8 marca 2011
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EXACT QFT IN CURVED BACKGROUNDS

(I) General setup, thm's

(II) Perturbation theory

(III) DeSitter space

1) Observables in QFT
2) Wightman's axioms
3) OPE
4) Axioms CST
5) General thm's
6) Pert. theory = Hochschild cohomolgy
7) Gauge theory

1) In QM:

* State $|\psi\rangle \in \mathcal{H}$
* Observable A self-adjoint operator on $\mathcal{H}$
* Dynamical Law $A(t) = e^{itH} A e^{-itH}$

Probabilty distr. of A in $|\psi\rangle$:

$$A = \int_{-\infty}^{\infty} a \underbrace{dP(a)}_{\text{eigenprojections}} \qquad \underbrace{p_\psi(a)}_{\text{prob. distr.}} da = \langle\psi| dP(a) |\psi\rangle$$

Prob of A between a_1 & a_2 $= \int_{a_1}^{a_2} p_\psi(a)\, da$

In practice can get moments

$$\int p_\psi(a) a^n da = \langle A^n \rangle_\psi = \langle \psi | A^n | \psi \rangle = \mu_n$$

reconstruct $p_\psi(a)$ from μ_n if $|\mu_n| \leq \text{cst.}^n\, n!$

In QFT $\quad A = \int_x \underbrace{\phi(x)}_{\text{quantum field}} f(x) \quad \cancel{f \in C_0^\infty(\mathbb{R}^{D-1,1})}$

$$\langle A^n \rangle_\psi = \int_{x_1 \dots x_n} f(x_1) \dots f(x_n) \underbrace{\langle \phi(x_1) \dots \phi(x_n) \rangle_\psi}_{n\text{-pt. fct.}}$$

you want this!

(i) Scattering amplitudes

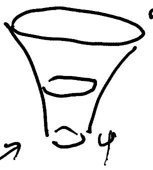
$$\langle p_1, \dots, p_n, \text{in} | p_1', \dots, p_{n'}', \text{out} \rangle$$

$$= \int_{x_1 \dots x_{n'}'} \prod_i u_{p_i}(x_i) \prod u_{p_i'}(x_i')^*$$

$$\times \prod_i (\Box_i - m^2) \prod_{i'} (\Box_i' - m^2)\, T \langle \phi(x_1) \dots \phi(x_{n'}') \rangle_\Omega$$

$$u_p(x) = \frac{1}{\sqrt{2\omega_p}} e^{+it\omega_p + i\vec{x}\vec{p}} \quad (\text{LSZ})$$

(ii)



recomb.

t

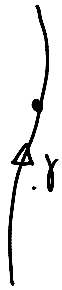
t_0

$$g = -dt^2 + a(t)^2 d\vec{x}^2$$

$$w(t, \vec{p}_1; \dots; t, \vec{p}_n) := \int \langle \phi(t, \vec{x}_1) \dots \phi(t, \vec{x}_n) \rangle_\psi \exp(i\vec{p}_1\vec{x}_1 + \dots + i\vec{p}_n\vec{x}_n)$$

$$\underbrace{w_{\{lm\}}}_{|C_{lm}|^2} = \int \underbrace{K_{\{lm\}}(\vec{p}_1, \dots, \vec{p}_n)}_{\text{Kernel}}\, w(t, \vec{p}_1, \dots, t, \vec{p}_n)$$

(iii) Unruh detectors



$\gamma(t)$

$$\mathcal{H}_{QFT} \otimes \mathcal{H}_{QM}$$

QM system (e.g. harm. oscillator)

Couple QFT & QM via

$$S_I = \lambda \int \phi(\gamma(t))\, X(t)\, dt$$

$X(t)$ Heisenb. pict. obs.

