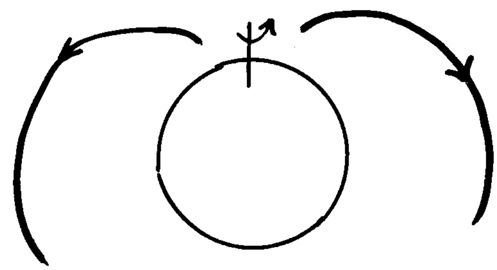
Spin – Gravity Coupling

Mathisson (1937), Papapetrou (1951), Dixon (1974)

$$F^{\alpha} = -\frac{1}{2} R^{\alpha}{}_{\beta\mu\nu}\, u^{\beta} S^{\mu\nu}$$

Linear Approximation of GR (spinning particle at rest, first-order term, etc.): $\mathbf{f} = -\boldsymbol{\nabla}(\mathbf{S}\cdot\boldsymbol{\Omega}_P)$

$$\mathcal{H}_{spin-gravity} = \mathbf{S}\cdot\boldsymbol{\Omega}_P$$



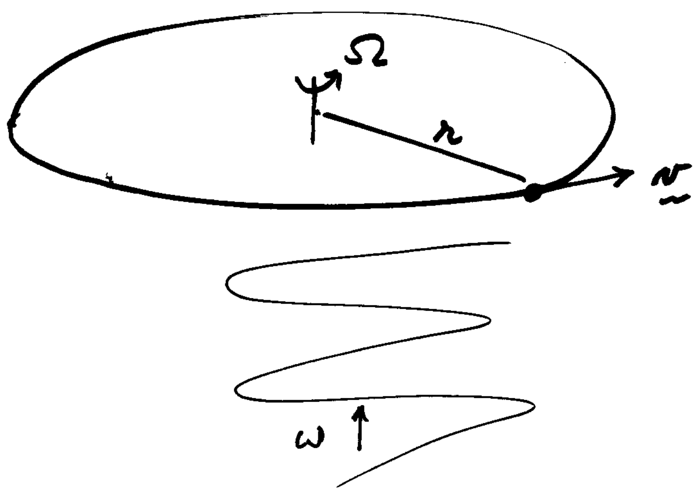
$$\boldsymbol{\Omega}_P = \frac{G}{r^5}\left[3(\mathbf{J}\cdot\mathbf{r})\mathbf{r} - \mathbf{J}\, r^2\right]$$

Gravitomagnetic Field

The Gravitational Larmor Theorem:

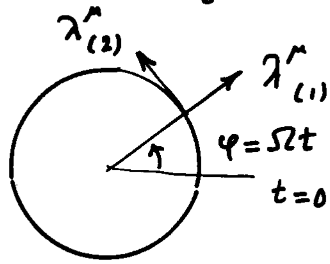
$$\mathcal{H}_{spin-rotation} = -\mathbf{S}\cdot\boldsymbol{\Omega}$$

Intrinsic Spin? "Stern-Gerlach" force ⇒ violation of universality of free fall!



$$\beta = \frac{v}{c} = \frac{1}{c} r\Omega$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}}$$



$$\tau = t/\gamma = t\sqrt{1-\beta^2}$$

$$\lambda^\mu{}_{(0)} = \gamma\,(1,\ -\beta\sin\varphi,\ \beta\cos\varphi,\ 0)$$

$$\lambda^\mu{}_{(1)} = (0,\ \cos\varphi,\ \sin\varphi,\ 0)$$

$$\lambda^\mu{}_{(2)} = \gamma\,(\beta,\ -\sin\varphi,\ \cos\varphi,\ 0)$$

$$\lambda^\mu{}_{(3)} = (0,\ 0,\ 0,\ 1)$$

(i) $F_{(\alpha)(\beta)}(\tau) = F_{\mu\nu}\,\lambda^\mu{}_{(\alpha)}\,\lambda^\nu{}_{(\beta)}$

(ii) Fourier Analyze $F_{(\alpha)(\beta)}(\tau) \Rightarrow \omega' = \gamma\,(\omega \mp \Omega)$

$$\omega'_{Doppler} = -k_\mu \lambda^\mu{}_{(0)} = \gamma\omega \quad , \quad \omega' = \omega'_{Doppler}\left(1 \mp \frac{\Omega}{\omega}\right)$$

$$\frac{\Omega}{\omega} = \frac{c/\omega}{c/\Omega} = \frac{\lambda\!\!\!^{-}}{\mathcal{L}} \ : \qquad \omega' \to \omega'_{Doppler} \text{ as } \frac{\lambda\!\!\!^{-}}{\mathcal{L}} \to 0 .$$

