Relatywistczna Mechanika Kwantowa

Równanie Kleina- Gordona

Równanie Kleina-Gordona i rozwiązania z $E<0$

$E^2 = p^2 + m^2$ $\qquad -\frac{\partial^2\phi}{\partial t^2} = -\nabla^2\phi + m^2\phi$

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi$$

r. K-G $\quad \begin{cases} \partial^\mu\partial_\mu\phi + m^2\phi = 0 \\ (\Box + m^2)\phi = 0 \end{cases}$ $\quad -i\phi^* \,|$

$\phi^*(\Box + m^2) = 0 \quad |\cdot i\phi$

$$\frac{\partial}{\partial t}\underbrace{\left[i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\right]}_{\rho} + \nabla\cdot\underbrace{\left[-i(\phi^*\nabla\phi - \phi\nabla\phi^*)\right]}_{\vec{j}} = 0$$

$\phi = N e^{-i(Et - \vec{p}\cdot\vec{x})} \longrightarrow E = \pm\sqrt{p^2+m^2}$ $\begin{cases} E>0 \\ i \\ E<0 \end{cases}$

$\phi^* = N^* e^{+i(Et - \vec{p}\cdot\vec{x})}$

$\partial_\mu j^\mu = 0$

$\uparrow$

$\rho = i(-2iE)|N^2| = 2E\,|N|^2$

$\vec{j} = -i(2i\vec{p})|N|^2 = 2\vec{p}\,|N|^2$

$j^\mu = (\rho, \vec{j}) = 2p^\mu |N|^2$

$= i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$

ρ jest $\propto E$ czasowa składowa 4-wektora.

niezmienniczość Lorentzowska: boost $d_3x \longrightarrow \frac{d_3x}{\gamma}$

$\rho\, d_3x \xrightarrow{\text{tr. L}} (\rho\gamma)\frac{d_3x}{\gamma}$

$\boxed{\text{Dla } E<0 \quad \rho<0}$ Wygląda jak nieszczęście

∞ wiele stanów o $E<0$

$\downarrow$

1927 r. Diraca - interpretacja: stany o $E<0$ są pełne.

1934 Pauli & Weisskopf $(-e)$ dla cząstek

$$j^\mu = -ie(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

$E<0 \longrightarrow \underline{E>0 \text{ i } e>0}$ antycząstki

Interpretacja Feynmana-Stueckelberga (1948/1941)

Elektron o $E>0$ i ładunku $-e$

$$j^\mu(e^-) = -2e|N|^2(E, \vec{p})$$

Pozyton o $(E, \vec{p})$, $E>0$

$$j^\mu(e^+) = +2e|N|^2(E, \vec{p}) = -2e|N|^2(-E, -\vec{p})$$

$\uparrow e^+$, $E>0$ $\quad\equiv\quad$ $\downarrow e^-$, $(-E)<0$ $\qquad \uparrow$ czas

pozyton $\equiv$ ujemna energia e^- poruszającego się do tyłu w czasie

bo

$$e^{-i(-E\cdot t)} = e^{-iEt}$$

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Cząstka K-G o ładunku $(-e)$: bezspinowy elektron
lub mion

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \quad \to \quad A^\mu = (A^0, \vec{A})$$

$$p^\mu \Rightarrow p^\mu + eA^\mu$$

$$i\partial^\mu \Rightarrow i\partial^\mu + eA^\mu$$

$$(\partial_\mu \partial^\mu + m^2)\phi = -V\phi$$

$$V(x) = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - \underbrace{e^2A^2}_{\text{do zaniedbania } O(\alpha)}$$

$$T_{fi} = -i\int d_4x\, \phi_f^* V(x)\, \phi_i =$$

$$= i\int d_4x\, \phi_f^*\, ie(A^\mu\partial_\mu + \partial_\mu A^\mu)\phi$$

całkowanie przez części

$$\int d_4x[\phi_f^*(\partial_\mu A^\mu)\phi + \phi_f^* A^\mu(\partial_\mu \phi)] = -\int d_4x(\partial_\mu \phi_f^*)A^\mu \phi_i$$

wobec tego

$$T_{fi} = -i\int d_4x\, j_{\mu,fi}\, A^\mu$$

$$j_\mu = -ie[\phi_f^*\partial_\mu\phi_i - (\partial_\mu\phi^*)\phi_i]$$

$$\phi_i = N_i e^{-ip_i\cdot x}$$

$$j_{\mu,fi} = -eN_iN_f^*(p_i+p_f)_\mu e^{i(p_f-p_i)x}$$

Kowariantna normalizacja

$$\rho = 2E|N|^2 \qquad \int_V \rho\, d_3x = 2E \quad \to \quad N = \frac{1}{\sqrt{V}}$$