$|1\rangle, |2\rangle \in \mathcal{H}_{QM}$, $|\psi\rangle \in \mathcal{H}_{QFT}$

$$\mathcal{M}_T := \sum \frac{\lambda^n}{!} \int \langle \phi(\gamma(t_1)) \dots \phi(\gamma(t_n)) \rangle_\psi$$

$$\mathcal{M}_T := \sum_{n\geq 0} \frac{\lambda^n}{n!} \int_{0<t_1<\dots<t_n<T} \langle \phi(\gamma(t_1))\cdots \phi(\gamma(t_n))\rangle_\psi \times \langle 1|X(t_1)\cdots X(t_n)|2\rangle \, dt_1 \cdots dt_n$$

$|\mathcal{M}_T|^2$ = prob. that $|1\rangle$ gets excited to $|2\rangle$ by the quantum field in state $|\psi\rangle$

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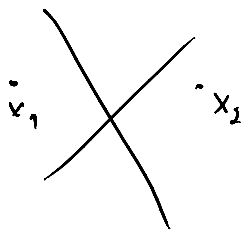
2)

1) $|\psi\rangle \in \mathcal{H}$ states
2) $\mathcal{H}$ carries a rep. of Poincaré group
3) $|\Omega\rangle$ vacuum inv. under Poincare
4) $U(g)\,\phi(x)\,U(g)^* = \phi(g^{-1}x)$
5) P generator of translatations
   spec$(P) \subset \bar{V}^+$
6) growth's of moments
7) $[\phi(x_1), \phi(x_2)]_\pm = 0$ if



Bose / Fermi

$\mathbb{R}^{D-1,1} \to (M, g$, orientation, time orientation)

7) ✓ rest not.

Not so bad ...

1) Really want $\langle \phi(x_1) \cdots \phi(x_n)\rangle_\psi$
   $(\langle A^* A\rangle_\psi \geq 0 \quad \forall A)$

Want abstract algebra of fields + expectation functional

2) general covariance ?

3) $|\Omega\rangle$?

4) —

5) μ-local spectrum condition

6) same

7) ✓

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3) $\to$ "operator product expansion" ('60's)

$$\text{"}\langle \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \rangle_\psi = \sum_b \underbrace{C^b_{a_1 \dots a_n}(x_1, \dots, x_n)}_{\text{OPE coefficients}} \cdot \langle \phi_b(x_n) \rangle_\psi \text{"}$$

ϕ_a all composite field

$$= \varphi, \nabla_\nu \varphi, \varphi^2, \varphi \nabla_\mu \varphi, \varphi^{27}, \dots, T_{\mu\nu}, \dots$$

- OPE coeff. sing. as $x_1, \dots, x_n \to \{pt\}$
- OPE coeff. indep. of ψ
- OPE coeff = structure 'const.' of QFT algebra

$$\varphi(x_1)\varphi(x_2) = \frac{1 + c_1 \log (x_1 - x_2)^2 + \dots}{(x_1 - x_2)^2} \mathbb{1}$$

$$+ \left(c_2 + c_3 \log \underbrace{(x_1 - x_2)^2}_{\sigma\text{ - geodesic dist}^2} + \dots\right) \varphi^2(x_2)$$

$$+ \dots$$

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4) C1) "Coordinate free notation"

$$V = \underbrace{\mathbb{C}}_{\substack{\text{identity} \\ \mathbb{1}}} \oplus \bigoplus_{\Delta > 0} \underbrace{V_\Delta}_{\text{fields of "dimensions" } \Delta} = \text{vector space of composite fields}$$