Intuitive Explanation for $\omega \mp \Omega$: "$-$" positive helicity $\uparrow\Uparrow \hbar$ (RCP)
"$+$" negative helicity $\uparrow\Downarrow -\hbar$ (LCP)

GPS : Phase Wrap-Up $\quad \gamma \ll 1,\ \Omega \ll \omega :\ \omega' \approx \omega \mp \Omega$

$$\frac{\omega}{2\pi} \sim 1\,\mathrm{GHz} \ , \quad \frac{\Omega}{2\pi} \sim 8\,\mathrm{Hz} \qquad (\text{N. Ashby})$$

Consequence of Exact Result $\gamma(\omega \mp \Omega)$: RCP can stand completely still by a mere rotation of the observer!

Spin-Rotation Coupling

$$\omega' = \gamma(\omega - M\Omega)$$

M = "magnetic quantum number"

$$M = 0, \pm 1, \pm 2, \ldots$$

Field: scalar, vector, ...

Dirac: $M \mp \frac{1}{2} = 0, \pm 1, \pm 2, \ldots$

ω' could be negative, zero, positive!

JWKB: $E' = \gamma(E - \underset{\sim}{\Omega} \cdot \underset{\sim}{J})$, $\underset{\sim}{J} = \underset{\sim}{r} \times \underset{\sim}{p} + \underset{\sim}{S}$

$$E' = \gamma(E - \underbrace{\underset{\sim}{\Omega} \cdot \underset{\sim}{r} \times \underset{\sim}{p}}_{\underset{\sim}{\Omega} \times \underset{\sim}{r} \cdot \underset{\sim}{p}} - \underset{\sim}{\Omega} \cdot \underset{\sim}{S}) = \gamma(E - \underset{\sim}{v} \cdot \underset{\sim}{p}) - \gamma \underset{\sim}{\Omega} \cdot \underset{\sim}{S}$$

L. Ryder: J. Phys. A $\underset{\sim}{31}$, 2465 (1998)

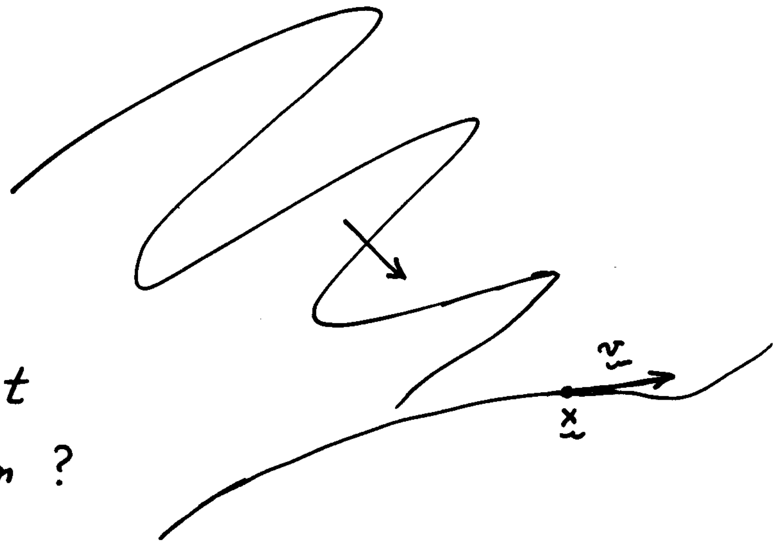
Spin-Rotation Coupling effect is fully relativistic.

Interpretation: Inertia of Intrinsic Spin

(m, s)

Basic Problem:

Frequency of Incident Radiation?



— A few periods of the wave must be received by the accelerated observer before an adequate determination would be possible.

Pointwise Determination of Field: $F_{(\alpha)(\beta)}(\tau) = F_{\mu\nu}\,\lambda^{\mu}_{(\alpha)}\,\lambda^{\nu}_{(\beta)}$

— Bohr + Rosenfeld: EM field cannot be measured at one point! Averaging is required.

