Mathisson's "New Mechanics": Its Aims and Realisation

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Lecture 1: Aims

It gives me great pleasure to be invited to speak at this meeting on the life and work of Myron Mathisson. I first came upon his work as a graduate student working on equations of motion in general relativity. I had come to this topic through the papers of Papapetrou published in 1951 under the title "*Spinning test-particles in general relativity*", a test particle being a body whose mass is, in some appropriate sense, negligible. Papapetrou considered a test particle sufficiently small that only the monopole and dipole moments of its energy-momentum tensor $T^{\alpha\beta}$ need be considered, a so-called pole-dipole particle. He showed that for the purpose of studying its motion, the monopole moment of such a particle could be described by a scalar M and the dipole moment by an antisymmetric tensor $S^{\alpha\beta}$, these representing its mass and spin, *i.e.* angular momentum. These were shown to satisfy the equations

$$\frac{\delta}{\mathrm{d}s}\left(Mv^{\alpha}+v_{\beta}\frac{\delta S^{\alpha\beta}}{\mathrm{d}s}\right)=\tfrac{1}{2}v^{\beta}S^{\gamma\delta}R^{\alpha}_{\bullet\beta\gamma\delta}$$

and

$$\frac{\delta S^{\alpha\beta}}{\mathrm{d}s}+v^{\alpha}v_{\gamma}\frac{\delta S^{\beta\gamma}}{\mathrm{d}s}-v^{\beta}v_{\gamma}\frac{\delta S^{\alpha\gamma}}{\mathrm{d}s}=0\,.$$

Here s is the proper time along the world line of the body, $v^{\alpha}\equiv \mathrm{d}x^{\alpha}/\mathrm{d}s$ is its four-velocity, $\delta/\mathrm{d}s$ denotes the absolute derivative with respect to s and $R^{\alpha}_{\bullet\beta\gamma\delta}$ is the curvature tensor of the spacetime.

The treatment by Papapetrou was extremely non-covariant. The quantities M and $S^{\alpha\beta}$ were constructed from non-covariant parts and had to be proved to be a scalar and tensor respectively. Similarly the equations as originally derived had to be manipulated into the above covariant forms. To my mind these aspects were unsatisfactory. Indeed it was this unsatisfactory nature of the derivation that led me to study the problem further. But the final equations were covariant and, so it seemed, new. Papapetrou pointed out that in a flat spacetime they agreed with the results of a study of the pole-dipole particle in special relativity by Mathisson (1937).

In due course I decided, for completeness, to look up the 1937 paper of Mathisson. I discovered to my astonishment that not only was the treatment actually in general relativity but it was also covariant. Mathisson derived essentially the same equations of motion as Papapetrou, including the spin-curvature interaction term that is arguably the most important discovery of the work. Indeed, he went further by considering the quadrupole terms. He showed that the equation of motion for the spin then gains additional terms representing a torque exerted by the gravitational field, expressed in terms of the quadrupole moment and the curvature tensor. On top of all this, the title of the paper was, in translation from its original German, "*A New Mechanics of Material Systems*", a far more all-embracing and inspiring title than that of Papapetrou.

The time was 1964. I felt compelled to find out what further work Mathisson had done on this "new mechanics". I discovered, to my great sadness, that he had died in 1940 while still working on this topic. I decided to continue Mathisson's programme of work as I understood it to be. This came to dominate the next ten years of my life.

To see the aims of Mathisson's work we need to look at his characterisation of the multipole moments of a body. In contrast to Papapetrou, Mathisson treated an extended body and defined an infinite set of covariant multipole moments for it. We consider a body occupying a world tube of finite spatial extent and choose a timelike world line L within this world tube. For present purposes L is otherwise arbitrary, but at a later stage we will wish to impose conditions that restrict it to be a suitably defined mass centre. Precisely what conditions are best suited for this is in fact one of the big questions of this subject.

Mathisson showed that there exists an infinite set of multipole moments $m^{\alpha\beta}$, $m^{\gamma\alpha\beta}$, $m^{\gamma\delta\alpha\beta}$,... such that

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = \int_L (m^{\alpha\beta}\varphi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\varphi_{\alpha\beta} + \tfrac{1}{2!}m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta} + \cdots)\,\mathrm{d}s$$

for all symmetric tensor fields $\varphi_{\alpha\beta}$ of compact support. Here $T^{\alpha\beta}$ is the energy-momentum tensor of the body, which is taken to be symmetric. The integral on the left extends over all space, with $\mathrm{D}X \equiv \sqrt{-g}\;\mathrm{d}^4x$ being the spacetime volume element. That on the right is over the world line L on which s is proper time, ∇_α denotes covariant differentiation and $\nabla_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta$. The moments are not uniquely determined by this but they become so if we require in addition that

(a) they are symmetric on their last two indices, *i.e.* those contracted with $\varphi_{\alpha\beta}$, and separately are symmetric on all their other indices, *i.e.* those contracted with the derivative operators, and

(b) they are orthogonal to the four-velocity $v^\alpha \equiv \mathrm{d}x^\alpha/\mathrm{d}s$ of L on all indices except the last two.

The infinite set of these moments was called by Mathisson the *gravitational skeleton* of the body, or in a later paper its *dynamical skeleton*. If we take

$$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}$$

where ω_α is an arbitrary vector field of compact support and round brackets around indices denote symmetrisation then the left side of the defining equation vanishes as a consequence of the conservation equation

$$\nabla_\beta T^{\alpha\beta} = 0.$$

The moments therefore satisfy

$$\int_L (m^{\alpha\beta}\varphi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_\gamma\varphi_{\alpha\beta} + \tfrac{1}{2!}m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta} + \cdots)\,\mathrm{d}s = 0$$

for all such $\varphi_{\alpha\beta}$. Mathisson called this the *variational equation of dynamics*, the variation in question being the ability to vary ω_α arbitrarily. It is a constraint on the gravitational skeleton and is the central equation of his programme of work.