Cząstka naładowana w równaniu K-G

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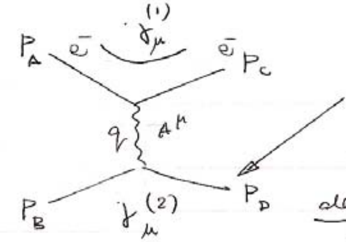

$$\Box^2 A^\mu = j^\mu_{(2)}$$

$$j^\mu_{(2)} = -eN_BN_D^*(p_D+p_B)^\mu e^{i(p_D-p_B)x}$$

ale

$$\Box^2 e^{iq\cdot x} = -q^2 e^{iq\cdot x}$$

$$A^\mu = -\frac{1}{q^2}j^\mu_{(2)} \qquad q = p_D - p_B$$

$$T_{fi} = -i\int d_4x\, j^{(1)}_\mu(x)\left(-\frac{1}{q^2}\right)j^\mu_{(2)}(x)$$

$$= -iN_AN_C^*N_BN_D^*(2\pi)^4\delta_4(p_D+p_C-p_A-p_B)\mathcal{M}$$

tensor metryczny

$$-i\mathcal{M} = (ie(p_A+p_C)^\mu)\left(-i\frac{g_{\mu\nu}}{q^2}\right)(ie(p_B+p_D)^\nu)$$

niezmiennicza amplituda

propagator fotonu

Przekrój czynny $A + B \to C + D$

$$W_{fi} = \frac{|T_{fi}|^2}{T\cdot V} = (2\pi)^4\frac{\delta_4(p_C+p_D-p_A-p_B)\,|\mathcal{M}|^2}{V^4}$$

$$\sigma = \frac{W_{fi}}{\text{strumień padający}} \text{ (liczba stanów końcowych)}$$

$$\text{(liczba stanów końcowych)} = \frac{V\, d_3p_D}{(2\pi)^3\, 2E_D}\;\frac{V\, d_3p_C}{(2\pi)^3\, 2E_C}$$

strumień padający w LAB

$|\vec{v}_A|\, 2E_A/V$ = # cząstek padających / jednostkę powierzchni / time

$\frac{2E_B}{V}$ liczba cząstek tarczy /V

strumień padający $= |\vec{v}_A|\,\frac{2E_A}{V}\,\frac{2E_B}{V}$

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Przekrój czynny

$$d\sigma = \underbrace{\frac{\cancel{\hbar}^2}{|\vec{v}_A|\, 2E_A\, 2E_B}}_{\text{Flux}} \frac{1}{\cancel{\hbar}^4} |\mathcal{M}|^2 \underbrace{\frac{(2\pi)^4}{(2\pi)^6} \delta_4(p_C + p_D - p_A - p_B) \frac{\cancel{\hbar}^2 d_3 p_C}{2E_C} \frac{\cancel{\hbar}^2 d_3 p_D}{2E_D}}_{dLIPS}$$

$$d\sigma = \frac{|\mathcal{M}|^2}{\text{Flux}}\, dLIPS$$

$$\text{Flux} = |\vec{v}_A|\, 2E_A\, 2E_B \qquad |\vec{v}_A| = \frac{|\vec{p}_A|}{E_A} \quad \text{w LAB}$$

$$\text{Flux} = |\vec{v}_A - \vec{v}_B|\, 2E_A\, 2E_B = 4\left(|\vec{p}_A| E_B + |\vec{p}_B| E_A\right) = 4\left[(p_A \cdot p_B)^2 - m_A^2\right.$$

CMS A + B

$$dLIPS = \frac{1}{4(4\pi)^2} \frac{p_f}{4\sqrt{s}}\, d\Omega$$

$$\text{Flux} = 4 p_i \sqrt{s}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{p_f}{p_i} |\mathcal{M}|^2$$

Cząstka Diraca

Równanie Diraca

PAM Dirac 1) r.r powinno być liniowe w $\frac{\partial}{\partial t}$ → zachowanie prawdop. jak w r. Schrödingera

2) Niezmienniczość rel. → liniowość w $\frac{\partial}{\partial x_i}$ $\vec{p} = -i\partial_i$

$$i\frac{\partial\psi}{\partial t} = \left[-i\left[\alpha_1\frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_3\frac{\partial}{\partial x_3}\right] + \beta m\right]\psi$$