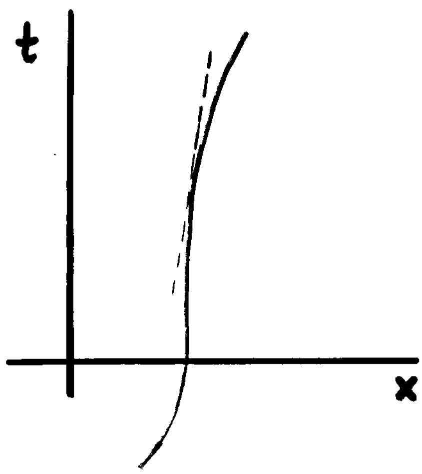
Assumptions Underlying G.R. :

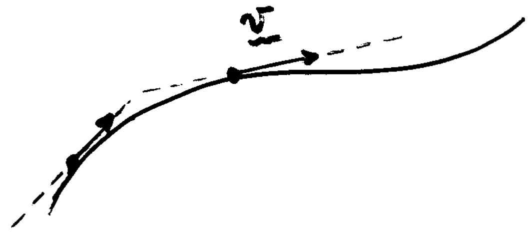
(i) Lorentz Invariance

Measurements of Ideal "Inertial Observers"

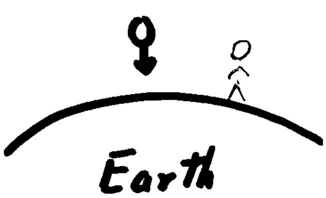
All Actual Observers are Accelerated.

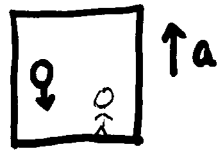
(ii) Hypothesis of Locality





(iii) Einstein's Principle of Equivalence





$$m_i \frac{d^2\vec{x}}{dt^2} = m_g \vec{g}$$

$$m_i = m_g$$

$$m_i \left(\frac{d^2\vec{x}}{dt^2} + \vec{a} \right) = 0$$

Gravitation = Spacetime Curvature

(iv) Field Equations $\quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ (Riemann)

Correspondence with Newton's theory!

stationary state of motion, to the actual velocity. Particularly, in the application of the formula (311), it has been presupposed that in a curvilinear motion the electron constantly has its short axis along the tangent to the path, and that, while the velocity changes, the ratio between the axes of the ellipsoid is changing at the same time.

Strictly speaking, it is not absolutely necessary for our results that the orientation and shape of the electron should follow instantaneously the alterations in direction and velocity of its translation; they may be supposed to lag somewhat behind. But it is clear that, at all events, if we want to apply the values of m' and m'' to optical phenomena, as we have done, the time of lagging must be small in comparison with the period of the vibrations of light.

Now, if we choose the latter of the alternatives that presented themselves in the last paragraph, we may as well simply assert that there is no lagging at all. But we must not proceed in this summary manner if we prefer the first alternative. If the form and the orientation of the electron are determined by forces, we cannot be certain that there exists at every instant a state of equilibrium. Even while the translation is constant, there may be small oscillations of the corpuscle, both in shape and in orientation, and under variable circumstances, i. e. when the velocity of translation is changing either in direction or in magnitude, the lagging behind of which we have just spoken cannot be entirely avoided. The case is similar to that of a pendulum bob acted on by a variable force, whose changes, as is well known, it does not instantaneously follow. The pendulum may, however, approximately be said to do so when the variations of the force are very slow in comparison with its own free vibrations. Similarly, the electron may be regarded as being, at every instant, in the state of equilibrium corresponding to its velocity, provided that the time in which the velocity changes perceptibly be very much longer than the period of the oscillations that can be performed under the influence of the regulating forces. If these vibrations are much more rapid than those of light, the values (313) of the masses m' and m'' may be confidently applied to the electrons in a body traversed by a beam of light, and with even more right to free electrons that are deflected from their line of motion by a magnetic or an electric field.

Of course, since we know next to nothing of the structure of an electron, it is impossible to form an opinion about the period of its free oscillations, but perhaps we shall not be far from the mark if we suppose it to correspond to a wave-length of the same order of magnitude as the diameter.

