \theta^3\theta^1 = i\varepsilon(\theta^2\theta^3 - \theta^3\theta^2)$$

$$\theta^2\theta^1\theta^3 + \theta^2\theta^3\theta^1 = -i\varepsilon\,\theta^2\theta^3\theta^2$$

$$\theta^1\theta^3\theta^1 = \theta^2\theta^3\theta^2$$

$$\theta^3\theta^1\theta^3 = 0 \,, \quad \theta^3\theta^2\theta^3 = 0$$

$$\theta^1\theta^2\theta^3 = -\theta^2\theta^1\theta^3 = \ldots = i\,\frac{\varepsilon^2-1}{2\varepsilon}\,\theta^2\theta^3\theta^2$$

...

$\varepsilon = 1: \quad \theta^1\theta^2\theta^3 = 0 \,,$

volume form $\sim \theta^2\theta^3\theta^2$ 1-dim vector space

<u>connection</u>

$$\omega_{\alpha\beta\gamma} = \frac{1}{2}\left(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}\right)$$

$$\omega_{12} = \mu\left(\frac{1}{2} - 2\mu z\right)\theta^3$$

$$\omega_{13} = \frac{\mu}{2}\,\theta^2 + 2\mu^2 x\,\theta^3$$

$$\omega_{23} = -\frac{\mu}{2}\,\theta^1 + 2\mu^2 y\,\theta^3$$

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} \,, \quad \Omega = d\omega + \omega\wedge\omega$$

$$R = R_{\alpha\beta}\,\eta^{\alpha\beta}$$

$$= \frac{11}{4}\mu^2 - 2\mu^2\left(2\mu z - \frac{1}{2}\right) - 4\mu^2\left(x^2+y^2\right)$$

Relation to the GW model

$$S = \int \frac{1}{2}\left(1-\frac{\Omega^2}{2}\right)(\partial_\mu\phi)(\partial^\mu\phi) + \frac{m^2}{2}\phi^2$$

$$+ \frac{\Omega^2}{2}\left(x^2+y^2\right)\phi^2 + \frac{\lambda}{4!}\phi^4$$

2d ; NC $\quad [x,y] = i\frac{\varepsilon}{\mu^2}$

trunc. Heis. $z=0$ $(N\to\infty)$

$R \sim (x^2+y^2) + \text{const}$

$$S' = \int \frac{1}{2}(e_\alpha\phi)(e^\alpha\phi) + \frac{M^2}{2}\phi^2 - \frac{\xi}{2}R\phi^2 + \frac{\lambda}{4!}\phi^4 \,)$$

$$S = \varkappa S' \qquad \varkappa = \left(1 - \frac{\Omega^2}{2}\right)$$

$$m^2 = \varkappa\left(M^2 - \xi \cdot \frac{15\mu^2}{2}\right)$$

$$\frac{\Omega^2\mu^4}{\varepsilon^2} = \varkappa\xi \cdot 8\mu^4$$

$$\lambda = \varkappa\Lambda$$

Define spinor & gauge fields?

gauge $U(1)$

$\to A = A_\alpha \theta^\alpha$

$\to F = dA + A^2 = \frac{1}{2} F_{\alpha\beta} \underbrace{P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta}_{=\theta^\alpha\theta^\beta}$

$\downarrow \qquad \downarrow$

[illegible]

$$F_{\xi\eta} = e_{[\xi}A_{\eta]} - A_\alpha C^\alpha{}_{\xi\eta} + [A_\xi, A_\eta]$$

$$+ 2i\varepsilon\,(e_\beta A_\gamma)\,Q^{\beta\gamma}{}_{\xi\eta} + 2i\varepsilon\, A_\beta A_\gamma\, Q^{\beta\gamma}{}_{\xi\eta}$$

'Covariant coordinates'

$$\theta = -p_\alpha \theta^\alpha$$ "Dirac operator"

$$A - \theta = X$$ transform covari

$$\updownarrow$$

$$F$$

under the gauge grm.

(adjoint rep.)

$$X_\alpha = p_\alpha + A_\alpha$$

$$\updownarrow$$

$$(D_\alpha = \partial_\alpha + A_\alpha)$$

YM model

matrices

$\int \to \mathrm{Tr}$

volume form $\int dV$

$^* F$

$$S = \mathrm{Tr}\, \cancel{\theta} \left({}^* F F + F {}^* F \right)$$

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3 d space

$\to$ dimen reduce to 2

$z = 0$, $p_3 = \mathrm{const}$; $e_3 f = [p_3, f] = 0$

KK red.

$A_3 = \phi$ $\quad [e_3 f] = 0$ scalar

A_1, A_2 gauge fields

$$S_{YM} = \frac{1}{2}\int (F_{12})^2 + F_{12}\phi + \phi^2$$

$$\ldots + (D_\mu \phi)^2 + \underbrace{\{p_\mu + A_\mu, \phi\}}_{X_\mu}{}^2 + F\phi^2$$

"geometrically" natural YM on such space

$\phi = 0 \quad A = 0$

$\phi = \text{const} \quad X_\alpha = 0$

$$\mathcal{F}_{12} = F_{12} - \mu\phi$$

$$\mathcal{F}_{13} = D_1\phi - i\varepsilon\underbrace{\{p_2 + A_2, \phi\}}_{X}$$

$$= [X_1, \phi] - i\varepsilon\{X_2, \phi\}$$