3) $E^2 = p^2 + m^2$ | r. ruchu: daje nam r. operatorowe

$$-\frac{\partial^2\psi}{\partial t^2} = (-\nabla^2 + m^2)\psi \qquad i\frac{\partial}{\partial t} = -i\alpha_i\partial_i + \beta m$$

wstawiając

$$\frac{\partial^2\psi}{\partial t^2} = \left[-i\frac{\partial}{\partial t}\right]i\frac{\partial\psi}{\partial t} = [i\alpha_i\partial_i - \beta m][-i\alpha_i\partial_i + \beta m]\psi$$

$$= \alpha_i^2\frac{\partial^2\psi}{\partial x_i^2} + \sum_{j>i}(\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2\psi}{\partial x^i\partial x^j} + im(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} +$$

$$-\beta^2 m^2\psi \qquad \boxed{\begin{aligned}\alpha_i\alpha_j + \alpha_j\alpha_i &= 0\\ \alpha_i\beta + \beta\alpha_i &= 0\\ \alpha_i^2 = \beta^2 &= 1\end{aligned}}$$

4) $m = 0$ $\quad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ jak dla macierzy Pauliego

Przyjmujemy np: $\alpha_i = -\sigma_i$

$$i\frac{\partial\psi}{\partial t} = \vec{\sigma}\cdot\vec{p}\,\psi \quad\to\quad \psi = \chi = \begin{pmatrix}a\\ b\end{pmatrix}$$

5) $m \neq 0$ – β nie jest trywialne $\{\alpha_i, \beta\} = 0$
– nie ma rozwiązań 2×2 bo $\vec{\sigma}$ wyczerpują algebrę su(2)

Rozw. 4×4

$$\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} \qquad \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \gamma^i = \begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix}$$

Macierze γ

$$\gamma^i = \beta\alpha_i \qquad \gamma^0 = \beta \qquad \gamma^\mu = (\gamma^0, \gamma^i)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (\gamma^i)^2 = -1, \quad (\gamma^0)^2 = +1$$

r. Diraca

$$i\frac{\partial\psi}{\partial t} = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \quad |\ \beta\cdot \qquad \boxed{\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\psi = 0}$$

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\vec{\gamma}\cdot\nabla\psi - m\psi = 0$$

Skalary spinowe:

$$\psi^\dagger\gamma_\mu\psi = (\ \ )(\ \ )4(\ \ )4$$

$$\bar{\psi} \stackrel{df}{=} \psi^\dagger\gamma^0 \quad\to$$ wyrażenia typu $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\psi$ są wtedy też spinowymi skalarami

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Tr. Lorentza przekształca tj. skalarów spinowych:

$\bar{\psi}\psi$	skalar	$\bar{\psi}\gamma^5\psi$	pseudoskalar
$\bar{\psi}\gamma^\mu\psi$	wektor	$\bar{\psi}\gamma^5\gamma^\mu\psi$	pseudowektor

Sprzężenie hermitowskie i prądy

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma^k\frac{\partial\psi}{\partial x^k} - m\psi = 0 \to \left| -i\frac{\partial\psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial\psi^\dagger}{\partial x^k}\gamma^{k\dagger} - m\psi^\dagger = 0 \right| \cdot\gamma^0$$

$$\boxed{\begin{aligned}&\gamma^{0\dagger} = \gamma^0 \qquad (\gamma^0)^2 = 1 \qquad (\gamma^k)^2 = -1\\ &\gamma^{k\dagger} = (\beta\alpha^k)^\dagger = \alpha^{k\dagger}\beta = \alpha^k\beta = -\beta\alpha^k = -\gamma^k\end{aligned}}$$

$$i\frac{\partial\bar{\psi}}{\partial t}\gamma^0 + i\frac{\partial\bar{\psi}}{\partial x^k}\underbrace{\gamma^k}_{-\gamma^{\dagger k}} + m\bar{\psi} = 0 \qquad \left| i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi} = 0 \right|$$

$$\gamma^{k\dagger}\gamma^0 = -\gamma^0\gamma^{k\dagger} = +\gamma^0\gamma^k \qquad F\ \nabla^\mu$$

$$\left| \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \right|$$

Zad 1 Udowodnij że dla cząstki swobodnej o p^μ zachodzi

$$\bar{\psi}\gamma^\mu\psi = \frac{p^\mu}{m}\bar{\psi}\psi$$

Zinterpretuj wynik

Lewo- i Prawoskrętne fermiony

R. Diraca c.d (3)

Rozwiązanie swobodne; Fermiony lewo- i prawoskrętne

inna reprezentacja γ^μ: $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$

w której γ^5 jest diagonalna $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Poszukujemy rozwiązania w postaci $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ $\quad \psi_{R,L}\}$ 2-komp spinory