It appears from these considerations that the idea of a deformability of the electrons would give rise to several new problems. One

H. A. Lorentz, The Theory of Electrons
(Dover, New York, 1952)

A. Einstein, The Meaning of Relativity
(Princeton University Press, 1950)

diameter do not experience this contraction (along their lengths!).* It therefore follows that

$$\frac{U}{D} > \pi.$$

It therefore follows that the laws of configuration of rigid bodies with respect to K' do not agree with the laws of configuration of rigid bodies that are in accordance with Euclidean geometry. If, further, we place two similar clocks (rotating with K'), one upon the periphery, and the other at the centre of the circle, then, judged from K, the clock on the periphery will go slower than the clock at the centre. The same thing must take place, judged from K', if we define time with respect to K' in a not wholly unnatural way, that is, in such a way that the laws with respect to K' depend explicitly upon the time. Space and time, therefore, cannot be defined with respect to K' as they were in the special theory of relativity with respect to inertial systems. But, according to the principle of equivalence, K' is also to be considered as a system at rest, with respect to which there is a gravitational field (field of centrifugal force, and

* These considerations assume that the behavior of rods and clocks depends only upon velocities, and not upon accelerations, or, at least, that the influence of acceleration does not counteract that of velocity.

and future, there lies the fundamental importance of Römer's discovery of the finite velocity of light to which Einstein's Principle of Relativity first gave full expression. The plane $t = 0$ passing through O in an allowable co-ordinate system may be placed so that it cuts the light-cone $Q(x) = 0$ only at O and thereby separates the cone of the active future from the cone of the passive past.

For a body moving with uniform translation it is always possible to choose an allowable co-ordinate system (= normal co-ordinate system) such that the body is at rest in it. The individual parts of the body are then separated by definite distances from one another, the straight lines connecting them make definite angles with one another, and so forth, all of which may be calculated by means of the formulæ of ordinary analytical geometry from the space-co-ordinates x_1, x_2, x_3 of the points under consideration in the allowable co-ordinate system chosen. I shall term them the **static measures** of the body (this defines, in particular, the **static length** of a measuring rod). If this body is a clock, in which a periodical event occurs, there will be associated with this period in the system of reference, in which the clock is at rest, a definite time, determined by the increase of the co-ordinate x_0 during a period; we shall call this the "proper time" of the clock. If we push the body at one and the same moment at different points, these points will begin to move, but as the effect can at most be propagated with the velocity of light, the motion will only gradually be communicated to the whole body. As long as the expanding spheres encircling each point of attack and travelling with the velocity of light do not overlap, the parts surrounding these points that are dragged along move independently of one another. It is evident from this that, according to the theory of relativity, there cannot be rigid bodies in the old sense; that is, no body exists which remains objectively always the same no matter to what influences it has been subjected. How is it that in spite of this we can use our measuring rods for carrying out measurements in space? We shall use an analogy. If a gas that is in equilibrium in a closed vessel is heated at various points by small flames and is then removed adiabatically, it will at first pass through a series of complicated stages, which will not satisfy the equilibrium laws of thermo-dynamics. Finally, however, it will attain a new state of equilibrium corresponding to the new quantity of energy it contains, which is now greater owing to the heating. We require of a rigid body that is to be used for purposes of measurement (in particular, a linear **measuring rod**) that, **after coming to rest in an**

H. Weyl, Space-Time-Matter (Dover, New York, 1952)

allowable system of reference, it shall always remain exactly the same as before, that is, that it shall have **the same static measures** (or **static length**); and we require of a **clock** that goes correctly **that it shall always have the same proper-time when it has come to rest** (as a whole) **in an allowable system of reference.** We may assume that the measuring rods and clocks which we shall use satisfy this condition to a sufficient degree of approximation. It is only when, in our analogy, the gas is warmed sufficiently slowly (strictly speaking, infinitely slowly) that it will pass through a series of thermo-dynamic states of equilibrium; only when we move the measuring rods and clocks steadily, without jerks, will they preserve their static lengths and proper-times. The limits of acceleration within which this assumption may be made without appreciable errors arising are certainly very wide. Definite and exact statements about this point can be made only when we have built up a **dynamics** based on physical and mechanical laws.