Mathisson's "new mechanics" consists of determining the consequences of this variational equation, so providing a description of an extended body and its motion in terms of parameters similar to those of Newtonian rigid body mechanics, rather than the description provided by the energy-momentum tensor which corresponds more to that of Newtonian continuum mechanics. However, he was only able to work with this equation by truncating the series after the first few terms, the justification being that successively higher moments should have less and less effect on the motion if the gravitational field varies only slowly across the body. The truncated equation then leads both to restrictions on the form of the moment tensors and to differential equations of motion that they must satisfy. When only the first two moments are retained, he found that they are determined by a scalar *M* and an antisymmetric tensor $S^{\alpha\beta}$ such that

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} + \frac{\delta}{\mathrm{d}s}\left(n^{(\alpha}v^{\beta)}\right)$$

and

$$m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)} + v^{\gamma}n^{(\alpha}v^{\beta)}$$

where

$$p^{\alpha} = Mv^{\alpha} + v_{\beta}\frac{\delta S^{\alpha\beta}}{\mathrm{d}s} \quad \text{and} \quad n^{\alpha} = S^{\alpha\beta}v_{\beta}\,.$$

These satisfy the equations of motion given above in connection with Papapetrou's work. With use of the auxiliary vector p^{α} these equations of motion can also be written in the form

$$\frac{\delta p^{\alpha}}{\mathrm{d}s} = \tfrac{1}{2}v^{\beta}S^{\gamma\delta}R^{\alpha}_{\bullet\beta\gamma\delta}\,, \quad \frac{\delta S^{\alpha\beta}}{\mathrm{d}s} = 2p^{[\alpha}v^{\beta]}$$

where square brackets around indices denote antisymmetrisation. Indeed the equations in this form imply that p^{α} has the form given above, where *M* is now defined by $M = p^{\alpha}v_{\alpha}$, so that alternatively p^{α} and $S^{\alpha\beta}$ can be regarded as giving the primary description with *M* being the auxiliary variable. This is perhaps a more elegant view of the results.

Mathisson identified n^{α} with the static mass dipole moment of the body, which vanishes in Newtonian mechanics when taken about the centre of mass. He therefore took $n^{\alpha} = 0$ as the condition to characterise *L* as the world line of the centre of mass. He did this *before* he exploited the variational equation, so that his equations of motion were only derived for this special case. This is why I said that Mathisson's equations were only *essentially* the same as those of Papapetrou, as Papapetrou did not impose this condition. He was aware that in fact it does not determine *L* uniquely but he offered no alternative of general applicability. In a paper of 1959, Tulczyjew identified p^{α} as defined above as the momentum of the body and he then adopted

$$p_{\beta}S^{\alpha\beta} = 0$$

as the appropriate condition. He showed that in a flat spacetime this does determine a unique *L*. He then adopted it in a curved spacetime "owing to the lack of another definition".

If the above expressions for $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ are put back into the defining relation for the moments, the terms involving n^{α} combine into a single derivative that integrates to zero. We could therefore use instead the simpler expressions

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} \quad \text{and} \quad m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)}$$

which no longer satisfy the condition (*b*). This is a first indication that the orthogonality condition (*b*) is not the most appropriate addition to (*a*) to ensure uniqueness of the moments.

We shall see in the next lecture that modification of this orthogonality condition is one of the key steps towards solving the variational equation exactly.

Let us now take a step backwards, to get an overall view. Mathisson's approach to equations of motion falls into two distinct parts. The first part is showing that moments exist which satisfy the defining equation and any supplementary conditions such as (*a*) and (*b*) above. The second part is exploiting the consequences of the variational equation. In his 1937 paper his treatment of the variational equation was approximate, in that it required a truncation. In contrast his existence proof for the moments dealt with the infinite set of moments without approximation. However, it left the uniqueness question unanswered, as the orthogonality condition (*b*) was imposed only on the moments retained in the truncated variational equation. An improved and more detailed proof was given by Bielecki, Mathisson and Weyssenhoff in 1939. This imposed both (*a*) and (*b*) from the outset and proved both the existence and uniqueness of the moments. Or at least it did so subject to one explicit proviso, namely that infinite series converge and functions are analytic to whatever extent is required.

This proviso is important, as the proof requires the moments at any point $z \in L$ to be determined by the value of $T^{\alpha\beta}$ on a hypersurface through z that has some freedom in its specification. This implies some form of analyticity of $T^{\alpha\beta}$ that is not physically reasonable. It is permissible to require $\varphi_{\alpha\beta}$ to be analytic as this is just an auxiliary field introduced for convenience, but the analyticity must not apply also to $T^{\alpha\beta}$. The problem originates not in the proof but in the moment definitions themselves. Mathisson studied this issue again in 1940, this time in the simpler case of special relativity, in a paper simply titled "*The Variational Equation of Relativistic Dynamics*". He was there able to give expressions for the moments as explicit integrals of $T^{\alpha\beta}$ over hyperplanes orthogonal to the chosen world line *L*. These were shown to satisfy the defining equations but they were not deduced from it. Indeed, the flawed nature of the defining equations makes this task impossible.

These problems suggest that it may be preferable to abandon the Mathisson approach and instead seek a covariant version of the Papapetrou method. Seek a set of multipole moments defined from the outset as explicit tensor-valued integrals over cross-sections of the body and study their properties directly. This would by-pass the difficulties associated with Mathisson's moment defining equation and leave only a study of the consequences for these moments of the energy-momentum conservation equation. This study would correspond to the solving of Mathisson's variational equation. One would expect to have to retain only the first few moments in this study, but that would correspond to Mathisson's need to truncate the variational equation. This approach was adopted by Tulczyjew and Tylczyjew (1962) and later, with different definitions, by myself in Dixon (1964). But once one sees how to construct such tensor-valued integrals it becomes clear that there is an infinite number of possibilities to choose from, no one choice being more natural, in some sense, than the rest.

Perhaps it does not matter which we choose, as the quantities such as p^{α} and $S^{\alpha\beta}$ that appear in the final equations of motion are constructs from these moments rather than being moment tensors themselves. But the goal must be to avoid the need to neglect the higher moments, as this truncation itself is fundamentally flawed. We have only to look at Papapetrou's original form for the equations of motion of a pole-dipole particle to see that even in a flat spacetime the dipole construct $S^{\alpha\beta}$ enters the equation governing the monopole construct M and the velocity v^{α}. If higher moments were retained we would expect further contributions to this equation. We have seen that p^{α} provides a preferable description of the monopole structure in that the spin $S^{\alpha\beta}$ then only occurs in the monopole equation of motion in combination with the curvature tensor. But what happens to this if we retain higher moments? It is not even obvious that analogues of p^{α} and $S^{\alpha\beta}$ will continue to exist.

So how might we avoid truncation. We can hope that by a judicious choice of our original moment definitions the consequences of the energy-momentum conservation equation might be sufficiently systematic that we can handle them to all orders. But we need a guide to help towards this judicious choice. There really is only one guide available. It is to return to Mathisson's approach, with its implicit definition of the moments through a defining equation, and to use the variational equation as our guide. That equation has a deceptive simplicity. Our goal is to find our way through that deception.

So in the 1960's I was led back to Mathisson's new mechanics of 1937 to provide the way forward. The aims were two-fold.