$\boxed{\psi_{sw} = u\, e^{-i p_\mu x^\mu}}$ $\quad \boxed{\bar{u} u = 2m}$

$\boxed{(\gamma^\mu p_\mu - m)\psi_{sw} = 0}$ $\quad p_\mu = -i(\partial_0, +\nabla)$

$$\begin{pmatrix} -m & p_0 + \vec{\sigma}\cdot\vec{p} \\ p_0 - \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0 \quad \begin{cases} (p_0 + \vec{\sigma}\cdot\vec{p})\psi_L = m\psi_R \\ (p_0 - \vec{\sigma}\cdot\vec{p})\psi_R = m\psi_L \end{cases} \quad \text{lub} \begin{pmatrix} u_R \\ u_L \end{pmatrix}$$

skrętność ≡ rzut spinu na kier. pędu

lub
$$\begin{cases} \psi_L = \left(\dfrac{p_0 - \vec{\sigma}\cdot\vec{p}}{m}\right)\psi_R \\ \psi_R = \left(\dfrac{p_0 + \vec{\sigma}\cdot\vec{p}}{m}\right)\psi_L \end{cases}$$

a) Rozwiązania dla $p_0 > 0$ i $p_0 < 0$ istnieją i mogą być zamieniane $\psi_L \Leftrightarrow -\psi_R$

b) dla $m = 0$ ψ_L i ψ_R separują się!

$E>$ righthanded $\rightarrow$ $p_0 > 0$, $\vec{p}\cdot\vec{\sigma} > 0$; $p_0 < 0$ $\vec{p}\cdot\vec{\sigma} < 0$ $\}$ ψ_R jest wielkie, $\psi_L \ll \psi_R$ $\qquad \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0} \approx \vec{\sigma}\cdot\hat{p}$ $\quad \dfrac{\vec{p}}{p_0} = \dfrac{\vec{p}}{p}$

$E_0 > 0$ lewoskrętne $\rightarrow$ $p > 0$ $\vec{\sigma}\cdot\vec{p} < 0$; $p < 0$ $\vec{\sigma}\cdot\vec{p} > 0$ $\}$ ψ_L jest wielkie, $\psi_R \ll \psi_L$

c) dla $m \neq 0$ równania nie separują się.
Człon masowy $\Leftrightarrow$ oddziaływanie między ψ_L i ψ_R

(4)

Przyp.: $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Def: $P_L = \dfrac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad P_R = \dfrac{1+\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Operatory rzutowe: $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L + P_R = 1$, $P_L P_R = 0$

$$\begin{cases} u_L \overset{df}{=} P_L u \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_R \\ u_L \end{pmatrix} = \begin{pmatrix} 0 \\ u_L \end{pmatrix} \\ \text{podobnie} \quad u_R = P_R u = \begin{pmatrix} u_R \\ 0 \end{pmatrix} \end{cases}$$

Rozwiązania r. D dla cząstek swobodnych (1

$\psi = u(\vec{p})\, e^{-ip\cdot x}$ $\quad u$ – 4-spinor, nie zależny od x!

$(\not{p} - m)u = 0 \qquad \not{p} = \gamma^\mu p_\mu$

Dla stanów o określonej energii $p_0 = E$:

$Hu = (\vec{\alpha}\cdot\vec{p} + \beta m)u = Eu$

4 niezależne rozwiązania: dwa $E>0$, dwa $E<0$

I. Cząstka w spoczynku: $\vec{p} = 0$

$$\beta m u = \begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u = E u$$

$E = m, m, -m, -m$

$$u = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

elektrony $q=-e$; pozytony $q=+e$ o $E>0$

II $\vec{p} \neq 0$

$$\begin{pmatrix} mI & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -mI \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\vec{\sigma}\cdot\vec{p}\; u_B = (E-m)\, u_A$$

$$\vec{\sigma}\cdot\vec{p}\; u_A = (E+m)\, u_B$$

$E>0$ $\quad u_A(s) = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\chi^{(1)}}$ lub $\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\chi^{(2)}}$

wtedy $u_B^{(s)} = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)}$ $\qquad u^{(s)} = N\begin{pmatrix}\chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)}\end{pmatrix}$ $\quad E>0$

dla $E<0$ $\qquad u_B(s) = \chi^{(s)}$

$$u_A(s) = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\, u_B(s) = -\frac{\vec{\sigma}\cdot\vec{p}\,\chi^{(s)}}{|E|+m}$$

$$u(s+2) = N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}\,\chi^{(s)}}{|E|+m} \\ \chi^{(s)} \end{pmatrix} \quad E<0$$