To get a clear picture of the Lorentz-Fitzgerald contraction from the point of view of Einstein's Theory of Relativity, we shall imagine the following to take place in a plane. In an allowable system of reference (co-ordinates t, x_1, x_2, one space-co-ordinate being suppressed), to which the following space-time expressions will be referred, there is at rest a plane sheet of paper (carrying rectangular co-ordinates x_1, x_2 marked on it), on which a closed curve **c** is drawn. We have, besides, a circular plate carrying a rigid clock-hand that rotates around its centre, so that its point traces out the edge of the plate if it is rotated slowly, thus proving that the edge is actually a circle. Let the plate now move along the sheet of paper with uniform translation. If, at the same time, the index rotates slowly, its point runs unceasingly along the edge of the plate: in this sense the disc is circular during translation too. Suppose the edge of the disc to coincide exactly with the curve **c** at a definite moment. If we measure **c** by means of measuring rods that are at rest, we find that **c** is not a circle but an ellipse. This phenomenon is shown graphically in Fig. 11. We have added the system of reference t', x'_1, x'_2 with respect to which the disc is at rest. Any plane $t' =$ const. intersects the light cone in this system of reference in a circle "that exists for a single moment". The cylinder above it erected in the direction of the t'-axis represents a circle that is at rest in the **accented** system, and hence marks off that part of the world which is passed over by our disc. The section of this cylinder and the plane $t = 0$ is not a circle but an ellipse. The right-angled cylinder constructed

8

Physical Origin of Locality:

Newtonian Mechanics : State of a Particle is determined by position and velocity

Hypothesis of Locality is valid if Inertial Effects can be neglected over the scale of Observation $\frac{\lambda}{\mathcal{L}} \ll 1$

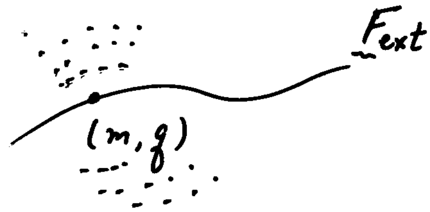
$\mathcal{L}$ = acceleration length : $\frac{c^2}{g}$, $\frac{c}{\Omega}$

$$\frac{c^2}{g_\oplus} \simeq 1 \text{ lyr}$$

$$\frac{c}{\Omega_\oplus} \simeq 28 \text{ AU}$$

Deviations from Locality $\left(\sim \frac{\lambda}{\mathcal{L}}\right)$ are generally very small.

(i) Radiation of Classical Charged Particle



- Accelerated Charged Particle

  Radiates $\lambda \sim \mathcal{L}$.

- Hypothesis of Locality Violated : $\frac{\lambda}{\mathcal{L}} \sim 1$.

- State of the Particle $\neq (\underset{\sim}{x}, \underset{\sim}{v})$:

$$m \frac{d\underset{\sim}{v}}{dt} - \frac{2}{3} \frac{q^2}{c^3} \frac{d^2 \underset{\sim}{v}}{dt^2} + \cdots = \underset{\sim}{F}_{ext}$$

(Abraham - Lorentz Equation)

(ii) Muon Decay $(\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu)$

$$\tau_\mu = \gamma \, \tau_\mu^0$$

Suppose muon decays from a high-energy Landau level in a constant magnetic field:

A. M. Eisele $\quad \tau_\mu \simeq \gamma \, \tau_\mu^0 \left[1 + \frac{2}{3} \left(\frac{\lambda}{\mathcal{L}} \right)^2 \right]$

$\lambda = \frac{\hbar}{mc}$ = Compton wavelength of the muon

$\mathcal{L} = \frac{c^2}{a}$, $a = \gamma^2 \frac{v^2}{r}$ = muon acceleration in storage ring

Muon Decay

J. Bailey et al. Nature 268 (1977) 301.

J. Bailey et al. Nucl. Phys. B150 (1979) 1.

muons, CERN

centripetal acceleration $a \sim 10^{18} g$

$$L = \frac{c^2}{a} \sim 1 \text{ cm}$$

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Very Accurate for muon lifetime!