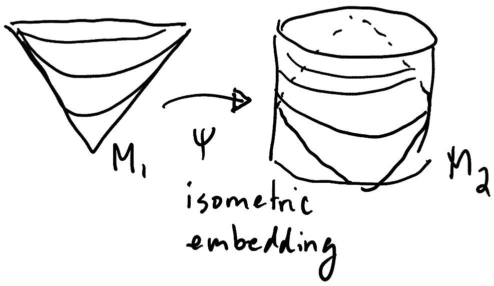
$$V_\Delta = \bigoplus_s \mathbb{C}^{n_{s,\Delta}} \otimes \underbrace{(SM)}^s$$

$$V_\Delta = \bigoplus_s \mathbb{C}^{n_{s,\Delta}} \otimes \underbrace{(SM)^s}_{\text{spin bundle}}$$

$$C(x_1, \dots, x_n) : V^{\otimes n} \longrightarrow V$$

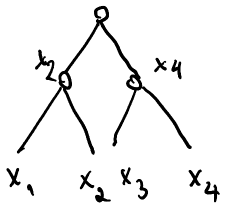
distributional in $x_1, \dots, x_n \in M$

C2) Covariance :



$$C_{M_1} = \psi^* C_{M_2}$$

C3) Factorization ("Associativity", "crossing symmetry")



x_i - spacelike

$d(x_1, x_2) < d(x_2, x_4)$

$d(x_3, x_4) < d(x_2, x_4)$

$\phi_{a_1} \quad \phi_{a_2} \quad \phi_{a_3} \quad \phi_{a_4}$

$$C(x_1, \dots, x_4) = C(x_2, x_4) \circ \big(C(x_1, x_2) \otimes C(x_3, x_4) \big)$$

$$\phi_{a_1} \phi_{a_2} \phi_{a_3} \phi_3 = \big((\phi_{a_1} \phi_{a_2})(\phi_{a_3} \phi_{a_4}) \big)$$

True for all trees T

C4) $WF(C) = \{ (x_1, \dots, x_n; k_1, \dots, k_n) \in T^* M^n \mid$

$\exists$ graph $G \subset M$, $Vert_G = \{x_1, \dots, x_n\}$
$Edge_G = \{\gamma_1, \dots, \gamma_E\}$

* γ_e geodesics oriented towards future
* $s(\gamma_e) < t(\gamma_e)$
* $\forall v \in Vert_G : k_v = \sum_{e:\, s(e)=v} \dot{\gamma}_e - \sum_{e:\, t(e)=v} \dot{\gamma}_e \}$

$WF(\delta_0) = \{ (0,k) \mid k \in \mathbb{R} \}$

$WF\left(\frac{1}{\quad}\right) = \{ (0,k) \mid k \in \mathbb{R} \}$

$$WF\left(\frac{1}{x \pm i0}\right) = \{ (0,k) \mid k \in \mathbb{R}^{\mp} \}$$

WFs condition $\Rightarrow$ analyticity

C5) If x_i, x_{i+1} spacelike

$$C(\dots, x_i, x_{i+1}, \dots) = \pm\, C(\dots, x_{i+1}, x_i, \dots) \circ (id^{i-1} \otimes P \otimes id^{n-i-1})$$

(Bose - Fermi)

C6) Condition relation dimension Δ of field to singularity degree of C.

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a state called $\langle \dots \rangle_\psi$ is a hierarchy of fct's

$$w_{a_1 \dots a_n}(x_1, \dots, x_n) = \langle \phi_{a_1}(x_1) \dots \phi_{a_n}(x_n) \rangle \text{ s.t.}$$

S1) Pos. defn. $\langle A^* A \rangle_\psi \geq 0$

S2) $\langle \mathbb{1} \rangle_\psi = 1$

S3) The OPE holds for spacelike related points $x_1, \dots, x_n$ that are suff. close.

Related VOA (in 2d CFT)

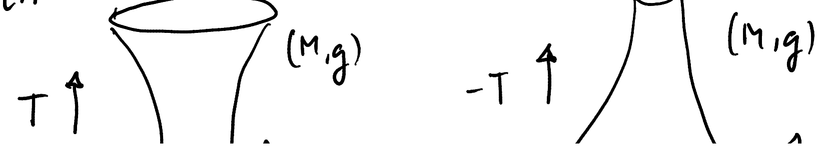
5) Consequences (Thm.) of S and C axioms.