Swobodna cząstka Diraca

Gdy (3

$\vec{p} = p(\sin\theta, 0, \cos\theta) \qquad \chi = \begin{pmatrix}\chi_1\\\chi_2\end{pmatrix}$

$$\frac{1}{2}\left[\begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}\right]\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix}$$

$\chi_1\cos\theta + \chi_2\sin\theta = \chi_1 \longleftarrow \begin{cases}\chi_1(\cos\theta - 1) = -\chi_2\sin\theta \\ \chi_2 = \chi_1\dfrac{1-\cos\theta}{\sin\theta}\end{cases}$

$\chi_1\sin\theta - \chi_2\cos\theta = \chi_2$

$\chi_1\sin\theta = \chi_2(1+\cos\theta)$

$$\chi_1\sin\theta = \chi_1\frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta} \Rightarrow \chi_1 = \frac{1-\cos^2\theta}{\sin^2\theta} = 1$$

$$\chi_2 = \frac{1-\cos\theta}{\sin\theta}$$

$$\cos\theta + \frac{1-\cos\theta}{\sin\theta}\sin\theta = 1$$

$$\sin\theta - \frac{1-\cos\theta}{\sin\theta}\cdot\cos\theta = \frac{1-\cos\theta}{\sin\theta}$$

$$\frac{\sin^2\theta - \cancel{1}\cos\theta + \cos^2\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}\ !$$

$$\chi = \begin{pmatrix} 1 \\ \frac{1-\cos\theta}{\sin\theta}\end{pmatrix} N \qquad N^2\left(1 + \frac{(1-\cos\theta)^2}{\sin^2\theta}\right) = 1$$

$$N^2\,\frac{\sin^2\theta + 1 + \cos^2\theta - 2\cos\theta}{\sin^2\theta} = 1$$

$$N^2 = \left(\frac{2-2\cos\theta}{\sin^2\theta}\right)^{-1} = \left[2\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\right]^{-1}$$

Swobodna cz. Diraca cd.

$$\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1$$

$$\chi = \begin{pmatrix} -\dfrac{1}{\cos^2\frac{\theta}{2}} \\[2ex] \dfrac{\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}} \end{pmatrix}$$

$$-\cos\theta \, \frac{1}{\cos^2\frac{\theta}{2}} + \frac{\sin\theta \sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}} = -\frac{1}{\cos^2\frac{\theta}{2}}$$

$$-\sin\theta \, \frac{1}{\cos^2\frac{\theta}{2}} - \frac{\cos\theta \sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}} = \frac{\sin\theta/2}{\cos^3\frac{\theta}{2}}$$

$$\frac{-\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}}$$

$$= -\frac{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos^3\frac{\theta}{2}}$$

Antycząstki

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Antycząstki

$u^{(1)}(\vec p)$ i $u^{(2)}(\vec p)$ – swobodny elektron o $E>0$ i pędzie $\vec p$

$u^{(3)}(\vec p)$ $u^{(4)}(\vec p)$ – swobodny pozytron o $-E>0$ i pędzie $-\vec p$

$$u^{(3,4)}(\vec p)\, e^{-i(-p\cdot x)} \equiv v^{(2,1)}(\vec p)\, e^{i\vec p\cdot\vec x} \quad i \quad p_0 = E > 0$$