A.M. Eisele (+ N. Straumann)

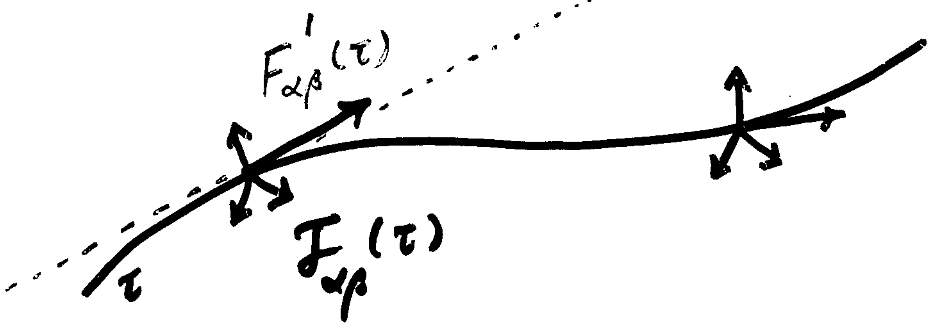
Helv. phys. Acta 60 (1987) 1024.

Muon in a Landau level with very high quantum number!

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} \left[1 + (\text{const.}) \left(\frac{\hbar/mc}{c^2/a} \right)^2 \right]$$

VERY SMALL BUT NONZERO! $\approx 10^{-25}$

$$F'_{\alpha\beta}(\tau) = F_{\mu\nu}\,\lambda^{\mu}_{(\alpha)}\,\lambda^{\nu}_{(\beta)}$$



Hypothesis of Locality : $\mathcal{F}_{\alpha\beta}(\tau) = F'_{\alpha\beta}(\tau)$

Nonlocal Theory :

$$\mathcal{F}_{\alpha\beta}(\tau) = F'_{\alpha\beta}(\tau) + \int_{\tau_0}^{\tau} K_{\alpha\beta}{}^{\gamma\delta}(\tau, \tau')\, F'_{\gamma\delta}(\tau')\, d\tau'$$

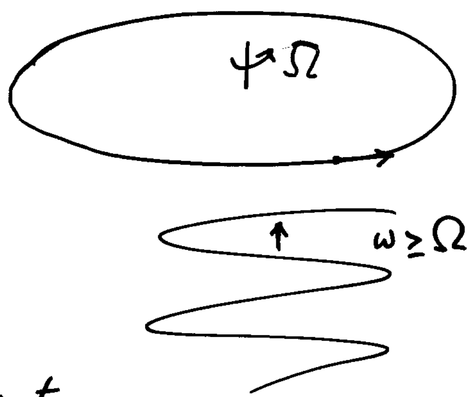
Volterra : *Continuous Fields* $(F_{\mu\nu}, \mathcal{F}_{\alpha\beta})$ *unique*

$$K \sim g \quad , \quad F \sim e^{-i\omega\tau}$$

$$\int K F' d\tau' \sim \frac{g}{\omega} F \quad , \quad \frac{g}{\omega} \sim \frac{\lambda\!\!\!^{-}}{\mathcal{L}}$$

JWKB $\Rightarrow$ *Hypothesis of Locality*

Consequences of Nonlocality:



I. Amplitude of "+" component

is enhanced by a factor of $\frac{1}{1-\frac{\Omega}{\omega}}$

Amplitude of "−" component

is diminished by a factor of $\frac{1}{1+\frac{\Omega}{\omega}}$

II. For $\omega = \Omega$ in the "+" case, the field is not

static; instead, it varies as t ("resonance").

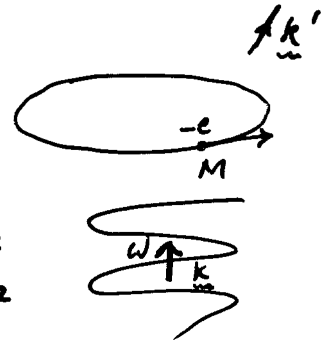
How to Verify the New Effects?

Problems of Principle:

I. Bialynicki-Birula and Z. Bialynicka-Birula:
Phys. Rev. Lett. 78, 2539 (1997).

Idea: Study Quantum Mechanics in Correspondence Regime

Rydberg States of Hydrogen Atom



"Circular Orbits": $n > 1$, $l = n-1$, $m = l$.

Bohr: $r_n = n^2 \dfrac{\hbar^2}{Me^2}$, $v_n = c\dfrac{\alpha}{n}$, $E_n = \dfrac{Me^4}{2\hbar^2 n^2}$

$v_n = r_n \Omega_n \Rightarrow \Omega_n = \dfrac{2E_n}{\hbar n}$ $\qquad \hbar\omega \ll Mc^2$

– Analog of the Hypothesis of Locality in Quantum Theory:

Impulse Approximation : E. Fermi, Ric. Sci. VII-11, 13 (1936.