1. To remove all assumptions of convergence and analyticity from the statement of the moment defining equation and its associated existence and uniqueness proof.
2. To solve the variational equation exactly, and only then to truncate the result to provide equations of motion to any desired order of approximation.

I regard my achievement of these two aims as the realisation of Mathisson's new mechanics.

Realisation of (1) led to explicit and unique expressions for the moment tensors as integrals of $T^{\alpha\beta}$. Realisation of (2) led to exact analogues of p^{α} and $S^{\alpha\beta}$ that have a natural identification as the momentum and spin of the body. This enables a third aim to be added.

3. To show that the Tulczyjew condition

$$p_{\beta}S^{\alpha\beta} = 0$$

determines *L* uniquely and that this *L* has properties which enable it to be identified naturally as the centre of mass line in general relativity.

Through the work of Ehlers and collaborators, this too has been achieved.

Lecture 2: Realisation

The first clue towards realising the aims of Mathisson's new mechanics can be found by considering the simpler environment of special relativity. Let us use a rectangular coordinate system so that the components $g_{\alpha\beta}$ of the metric tensor are constant, but not necessarily diagonal. The coordinates x^α can then be treated as components of a position vector.

Recall Mathisson's defining equation for the moments, which we now write in full as

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = \int_L \sum {}^1\!/_{n!}\, m^{\delta\cdots\gamma\alpha\beta}(s)\nabla_{\delta\cdots\gamma}\varphi_{\alpha\beta}(z(s))\mathrm{d}s\,.$$

Here *L* is parametrised as $z^\alpha(s)$, where for generality *s* is not necessarily proper time. Here and throughout, *n* is the number of indices in the set marked with dots, in this case $\delta\cdots\gamma$, and we allow the possibilities *n* = 0 and *n* = 1. We seek to avoid the requirement for $\varphi_{\alpha\beta}$ to be analytic. To do so we introduce the Fourier transform $\tilde{\varphi}_{\alpha\beta}$ defined by

$$\tilde{\varphi}_{\alpha\beta}(k) = \int \varphi_{\alpha\beta}(x)\exp(\mathrm{i}k\cdot x)\mathrm{D}x$$

where $k\cdot x \equiv k_\alpha x^\alpha$. For more complicated cases we shall also write the Fourier transform as $\mathrm{F}(\varphi_{\alpha\beta})$

If we express $\varphi_{\alpha\beta}$ in terms of $\tilde{\varphi}_{\alpha\beta}$ in the defining equation, we get

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = {}^1\!/_{(2\pi)^4}\int_L \mathrm{d}s\sum\int \mathrm{D}k\,{}^{(-\mathrm{i})^n}\!/_{n!}\, k_\delta\cdots k_\gamma m^{\delta\cdots\gamma\alpha\beta}(s)\tilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}k\cdot x)\,.$$

If we now exchange the order of the summation and the *k*-space integration we get

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = M^{\alpha\beta}[\Phi_{\alpha\beta}]$$

where $$M^{\alpha\beta}[\Phi_{\alpha\beta}] = {}^1\!/_{(2\pi)^4}\int \mathrm{d}s\int \mathrm{D}k\,\tilde{m}^{\alpha\beta}(s,k)\tilde{\Phi}_{\alpha\beta}(z(s),k)$$

with $$\tilde{m}^{\alpha\beta}(s,k) = \sum {}^{(-\mathrm{i})^n}\!/_{n!}\, k_\delta\cdots k_\gamma m^{\delta\cdots\gamma\alpha\beta}(s)$$

and $$\tilde{\Phi}_{\alpha\beta}(z,k) = \tilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}\,k\cdot z)\,.$$

Note that $\tilde{\Phi}_{\alpha\beta}(z,k)$ is simply the Fourier transform of $\varphi_{\alpha\beta}$ about *z* as origin.

In this form the moment defining equation no longer requires $\varphi_{\alpha\beta}$ to be analytic. We have, of course, achieved this by exchanging a summation and an integration that cannot in general be validly exchanged. We adopt this new form as an improved defining equation in special relativity.

To extend this definition to a curved spacetime, note that since the moments $m^{\delta\cdots\gamma\alpha\beta}$ are tensors at *z*(*s*) on *L*, k_α must be considered as a vector at the same point. This makes $\tilde{m}^{\alpha\beta}(s,k)$ be a tensor field on the tangent space $T_z(M)$ to the spacetime manifold *M* at *z*(*s*), so $\tilde{\Phi}_{\alpha\beta}$ must be likewise. There is no problem in defining Fourier transforms on $T_z(M)$

since it is always a flat manifold, so we can take $\tilde{\Phi}_{\alpha\beta}$ to be the Fourier transform of a $\Phi_{\alpha\beta}$ that is also a tensor field on $T_z(M)$. This leaves us simply to decide how $\Phi_{\alpha\beta}$ is to be defined in terms of $\varphi_{\alpha\beta}$.

When M is flat we have a natural identification of $T_z(M)$ with M, which is why we were able to take $\Phi_{\alpha\beta} = \varphi_{\alpha\beta}$ in special relativity. In a curved spacetime we relate the two by means of the exponential map $\mathrm{Exp}_z : T_z(M) \to M$. If $X \in T_z(M)$ and $x = \mathrm{Exp}_z X$ then the derivative map $((\mathrm{Exp}_z)_*)_X$ of Exp_z at X maps $T_X(T_z(M))$ isomorphically onto $T_z(M)$. This mapping between tangent spaces has a unique extension to a mapping of the corresponding tensor algebras. We denote this by replacing the * by A in the notation. By letting X vary we get a map $(\mathrm{Exp}_z)_{\mathrm{A}}$ from tensor fields on $T_z(M)$ to tensor fields on M. If we have such a tensor field for each z, we can apply the corresponding map to each of them to obtain a family of tensor fields on M parametrized by z, *i.e.* a two-point function with scalar character at z and tensor character at the second point x. We let $\mathrm{Exp}_{\mathrm{A}}$ denote this overall map and $\mathrm{Exp}^{\mathrm{A}}$ denote its inverse. These can both be formalised as maps between appropriate vector bundles.

We see that $\mathrm{Exp}^{\mathrm{A}}$ acts on two-point tensors with scalar character at one point, say z. A special case of this is an ordinary tensor field treated as such a two-point tensor that is independent of z. We can therefore take

$$\Phi_{\alpha\beta} = \mathrm{Exp}^{\mathrm{A}}\, \varphi_{\alpha\beta}\,.$$

For each z, this defines $\Phi_{\alpha\beta}$ as a tensor field on $T_z(M)$. If M is flat and we identify each of its tangent spaces with M itself then this gives $\Phi_{\alpha\beta} = \varphi_{\alpha\beta}$ for all z, so recovering our starting point in special relativity.