(i) PCT - theorem (parity - charge - time)

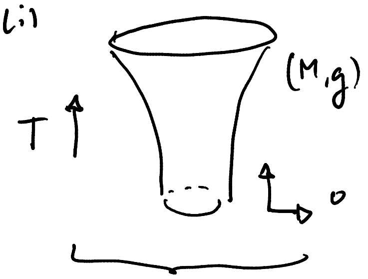
(ii) Spin & statistic

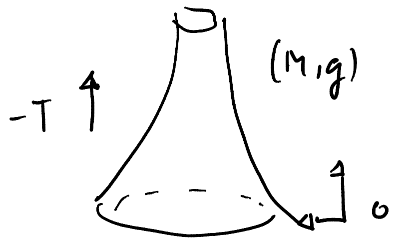
(iii) Coherence thm

(i)



(i)





$\mathcal{M} = (M, g, T, o)$
(M, e)

$\bar{\mathcal{M}} = (M, g, -T, o)$
$= (M, -e)$

$$C_{\mathcal{M}} \overset{?}{=} C_{\bar{\mathcal{M}}}$$

Thm:

$$C_{\bar{\mathcal{M}}}(x_1, \dots, x_n) = (-1)^F c \circ C_{\mathcal{M}}(x_n, \dots, x_1) \circ c^{\otimes n}$$

c – charge-conjugation $V \to V$ anti-linear

Proof: Curvature expansion, analytic cont.

(ii) Spin-Statistics ($D > 2$)

Bose fields $\to$ integer spin
Fermi —"— $\to$ ½-integer spin

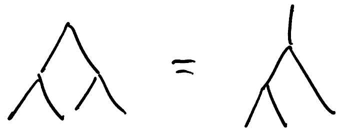
Thm: ✓

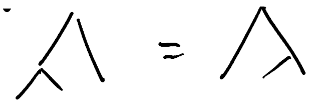
(iii) Coherence:

A - finite dim. algebra

$(ab)c = a(bc)$ (Assoc.)

$(ab)(cd) \overset{?}{=} ((ab)c)d$ (is consequence of Ass)



Factorization conditions (for trees T) follow from 

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6) General perturbation theory

$$C(x_1, x_2; \lambda) = \sum_{i=0}^{\infty} \underbrace{C_i(x_1, x_2)}_{} \lambda^i$$

λ – coupling parameter

$C_i(x_1, x_2)$ – i-th order perturbative correction

Normally have C_0 e.g. OPE of free QFT.

C_1 – 1^{st} correction

C_2 – 2^{nd} correction ...

Factorization: 

$$d(x_2, x_3) < d(x_1, x_2) < d(x_1, x_3)$$

$$C(x_1, x_3)(id \otimes C(x_2, x_3)) =$$
$$C(x_2, x_3)(C(x_1, x_2) \otimes id)$$

has to hold $\forall \lambda$

gives a constraint on $C_1, C_2, \ldots$

Cohomogly theory describes constraints

$$\Omega^n = \{ f(x_1, \ldots, x_n) : V^{\otimes n} \to V \}$$

$$b : \Omega^n \to \Omega^{n+1}$$

$$(bf)(x_1, \ldots, x_n) := C_0(x_1, x_{n+1})(id \otimes f(x_2, \ldots, x_{n+1}))$$
$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, \hat{x}_j, \ldots, x_n)(id^{i-1} \otimes C_0(x_i, x_{i+1}) \otimes id^{n-i-1})$$

Lemma: $b^2 = 0$ (C_0 satisfies factorization)

$$H^n = \frac{\ker(\Omega^n \to \Omega^{n+1})}{\mathrm{Im}(\Omega^{n-1} \to \Omega^n)}$$

It turns out : $bC_1 = 0$, $C_1 = bz \Leftrightarrow$ field redefn.
1st order deformations $\in H2$
It turns out : Obstruction for n-th order pert. $\in H^3$ for any n.

H^2 = 1-1 corresp w. interaction Lagrangians