Spinory pozytronowe v = wygoda notacji

$$\boxed{(\not p + m)\, v(\vec p) = 0 \qquad p_0 = E > 0}$$

$\vec p, \vec\Sigma, \lambda$ $\qquad$ $-\vec p, \vec\Sigma, +\lambda$

Normalizacja spinorów $\displaystyle\int_{V=1} \rho\, dV = \int \psi^+\psi\, dV = u^+ u = 2E$

$$u^{(r)+} u^{(s)} = 2E\,\delta_{rs} \quad ; \quad v^{(r)+} v^{(s)} = 2E\,\delta_{rs} \qquad r,s = 1,2$$

$$u^{(s)+} u^{(s)} = |N|^2\left[1 + \left(\frac{\vec\sigma\cdot\vec p}{E+m}\right)^2\right] = |N|^2\,\frac{(E+m)^2 + \overbrace{(\vec\sigma\cdot\vec p)^2}^{p^2}}{(E+m)^2} = |N|^2\,\frac{2E}{E+m}$$

$$u^{(s)} = \sqrt{E+m}\begin{pmatrix}\chi^{(s)} \\ \frac{\vec\sigma\cdot\vec p}{E+m}\,\chi^{(s)}\end{pmatrix}$$

Podobnie:

$$v^{(s)} = \sqrt{E+m}\begin{pmatrix}\frac{\vec\sigma\cdot\vec p}{E+m}\,\chi^{(s+2)} \\ \chi^{(s)}\end{pmatrix}$$

$$\bar u^{(s)} = u^+\gamma^0 : \quad \begin{cases}\bar u^{(s)} u^{(s)} = 2m \\ \bar v^{(s)} v^{(s)} = -2m\end{cases}$$

Cząstka Diraca w polu e-m

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Cząstka Diraca w polu e-m

$$\psi = u(\bar{p})\, e^{-ip\cdot x} \qquad (\gamma^\mu p_\mu - m)\psi = 0$$

$$p^\mu \to p^\mu + eA^\mu$$

$$(\gamma_\mu p^\mu - m)\psi = \underbrace{\gamma^0 V}\psi \qquad \gamma^0 V = -e\gamma_\mu A^\mu$$

$$T_{fi} = -i\int \psi_f^\dagger(x)\gamma^0 V(x)\psi_i(x)\, d_4x = ie\int \bar{\psi}_f \gamma_\mu A^\mu \psi_i\, d_4x$$

$$= -i\int j_{fi,\mu} A^\mu\, d_4x \qquad j_{fi,\mu} = -e\bar{\psi}_f\gamma_\mu\psi_i$$

$$= -e\,\bar{u}_f\gamma_\mu u_i\, e^{+i(p_f-p_i)\cdot x}$$

Dla R. K-G otrzymaliśmy podobny prąd

(oddziaływanie tylko ładunku)

$$(j_\mu^{fi})_{K-G} = -e(p_f+p_i)_\mu\, e^{i(p_f-p_i)\cdot x}$$

$$\boxed{u_f\gamma^\mu u_i = u_f\,\frac{(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu}{2m}\,u_i}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

W granicy nierelatywistycznej

$$\int\psi_f^\dagger\Big[-\frac{e}{2m}\, i\sigma^{\mu\nu}(p_f-p_i)_\nu\Big]A_\mu\psi_i = \int\psi_f^\dagger\frac{e}{2m}(\vec{\sigma}\cdot\vec{B})\psi_i$$

Relacje zupełności

$$\sum_{s=1,2} u^{(s)}(\bar{p})\,\bar{u}^{(s)}(p) = \not{p} + m$$

$$\sum_{s=1,2} v^{(s)}(\bar{p})\,\bar{v}^{(s)}(p) = \not{p} - m$$

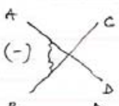
Rozpraszanie e^+e^-

Rozpraszanie $\overset{*}{e}\bar{e} \to \overset{*}{e}\bar{e}$ (Møllera)

$$T_{fi} = -i \int j^{(A)}_\mu(x)\left(-\frac{1}{q^2}\right) j^\mu_{(B)}(x)\, d_4x = \qquad q = p_A - p_C$$

$$= -i\,(-e\,\bar{u}_C \gamma_\mu u_A)\left(-\frac{1}{q^2}\right)\left(-e(\bar{u}_D \gamma^\mu u_B)\right)(2\pi)^4 \delta_4(p_A+p_B-p_C-p_D)$$

$$-i\,\mathcal{M} = i(e\,\bar{u}_C\gamma_\mu u_A)\left(-\frac{i g_{\mu\nu}}{q^2}\right)(+ie\,\bar{u}_D\gamma_\nu u_B) + (-)$$



$$= -e^2\frac{(\bar{u}_C\gamma_\mu u_A)(\bar{u}_D\gamma^\mu u_B)}{\underbrace{(p_A-p_C)^2}_{t}} + e^2\frac{(\bar{u}_D\gamma_\mu u_A)(\bar{u}_C\gamma^\mu u_B)}{\underbrace{(p_A-p_D)^2}_{u}}$$

identyczne cząstki i antysymet.