We have now done everything necessary to take our defining equation over into a curved spacetime. Note, however, that there is no longer any sense in which it is a transformation of Mathisson's original equation. We have made a real change in the definition of the moments. But we can now obtain explicit expressions for them as integrals without encountering problems of analyticity.

For the present we continue to adopt the orthogonality condition (*b*) of Lecture 1, namely that $m^{\delta\cdots\gamma\alpha\beta}$ is orthogonal to v_α on each of its indices $\delta\cdots\gamma$. Choose a Minkowskian coordinate system on $T_{z(s)}(M)$ such that $v_\alpha \neq 0$ only for $\alpha = 4$. Then $\tilde{m}^{\alpha\beta}(s,k)$ is independent of k_4. Recall now the result that for the Fourier transform $\tilde{f}(k)$ of a function $f(k)$ of a single variable, we have

$$\int \tilde{f}(k)\,\mathrm{d}k = 2\pi\, f(0)\,.$$

This enables us to perform the k_4 integration in the s-integrand of $M^{\alpha\beta}[\Phi_{\alpha\beta}]$ to show that its value for fixed s depends on $\Phi_{\alpha\beta}(z(s),X)$ only through its value on the hyperplane $X^4 = 0$, *i.e.* $X^\alpha v_\alpha = 0$. This hyperplane is mapped by Exp_z into the hypersurface $\Sigma(s)$ formed by all geodesics through $z(s)$ orthogonal to v_α. It follows that for each s, the s-integrand of $M^{\alpha\beta}[\Phi_{\alpha\beta}]$ depends on $\varphi_{\alpha\beta}$ only through its values on $\Sigma(s)$.

Now define a scalar function $\tau(x)$ by $\tau(x) = s$ if $x \in \Sigma(s)$ and let w^{α} be any vector field such that $w^{\alpha}\nabla_{\alpha}\tau = 1$. Then we have a corresponding decomposition

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = \int \mathrm{d}s \int_{\Sigma(s)} T^{\alpha\beta}\varphi_{\alpha\beta}w^{\gamma}\mathrm{d}\Sigma_{\gamma}$$

where $d\Sigma_{\alpha}$ is the surface element on $\Sigma(s)$. It follows that the two *s*-integrands are equal, so that

$$\int_{\Sigma(s)} T^{\alpha\beta}\varphi_{\alpha\beta}w^{\gamma}\mathrm{d}\Sigma_{\gamma} = {}^{1}\!\big/_{(2\pi)^4} \int \mathrm{D}k\, \tilde{m}^{\alpha\beta}(s,k)\tilde{\Phi}_{\alpha\beta}(z(s),k) .$$

It is now straightforward to express $\varphi_{\alpha\beta}$ in terms of $\tilde{\Phi}_{\alpha\beta}$ and so to identify $\tilde{m}^{\alpha\beta}(s,k)$ as an integral over $\Sigma(s)$. By expanding the resulting integrand as a series in k_{α} we may obtain explicit expressions for the moments $m^{\delta\cdots\gamma\alpha\beta}$ as integrals of $T^{\alpha\beta}$ over $\Sigma(s)$. Note that no use has been made here of v^{α} being tangent to *L*. We could equally well have taken it to be an arbitrary timelike vector field n^{α} along *L* and we shall make this change to the orthogonality conditions from now on. Without loss of generality we shall take n^{α} to be a unit vector.

We shall not give further detail here but instead turn to the variational equation. This is obtained by taking

$$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}$$

for an arbitrary vector field ω_{α}. We begin by investigating to what extent ω_{α} is determined if we only know $\varphi_{\alpha\beta}$. Since the difference of two solutions for the same $\varphi_{\alpha\beta}$ is a Killing vector field, ω_{α} is not unique if the spacetime admits Killing vectors.

It is easily shown that ω_{α} satisfies

$$\frac{\delta^2}{\mathrm{d}u^2}\omega_{\alpha} + \omega_{\beta}\dot{x}^{\gamma}\dot{x}^{\delta}R^{\beta}_{\bullet\gamma\delta\alpha} = \dot{x}^{\beta}\dot{x}^{\gamma}\nabla_{\{\beta}\varphi_{\alpha\gamma\}}$$

along all affinely parametrised geodesics $x^{\alpha}(u)$, where $\dot{x}^{\alpha} = \mathrm{d}x^{\alpha}/\mathrm{d}u$ and curly brackets around three indices are defined by

$$A_{\{\alpha\beta\gamma\}} = A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} .$$

The sign convention for the curvature tensor is such that

$$\nabla_{[\gamma\beta]}\omega_{\alpha} = -\tfrac{1}{2}R^{\delta}_{\bullet\alpha\beta\gamma}\omega_{\delta} .$$

By integrating the equation for ω_{α} along all geodesics through a fixed point *z*, we can therefore find ω_{α} everywhere if we only know ω_{α} and $\nabla_{\beta}\omega_{\alpha}$ at *z*, as these values determine the required initial conditions for any geodesic. But $\nabla_{(\beta}\omega_{\alpha)} = \varphi_{\beta\alpha}$, so in fact we only need

$$A_{\alpha}(z) = \omega_{\alpha}(z) \text{ and } B_{\alpha\beta}(z) = \nabla_{[\alpha}\omega_{\beta]}(z) .$$

Given a base point *z*, this construction will determine a vector field ω_{α} from an arbitrary tensor field $\varphi_{\alpha\beta}$, for any values of A_{α} and antisymmetric $B_{\alpha\beta}$ at *z*. We shall let $\lambda_{\alpha}(z,x)$ be the vector field so obtained when we take $A_{\alpha} = 0$ and $B_{\alpha\beta} = 0$. It is of course also a functional of the field $\varphi_{\alpha\beta}$, but we shall leave this dependence implicit. Similarly we let $\xi_\alpha(z,x)$ be the vector field so obtained when we take $\varphi_{\alpha\beta}=0$ but leave A_α and $B_{\alpha\beta}$ arbitrary, its dependence on these two tensors at *z* again being left implicit. If $\varphi_{\alpha\beta}$, A_α and $B_{\alpha\beta}$ are constructed as above from a given ω_α then we have

$$\omega_\alpha(x)=\lambda_\alpha(z,x)+\xi_\alpha(z,x)$$

for any *z*. Note that the equation satisfied by ξ_α is the equation of geodesic deviation.