σ_0 (photoionization) $\quad \propto |\hat{\psi}(\underset{\sim}{q})|^2 = |\int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}}\, \psi_{nlm}(\underset{\sim}{r})\, d^3\underset{\sim}{r}|^2$

= indep. of helicity of incident rad'n $\qquad \underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k}$

– Include Coulomb Interaction

Dipole Approximation $\quad k r_n \ll 1 \quad \Rightarrow \quad \dfrac{\Omega_n}{\omega} \gg \dfrac{1}{137 n}$

Cross Sections for photoionization $\sigma_\pm$:

$$\frac{\sigma_-}{\sigma_+} = \frac{3n^2(n-1) + \gamma^2(3n+1)}{4n^3\left[\gamma^2 + (n-1)^2\right]} \quad , \quad \frac{n}{\gamma} = \sqrt{\frac{\hbar\omega}{E_n} - 1} \ .$$

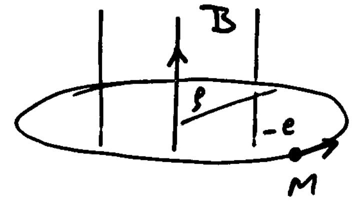
- $n = 1$, s-state $\quad \sigma_+ = \sigma_-$ by symmetry.
- $n > 1$, "circular orbits" $\quad \sigma_+ > \sigma_-$.

$$\frac{\sigma_-}{\sigma_+} \approx \frac{3}{4n(n-1)}\left[1 - \frac{n(2n+1)}{6(n-1)^2}\,\frac{\Omega_n}{\omega}\right] \quad \text{for} \quad \frac{\Omega_n}{\omega} \ll 1 \ .$$

"Circular Orbits" in a Uniform Magnetic Field

$\Omega_c = \dfrac{eB}{Mc}$ cyclotron frequency

$\rho_0 = \sqrt{\dfrac{2\hbar c}{eB}}$ magnetic length

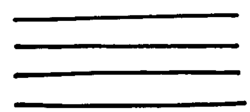


$\hbar\Omega_c \ll Mc^2$

$$\psi = N\, e^{im\varphi}\, e^{ik_z z} \left(\frac{\rho}{\rho_0}\right)^{|m|} e^{-\frac{1}{2}(\rho/\rho_0)^2}\, L_{n_\rho}^{|m|}\left(\frac{\rho^2}{\rho_0^2}\right)$$

$p_z = \hbar k_z$ — associated Laguerre polynomials

$$E = \frac{p_z^2}{2M} + \hbar\Omega_c\,(\underbrace{n_\rho + \frac{m+|m|}{2}}_{n} + \frac{1}{2})$$

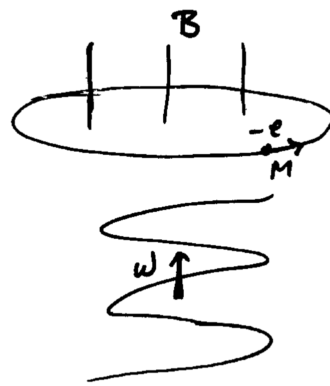


$L_z\psi = \hbar m\psi$; $\boldsymbol{l} = \boldsymbol{r} \times (\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})$

$E = \frac{1}{2}\Omega_c \langle l_z \rangle$ for $p_z = 0$.

Correspondence Principle : $m \gg 1$, $m \gg n_\rho$

– Electric Dipole Transitions



$\hbar\omega \ll Mc^2$

P = probability of transition to the next higher state

$$P = \frac{1}{\hbar^2}\,|\langle f\,|\,\mathcal{H}_{int}\,|\,i\rangle|^2\, t^2,$$

$P_+ \neq 0$, $P_- = 0$.

Nonlocal theory : $\mathcal{L} = c/\Omega_c$

$\hbar\omega = \hbar\Omega_c \Rightarrow \lambda\!\!\!^{-} = c/\omega = \mathcal{L}$

$$\frac{\text{Amplitude for RCP}}{\text{Amplitude for LCP}} = \frac{\omega + \Omega}{\omega - \Omega}$$

The general trends of the nonlocal results are similar to quantum mechanical expectations!