Uśrednienie spinu (niespolaryzowane)

$$|\mathcal{M}|^2 \to \overline{|\mathcal{M}|^2} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{\substack{\text{wszystkie}\\ \text{stany}\\ \text{spinowe}}} |\mathcal{M}|^2$$

uśrednianie po spinach początkowych

NIETRYWIALNE, jak to zrobić

$|\vec{p}| \to 0$

$$u^{(s)}_{A,B} = \sqrt{2m}\begin{pmatrix}\chi^{(s)}\\ 0\end{pmatrix} \qquad \bar{u}^{(s)}_{C,D} = \sqrt{2m}\,(\chi^{(s)+}, 0)$$

$$\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}$$

$$\bar{u}^{(s)}\gamma^\mu u^{(s)} = \begin{cases} 2m & \mu = 0\\ 0 & \mu \neq 0\end{cases} \qquad \bar{u}^{(s)}\gamma^\mu u^{(s')} = 0 \quad s \neq s'$$

Nierelatywistycznie, nie zmienia się kierunek spinu

(tylko oddz. elektryczne bo magn. siła Lorentza ≈ 0)

$$\mathcal{M}(\uparrow\uparrow \to \uparrow\uparrow) = \mathcal{M}(\downarrow\downarrow \to \downarrow\downarrow) = -e^2\, 4m^2\left(\frac{1}{t} - \frac{1}{u}\right)$$

$$\mathcal{M}(\uparrow\downarrow \to \uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow \to \downarrow\uparrow) = -e^2\, 4m^2\, \frac{1}{t} \quad \text{tylko}$$

$$\mathcal{M}(\uparrow\downarrow \to \downarrow\uparrow) = \mathcal{M}(\downarrow\uparrow \to \uparrow\downarrow) = +e^2\, 4m^2\, \frac{1}{u} \quad \text{tylko}$$

czyli

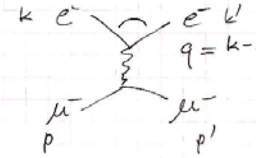
$$\overline{|\mathcal{M}|^2} = \frac{1}{4}(4m^2e^2)^2\, 2\left(\left(\frac{1}{t}-\frac{1}{u}\right)^2 + \frac{1}{t^2} + \frac{1}{u^2}\right)$$

w CMS $\quad t = 2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}$

$\qquad\quad u = 2p^2(1+\cos\theta) = -4p^2\cos^2\frac{\theta}{2}$

$$\frac{d\sigma}{d\Omega} = \frac{m^2\alpha^2}{16p^2}\left[\frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}} - \frac{1}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right] \qquad \alpha = \frac{e^2}{4\pi}$$

Rozpr. $e^-\mu^- \to e^-\mu^-$ (nie by było komplikacji z identycznymi cząstkami)



$$\mathcal{M} = -e^2\, \bar{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\bar{u}(p')\gamma_\mu u(p)$$

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{q^4} L_e^{\mu\nu} L^{\text{muon}}_{\mu\nu}$$

$$L_e^{\mu\nu} = \frac{1}{2}\sum_{\text{spiny } e}\left[\bar{u}^{s'}(k')\gamma^\mu u^{s}(k)\right]\left[\bar{u}^{s'}(k')\gamma^\nu u^{s}(k)\right]^* ; \quad \text{podobnie dla } L^{\text{muon}}_{\mu\nu}$$

$$\left[u^+(k')\gamma^0\gamma^\nu u(k)\right]^* = \left[u^+(k')\gamma^0\gamma^\nu u(k)\right]^+ = \left[u^+(k)\gamma^{\nu+}\gamma^0 u(k')\right]$$

$$= \left[\bar{u}(k)\gamma^\nu u(k')\right] \quad \text{bo} \quad \gamma^{\mu+}\gamma^0 = \gamma^0\gamma^\nu$$

$$L_e^{\mu\nu} = \frac{1}{2}\underbrace{\sum_{s'} u^{s'}_\sigma(k')\bar{u}^{s'}_\alpha(k')}_{(\not{k}'+m)_{\sigma\alpha}}\gamma^\mu_{\alpha\beta}\underbrace{\sum_s u^s_\beta(k)\bar{u}^s_\varrho(k)}_{(\not{k}+m)_{\beta\varrho}}\gamma^\nu_{\varrho\sigma}$$

$$= \frac{1}{2}\left[(\not{k}'+m)_{\sigma\alpha}(\gamma^\mu_{\alpha\beta})(\not{k}+m)_{\beta\varrho}\gamma^\nu_{\varrho\sigma}\right] =$$

$$= \frac{1}{2} M_{\sigma\sigma} = \frac{1}{2}\,\mathrm{Tr}\, M$$

Sumowanie po spinach

Sumowanie po spinach zatem sprowadza się do policzenia śladów iloczynów macierzy (8

$$L_{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left((\not{k}' + m) \gamma^{\mu} (\not{k} + m) \gamma^{\nu} \right)$$