To continue, we need to be able to differentiate fields such as $\Phi_{\alpha\beta}(z,X)$ in a covariant manner, with respect to both $z\in M$ and $X\in T_z(M)$. We give the required formulae for a field $\Psi^{\alpha}_{\bullet\beta}(z,X)$ so as to illustrate the terms that arise from both contravariant and covariant indices. We define

$$\nabla_{\alpha*}\Psi^{\beta}_{\bullet\gamma}=\frac{\partial}{\partial z^\alpha}\Psi^{\beta}_{\bullet\gamma}-\Gamma^{\varepsilon}_{\alpha\delta}X^\delta\frac{\partial}{\partial X^\varepsilon}\Psi^{\beta}_{\bullet\gamma}+\Gamma^{\beta}_{\alpha\delta}\Psi^{\delta}_{\bullet\gamma}-\Gamma^{\delta}_{\alpha\gamma}\Psi^{\beta}_{\bullet\delta}$$

and

$$\nabla_{*\alpha}\Psi^{\beta}_{\bullet\gamma}=\frac{\partial}{\partial X^\alpha}\Psi^{\beta}_{\bullet\gamma}$$

where the Levi-Civita connection $\Gamma^{\alpha}_{\beta\gamma}$ is evaluated at *z*. The first of these differs from the usual covariant derivative only through the addition of one term involving a derivative with respect to X^α. The second needs no connection terms since the components of X^α form a rectangular coordinate system on the flat tangent space. It can be shown that $\nabla_{\alpha*}$ commutes with Fourier transformation and that $\nabla_{\alpha*}X^\beta=0$.

With this notation it can be shown that

$$\operatorname{Exp}^{\mathrm{A}}\nabla_{(\alpha}\lambda_{\beta)}=\nabla_{*(\alpha}\mathrm{M}_{\beta)}-\Lambda^\gamma\nabla_{*\{\alpha}G_{\gamma\beta\}}$$

and

$$\Xi_{\alpha\beta}\equiv\operatorname{Exp}^{\mathrm{A}}\nabla_{(\alpha}\xi_{\beta)}=\tfrac{1}{2}A^\gamma\nabla_{\gamma*}G_{\alpha\beta}+\tfrac{1}{2}B^{\gamma\delta}X_\delta G_{\alpha\beta\gamma}-B^{\gamma\delta}\nabla_{*(\alpha}(G_{\beta)\gamma}X_\delta)$$

where

$$G_{\alpha\beta}=\operatorname{Exp}^{\mathrm{A}}g_{\alpha\beta},\quad G_{\alpha\beta\gamma}=\tfrac{1}{2}\nabla_{*\{\alpha}G_{\gamma\beta\}}$$

and

$$\Lambda^\alpha=\operatorname{Exp}^{\mathrm{A}}\lambda^\alpha,\quad \mathrm{M}_\alpha=\operatorname{Exp}^{\mathrm{A}}\lambda_\alpha=G_{\alpha\beta}\Lambda^\beta.$$

The A^α and $B^{\alpha\beta}$ are evaluated at *z* and are the values used in the construction of ξ^α. Indices of tensors on $T_z(M)$ are raised and lowered with $g_{\alpha\beta}(z)$, which is the flat metric on this tangent space, and is why we need different symbols for the lifts of the contravariant and covariant forms of the vector field $\lambda_\alpha(z,x)$. The field $\Xi_{\alpha\beta}(z,X)$ is defined by the above equation.

If $\varphi_{\alpha\beta}=\nabla_{(\alpha}\omega_{\beta)}$ as is used in the variational equation then these results give

$$\Phi_{\alpha\beta}+\tfrac{1}{2}\Lambda^\gamma\nabla_{*\{\alpha}G_{\gamma\beta\}}=\nabla_{*(\alpha}\mathrm{M}_{\beta)}+\Xi_{\alpha\beta}.$$

The right hand side is particularly simple as its first term only involves partial differentiation and its second term is completely determined by the parameters A^α and $B^{\alpha\beta}$ at *z*. The left hand side is well defined for a general field $\varphi_{\alpha\beta}(x)$ as this completely determines $\lambda_\alpha(z,x)$ and hence also $\Lambda^\alpha(z,X)$.

We capitalise on this simplicity by modifying the definition of our moments to take advantage of it. We change the defining equation to

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = M^{\alpha\beta}[\Phi_{\alpha\beta} + \tfrac{1}{2}\Lambda^\gamma\nabla_{*\{\alpha}G_{\gamma\beta\}}].$$

The additional term vanishes in a flat spacetime since $G_{\alpha\beta}$ is then constant, so we are only modifying the gravitational contribution to the moments.

The variational equation now becomes

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)} + \Xi_{\alpha\beta}] = 0.$$

Since Fourier transformation gives

$$F(\nabla_{*(\alpha}\mathrm{M}_{\beta)}) = -\mathrm{i}k_{(\alpha}\tilde{\mathrm{M}}_{\beta)}$$

we have

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)}] = {}^{1}\!\big/_{(2\pi)^4}\int \mathrm{d}s\int \mathrm{D}k\,\tilde{t}^{\,\alpha}\tilde{\mathrm{M}}_\alpha$$

where $$\tilde{t}^{\,\alpha}(s,k) \equiv -\mathrm{i}k_\beta\tilde{m}^{\alpha\beta}(s,k) = \sum {}^{(-\mathrm{i})^n}\!\big/_{n!}\; k_\gamma\cdots k_\beta t^{\gamma\cdots\beta\alpha}$$

with $$t^\alpha = 0 \quad\text{and}\quad t^{\delta\cdots\gamma\beta\alpha} = (n+1)m^{(\delta\cdots\gamma\beta)\alpha} \quad\text{for } n \geq 0.$$

For $n \geq 2$ we can decompose $m^{\delta\cdots\gamma\beta\alpha}$ through the use of symmetry operations into two tensors, one of which is $t^{\delta\cdots\gamma\beta\alpha}$. The other can be taken as the appropriate one from the set

$$J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha} \equiv \tfrac{1}{2}(m^{\zeta\cdots\varepsilon\delta[\beta\alpha]\gamma} - m^{\zeta\cdots\varepsilon\gamma[\beta\alpha]\delta}) \quad\text{for } n \geq 0.$$

For $n = 0$ and 1 the *m*'s and *t*'s are equivalent as either can be expressed in terms of the other. Recall that the orthogonality conditions (*b*) of lecture 1 which were imposed to ensure uniqueness of the moments are now

(*b*) $$n_\delta m^{\delta\cdots\gamma\beta\alpha} = 0 \quad\text{for } n \geq 1$$

since we generalised the original choice by using an arbitrary timelike unit vector field along *L* in place of the tangent vector. The set of these for $n \geq 2$ has a similar decomposition into the two sets

(*b1*) $$n_\zeta J^{\zeta\cdots\varepsilon\delta\gamma\beta\alpha} = 0 \quad\text{for } n \geq 1$$

and

(*b2*) $$n_\delta m^{\delta(\gamma\cdots\beta)\alpha} = 0 \quad\text{for } n \geq 2.$$

We see that although set (*b1*) involves only the *J*'s, set (*b2*) does not involve only the *t*'s. We can achieve this separation if we replace set (*b2*) by the set

(*b3*) $$n_\delta t^{\delta\gamma\cdots\beta\alpha} \equiv n_\delta m^{(\delta\gamma\cdots\beta)\alpha} = 0 \quad\text{for } n \geq 2.$$

These have the same symmetry properties as those of set (*b2*) and hence they impose the same number of constraints. They have the advantage, however, of constraining precisely those parts of the *m*'s that occur in the variational equation. We therefore make this change, which is of course a further change to the definition of the moments. We cannot extend *(b3)* to include $n = 1$ since for this case (*b*) is symmetric but (*b3*) is not, so this would increase the number of constraints and we have already shown that the original set (*b*) suffices to ensure uniqueness of the moments. We shall for the present omit any constraint for $n = 1$ and shall see in due course precisely what freedom this leaves in the definition of the moments.

We now show that as a consequence of the constraints (*b3*), the variational equation can be solved exactly. We first separate out the two terms in $\tilde{t}^{\alpha}(s,k)$ whose *t* tensors are not constrained by (*b3*). We integrate these over *k* and then express the result in terms of the *m*'s and ω_α instead of the *t*'s and λ_α. If we also use the expression for $\Xi_{\alpha\beta}$ given above, the variational equation can be put in the form

$$\int \mathrm{d}s \Big(m^{\beta\alpha}\nabla_{\beta}\omega_{\alpha}(z) + m^{\gamma\beta\alpha}\nabla_{\gamma\beta}\omega_{\alpha}(z) + A^{\gamma}F_{\gamma} + \tfrac{1}{2}B^{\lambda\delta}(K_{\gamma\delta} + L_{\gamma\delta}) +$$

$$1\big/(2\pi)^4 \int \mathrm{D}k\,\tilde{\mathrm{M}}_{\alpha} \sum_{n\geq 3} {(-\mathrm{i})^n}\big/{n!}\; k_{\gamma}\cdots k_{\beta}t^{\gamma\cdots\beta\alpha}\Big) = 0$$

where $$F_{\alpha} = 1\big/(2\pi)^4 \int \tilde{m}^{\beta\gamma}\nabla_{\alpha *}\tilde{G}_{\beta\gamma}\mathrm{D}k\,,$$

$$K_{\alpha\beta} = (-2\mathrm{i})\big/(2\pi)^4 \int \tilde{t}^{\gamma}\nabla_{*[\alpha}\tilde{G}_{\beta]\gamma}\mathrm{D}k \quad \text{and} \quad \mathrm{L}_{\alpha\beta} = 2\mathrm{i}\big/(2\pi)^4 \int \tilde{G}_{\gamma\delta[\alpha}\nabla_{*\beta]}\tilde{m}^{\gamma\delta}\mathrm{D}k$$

Let, as before, $\Sigma(s)$ be the hypersurface formed by all geodesics through *z*(*s*) that are orthogonal at *z* to $n_{\alpha}(s)$. The method used above in connection with the original orthogonality condition (*b*) then shows that the *k*-space integral in this form of the variational equation depends on $\lambda_{\alpha}(z(s),x)$ only through its values on $\Sigma(s)$. The construction of λ_α shows in turn that its values on $\Sigma(s)$ are determined by the values of $\omega_{\alpha}(x)$ on $\Sigma(s)$.

We will show that all the other terms in the *s*-integrand of the variational equation can also be put in a form with this restricted dependence and still dependent on ω_α only through the value of it and its first two derivatives at *z*. Once this is done, since the vector field ω_α is arbitrary it follows that the *s*-integrand must vanish separately for each value of *s*. But this integrand can be regarded as a generalised function, *i.e.* a continuous linear functional, on *M* itself rather than on $\Sigma(s)$. Its vanishing therefore implies that the infinite series in the *k*-space integral must be identically zero, as must be the coefficients of ω_α and its first two derivatives in the other terms of this integrand. Note that these terms correspond to terms in the infinite series that are constant, linear and quadratic in k_α and it is because such terms are absent that the series and the remainder of the integrand must vanish separately.

We now follow this through. At each point of *L* we define the projection operators

$$P_{\alpha}^{\bullet\beta} \equiv \chi n_{\alpha}v^{\beta} \quad \text{and} \quad Q_{\alpha}^{\bullet\beta} = A_{\alpha}^{\beta} - P_{\alpha}^{\bullet\beta}$$

where $\chi = 1/(n_{\alpha}\,v^{\alpha})$ and A_{α}^{β} is the unit tensor. Then the values at *z*(*s*) of

$$\omega_{\alpha}\,,\;\; Q_{\beta}^{\bullet\gamma}\nabla_{\gamma}\omega_{\alpha} \;\; \text{and} \;\; Q_{\gamma}^{\bullet\varepsilon}Q_{\beta}^{\bullet\delta}(\nabla_{(\varepsilon\delta)}\omega_{\alpha} + h_{\varepsilon\delta}n^{\zeta}\nabla_{\zeta}\omega_{\alpha})$$

can all be evaluated from knowledge of $\omega_{\alpha}(x)$ on $\Sigma(s)$. Here $h_{\beta\alpha}$ is symmetric and is the extrinsic curvature tensor (or second fundamental form) of $\Sigma(s)$, defined by

$$h_{\beta\alpha} = n_\beta n^\gamma \nabla_\gamma n_\alpha - \nabla_\beta n_\alpha$$

where n_α is the field of unit normals to $\Sigma(s)$, which of course agrees with $n_\alpha(s)$ on *L*. In a flat spacetime $\Sigma(s)$ is a hyperplane so $h_{\beta\alpha} = 0$ and in a Minkowskian coordinate system the quantities on which the projection operators act reduce to the first and second partial derivatives of ω_α. Since we have

$$P_\beta^{\bullet\gamma} \nabla_\gamma \omega_\alpha = \chi n_\beta \frac{\delta \omega_\alpha}{\mathrm{d}s}$$

any term in the *s*-integrand involving this can be reduced by integration by parts to one depending only on ω_α on *L*. We see similarly that if either or both of the projection operators *Q* in the last of the above list is replaced by the corresponding *P* then an integration by parts of any term involving this expression can be used to reduce it to one depending only on ω_α and $\nabla_\beta \omega_\alpha$ on *L*. These steps applied in the reverse order, *i.e.* treating the second derivatives first, will achieve the reduction of the entire *s*-integrand to one that has the required dependence only on the values of $\omega_\alpha(x)$ on $\Sigma(s)$.