Ślady można liczyć bez obliczania macierzowego, z reguł algebraicznych.

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \qquad \not{a} = \gamma_{\mu} a^{\mu}$$

Tr (nieparzysta l. macierzy γ) $= 0$

$$\mathrm{Tr}(\not{a}\not{b}) = 4 a\cdot b \qquad \mathrm{Tr}\, 1 = 4$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c) \right)$$

$$\mathrm{Tr}(\not{k}'\gamma^{\mu}\not{k}\gamma^{\nu}) = 4(k'^{\mu}k^{\nu} - k'\cdot k\, \eta^{\mu\nu} + k'^{\nu}k^{\mu})$$

$$\mathrm{Tr}\, \gamma_5 = 0$$

$$\mathrm{Tr}(\gamma_5 \not{a}\not{b}) = 0$$

$$\mathrm{Tr}(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = 4i\,\varepsilon_{\mu\nu\lambda\sigma} a^{\mu} b^{\nu} c^{\lambda} d^{\sigma}$$

Także:

$$\begin{cases} \gamma_{\mu}\gamma^{\mu} = 4 \\ \gamma_{\mu}\not{a}\gamma^{\mu} = -2\not{a} \\ \gamma_{\mu}\not{a}\not{b}\gamma^{\mu} = 4a\cdot b \\ \gamma_{\mu}\not{a}\not{b}\not{c}\gamma^{\mu} = -2\not{c}\not{b}\not{a} \end{cases}$$

$$2k'^{\mu}k^{\nu} \cdot 2p'_{\mu}p_{\nu} = 4(k'\cdot p')(k\cdot p)$$

4 czy 8 ?

$$L_e^{\mu\nu} = \frac{1}{2}\mathrm{Tr}[\not{k}'\gamma^{\mu}\not{k}\gamma^{\nu}] + \frac{m^2}{2}\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) =$$

$$= \frac{1}{2}\left[4k'^{\mu}k^{\nu} + 4k'^{\nu}k^{\mu} - 4k'\cdot k\,\eta^{\mu\nu} + 4m^2\eta^{\mu\nu} \right] =$$

$$L^{\text{muon}}_{\mu\nu} = \frac{1}{2}\left[4p'_{\mu}p_{\nu} + 4p'_{\nu}p_{\mu} - 4p'\cdot p\,\eta_{\mu\nu} + 4M^2\eta_{\mu\nu} \right]$$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') - m^2 p'\cdot p - M^2 k'\cdot k + 2m^2M^2 \right]$$

$$\rightarrow \frac{8e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p') \right] = 2e^2\,\frac{s^2+u^2}{t^2}$$

Anihilacja e^+e^-

(9

$$\left.\frac{d\sigma}{d\Omega}\right|_{CMS} = \frac{1}{64\pi^2 s}\frac{p'}{p}\overline{|\mathcal{M}|^2}$$

$$\left(\frac{k'}{k}\right)$$

Relacje krzyżowania. $e^+e^- \to \mu^+\mu^-$

Zamieniamy $k' \to -p$

$s = (k+p)^2 \approx 2k\cdot p \approx 2k'\cdot p' \longrightarrow t$

$t = (k-k')^2 = -2k\cdot k' = -2p\cdot p' \longrightarrow s$

$u = (k-p')^2 = -2k\cdot p' \approx -2k'p$

$$\overline{|\mathcal{M}|^2} = 2e^4\frac{t^2+u^2}{s^2} \qquad \alpha = \frac{e^2}{4\pi} \qquad e^4 = \overbrace{(4\pi)^2}^{16\pi^2}\alpha^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{1+\cos^2\theta}{2}\right)$$

$$\boxed{\sigma = \int d\phi\, d\theta\, \frac{d\sigma}{d\Omega}\sin\theta = \frac{4\pi\alpha^2}{3s} = \frac{87\,nb}{s}}$$