We perform this reduction in stages, drawing conclusions as we go along. We can immediately conclude that the infinite series in the *k*-space integral must be identically zero, so that

$$t^{\gamma \cdots \beta\alpha} = 0 \quad \text{for } n \geq 3 .$$

This is sufficient to give $K_{\alpha\beta} = 0$ as the terms that remain also evaluate to zero. The coefficient of $\nabla_{(\gamma\beta)} \omega_\alpha$ in the reduced expression is easily seen and its vanishing gives

$$Q_\varepsilon^{\bullet\gamma} Q_\delta^{\bullet\beta} m^{(\varepsilon\delta)\alpha} = 0$$

from which $m^{(\gamma\beta)\alpha}$ must have the form

$$m^{(\gamma\beta)\alpha} = \tfrac{1}{2} v^{(\gamma} S^{\beta)\alpha}$$

for some tensor $S^{\alpha\beta}$ that at this stage is not necessarily antisymmetric. This gives

$$m^{\gamma\beta\alpha} = \tfrac{1}{2}\left(S^{[\gamma\beta]} v^\alpha + S^{[\gamma\alpha]} v^\beta + S^{(\beta\alpha)} v^\gamma\right)$$

and

$$m^{\gamma\beta\alpha} \nabla_{\gamma\beta} \omega_\alpha = \tfrac{1}{2} S^{\beta\alpha} \frac{\delta}{\mathrm{d}s} \nabla_\beta \omega_\alpha + \tfrac{1}{2} \omega_\alpha v^\beta S^{[\gamma\delta]} R^\alpha_{\bullet\beta\gamma\delta} .$$

The first term on the right requires an integration by parts, following which the coefficient of $\nabla_\beta \omega_\alpha$ can be read off and its vanishing gives

$$Q_\gamma^{\bullet\beta} \left(m^{\gamma\alpha} - \tfrac{1}{2} \frac{\delta}{\mathrm{d}s} S^{\gamma\alpha} + \tfrac{1}{2} L^{\gamma\alpha} \right) = 0 .$$

The bracketed expression therefore has the form

$$m^{\beta\alpha} - \tfrac{1}{2} \frac{\delta}{\mathrm{d}s} S^{\beta\alpha} + \tfrac{1}{2} L^{\beta\alpha} = v^\beta p^\alpha$$

for some vector p^α. The symmetric and antisymmetric parts of this give

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} + \tfrac{1}{2}\frac{\delta}{\mathrm{d}s}S^{(\alpha\beta)}$$

and
$$\frac{\delta}{\mathrm{d}s}S^{[\alpha\beta]} = 2p^{[\alpha}v^{\beta]} + L^{\alpha\beta}\,.$$

It also shows that

$$\left(m^{\beta\alpha} - \tfrac{1}{2}\frac{\delta}{\mathrm{d}s}S^{\beta\alpha} + \tfrac{1}{2}L^{\beta\alpha}\right)\nabla_\beta\omega_\alpha = p^\alpha\frac{\delta}{\mathrm{d}s}\omega_\alpha\,.$$

This requires another integration by parts before the coefficient of ω_α can be read off and its vanishing seen to give

$$\frac{\delta}{\mathrm{d}s}p^\alpha = \tfrac{1}{2}v^\beta S^{[\gamma\delta]}R^{\alpha}_{\bullet\beta\gamma\delta} + F^\alpha\,.$$

This completes the solution of the variational equation but it has left us with some freedom in the definitions of $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$, as we expected when we left one orthogonality condition unspecified. The value of $S^{(\alpha\beta)}$ is arbitrary, since if we substitute the above expressions for $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$ into the moment defining equation then the two contributions from $S^{(\alpha\beta)}$ combine into a total *s*-derivative that integrates to zero. We saw in lecture 1 that Mathisson's orthogonality conditions led to contributions of this form in his pole-dipole approximation. We shall make the simplest choice, $S^{(\alpha\beta)} = 0$. This replaces the case $n = 1$ of condition (*b*) above which was left outstanding by our adoption in its place of (*b1*) and (*b3*) and, being symmetric, it imposes the same number of constraints.

We now have the monopole and dipole moments of the body given by

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} \quad \text{and} \quad m^{\alpha\beta\gamma} = S^{\alpha(\beta}v^{\gamma)}$$

in terms of the vector p^α and antisymmetric tensor $S^{\alpha\beta}$, which are now precisely defined quantities that arose naturally in the development rather than being constructs within some approximation. The quadrupole and higher moments are determined by the *J*'s, which have the symmetry and orthogonality properties

$$J^{\alpha\cdots\beta\gamma\delta\varepsilon\zeta} = J^{(\alpha\cdots\beta)[\gamma\delta][\varepsilon\zeta]} \quad \text{and} \quad J^{\alpha\cdots\beta\gamma[\delta\varepsilon\zeta]} = 0 \quad \text{for } n \geq 0\,,$$

$$J^{\alpha\cdots[\beta\gamma\delta]\varepsilon\zeta} = 0 \quad \text{and} \quad n_\alpha J^{\alpha\cdots\beta\gamma\delta\varepsilon\zeta} = 0 \quad \text{for } n \geq 1\,.$$

and which determine the *m*'s with four or more indices by

$$m^{\alpha\cdots\beta\gamma\delta\varepsilon} = \frac{4(n-1)}{n+1}J^{(\alpha\cdots\beta|\delta|\gamma)\varepsilon} \quad \text{for } n \geq 1\,.$$

Mathisson took the set of *m*'s as his *gravitational skeleton* of the body, despite the existence of an infinite number of relations between them. It seems preferable now to take the gravitational skeleton as being p^α, $S^{\alpha\beta}$ and the set of *J*'s, as the only relations between them implied by the conservation equation for the energy-momentum tensor are the two equations of motion

$$\frac{\delta}{\mathrm{d}s}p^\alpha = \tfrac{1}{2}v^\beta S^{\gamma\delta}R^{\alpha}_{\bullet\beta\gamma\delta} + F^\alpha$$

and
$$\frac{\delta}{\mathrm{d}s} S^{\alpha\beta} = 2p^{[\alpha}v^{\beta]} + L^{\alpha\beta}.$$

These are the familiar pole-dipole equations with the addition of a force F^{α} and torque $L^{\alpha\beta}$ produced by the quadrupole and higher moments. Exact expressions were given above for these in terms of $\tilde{m}^{\alpha\beta}$, which can be considered as a moment generating function. The contributions to F^{α} and $L^{\alpha\beta}$ from p^{α} and $S^{\alpha\beta}$ vanish identically, so the force and torque are expressible in terms solely of the *J*'s.

By exchanging the order of the summation in $\tilde{m}^{\alpha\beta}$ and the *k*-space integration we can obtain a formal multipole expansion of the force and torque which can be truncated to give a multipole approximation to any desired order. The full infinite series is only a formal one, as the exchange of summation and integration is not actually valid. The multipole expansion is most easily expressed in terms of the *extensions* $g_{\alpha\beta,\gamma\cdots\delta}$ of the metric tensor $g_{\alpha\beta}$ as defined by Veblen and Thomas (1923). These are the tensors that reduce at the pole of a normal coordinate system to the partial derivatives of the metric tensor. This is a natural construction in our context as the image under exponential map of a rectangular coordinate system in the tangent space $T_z(M)$ is a normal coordinate system on *M* with *z* as its pole. We obtain

$$F_\alpha = \tfrac{1}{2}\sum_{n\geq 2} {}^{1}\!/_{n!}\, m^{\varepsilon\cdots\delta\gamma\beta}\, \nabla_\alpha g_{\gamma\beta,\varepsilon\cdots\delta}$$

and
$$L^{\alpha\beta} = \sum_{n\geq 1} {}^{1}\!/_{n!}\, g^{\gamma[\alpha} m^{\beta]\varepsilon\cdots\delta\zeta\eta}\, g_{\{\gamma\eta,\zeta\}\varepsilon\cdots\delta}\,.$$

The extensions are easily evaluated in terms of the curvature tensor and the lowest two are

$$g_{\alpha\beta,\gamma\delta} = \tfrac{4}{3} R_{\alpha(\gamma\delta)\beta}, \quad g_{\alpha\beta,\gamma\delta\varepsilon} = \tfrac{1}{3}\nabla_{(\gamma} R_{|\alpha|\delta\varepsilon)\beta}\,.$$

To complete the proof of the above results we still need to show that moments do exist that satisfy our new moment defining equation and with the symmetry and orthogonality conditions we have required. This can be done and it leads to explicit expressions for p^{α}, $S^{\alpha\beta}$ and the *J*'s as integrals of the energy-momentum tensor over $\Sigma(s)$. It is interesting to note that the surface of integration is still uniquely determined even though the orthogonality conditions now first arise with the octopole moment.

This completes the realisation of the first two aims laid down in lecture 1. We turn now to the third aim. The explicit expressions for p^{α} and $S^{\alpha\beta}$ are simple and they show that their values at $z(s) \in L$ depend on *L* and the vector field n^{α} along it only through the point *z* and the value of n^{α} at this point. The expressions for the *J*'s are significantly more complicated. They depend not only on *z* and n^{α} but also on their first derivatives along *L*. This dependence arises through the vector field w^{α} introduced above in the splitting of an integral over *M* into one over $\Sigma(s)$ followed by one over *L*. It turns out, perhaps unexpectedly, that the expressions for p^{α} and $S^{\alpha\beta}$ do not involve w^{α} and so are free of this derivative dependence.

The expressions for p^{α} and $S^{\alpha\beta}$ can therefore be considered as functions of a general point $z \in M$ and timelike unit vector n^{α} at *z*. In a flat spacetime the function $p^{\alpha}(z,n)$ is independent of n^{α} so at any point *z* we can choose n^{α} to be parallel to $p^{\alpha}(z)$. This suggests that in a curved spacetime we can find an n^α that is parallel to $p^\alpha(z,n)$ even though this is now an implicit equation, *i.e.* we can choose

$$p^\alpha(z,n) = M(z)n^\alpha$$

for some scalar $M(z)$. With this choice of n^α we can consider p^α and $S^{\alpha\beta}$ to be functions only of *z*. The Tulczyjew condition

$$p_\beta S^{\alpha\beta} = 0$$

is then well defined and can be expected to determine *L* uniquely It has been proved by Schattner (1979) that subject to mild restrictions on the strength of the gravitational field, these conditions do determine the field n^α and the line *L* uniquely.

With these conditions we have completely specified the line *L* and the fields *M*, p^α and $S^{\alpha\beta}$ along it. We take *M*, p^α and $S^{\alpha\beta}$ as defining the mass, momentum and spin of the body and *L* as being the centre-of-mass line. We now seek a relationship between p^α and the four-velocity $v^\alpha \equiv \mathrm{d}z^\alpha / \mathrm{d}s$ of *L*. First recall that we have not required *s* to be proper time along *L*. A change of parametrisation changes the *J*'s by an overall scale factor but it leaves p^α and $S^{\alpha\beta}$ unchanged. We use this final freedom to choose *s* so that

$$n_\alpha v^\alpha = 1 \quad \text{as well as} \quad n_\alpha n^\alpha = 1.$$

As distinct from our other choices in this development, there is no compelling reason for this. It is purely for convenience as it simplifies subsequent developments. If we define a vector h^α by

$$Mh^\alpha = p^\alpha + L^{\alpha\beta} n_\beta$$

it can then be shown that

$$(M^2 + \tfrac{1}{4} R_{\beta\gamma\delta\varepsilon} S^{\beta\gamma} S^{\delta\varepsilon})(v^\alpha - h^\alpha) = S^{\alpha\beta}(F_\beta + \tfrac{1}{2} R_{\beta\gamma\delta\varepsilon} h^\gamma S^{\delta\varepsilon}).$$

This is the momentum-velocity relation of the theory, due to Ehlers and Rudolph (1977). The velocity v^α it determines satisfies the above normalisation condition since $n_\beta S^{\alpha\beta} = 0$.

The simplest special case is a flat spacetime, where the curvature and the gravitational force and torque are all zero and it shows that the momentum and velocity are parallel. However, the less trivial case of a spacetime of constant curvature can also be treated exactly. In this case the auxiliary field ξ_α is a Killing vector for all values of the parameters A_α and $B_{\alpha\beta}$, from which we get the vanishing of $\Xi_{\alpha\beta}$ and hence also of the force F_α and torque $L^{\alpha\beta}$. On using the expression

$$R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

for the curvature tensor in this case, we find from the momentum-velocity relation that $v^\alpha = n^\alpha$. The equation of motion for p^α then shows that *L* is a geodesic. This supports our identification of *L* as the centre-of-mass line in general relativity, as it is what we would expect of this line from the symmetry of the spacetime. This completes the realisation of our third and final aim of Mathisson's new mechanics.

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