Mathisson's "New Mechanics": Its Aims and Realisation

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Part 1: Aims

The Mathisson-Papapetrou equations

$$\frac{\delta}{\mathrm{d}s}\left(Mv^{\alpha}+v_{\beta}\frac{\delta S^{\alpha\beta}}{\mathrm{d}s}\right)=\tfrac{1}{2}v^{\beta}S^{\gamma\delta}R^{\alpha}_{\bullet\beta\gamma\delta}$$

and

$$\frac{\delta S^{\alpha\beta}}{\mathrm{d}s}+v^{\alpha}v_{\gamma}\frac{\delta S^{\beta\gamma}}{\mathrm{d}s}-v^{\beta}v_{\gamma}\frac{\delta S^{\alpha\gamma}}{\mathrm{d}s}=0$$

originated from Mathisson's defining equation

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X=\int_{L}(m^{\alpha\beta}\varphi_{\alpha\beta}+$$

$$m^{\gamma\alpha\beta}\nabla_{\gamma}\varphi_{\alpha\beta}+{}^{1}\!/_{2!}\,m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta}+\cdots)\,\mathrm{d}s$$

with symmetry and orthogonality conditions

(*a*) $$m^{\gamma\cdots\delta\alpha\beta}=m^{(\gamma\cdots\delta)(\alpha\beta)}\text{ and}$$

(*b*) $$v_{\gamma}m^{\gamma\cdots\delta\alpha\beta}=0.$$

If $$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}$$

then as a consequence of the conservation equation

$$\nabla_{\beta} T^{\alpha\beta} = 0$$

we get

$$\int_L (m^{\alpha\beta}\varphi_{\alpha\beta} + m^{\gamma\alpha\beta}\nabla_{\gamma}\varphi_{\alpha\beta} + \\ {}^1\!/_{2!}\, m^{\gamma\delta\alpha\beta}\nabla_{\gamma\delta}\varphi_{\alpha\beta} + \cdots)\,\mathrm{d}s = 0$$

for all such $\varphi_{\alpha\beta}$. This is Mathisson’s *variational equation of dynamics.*

In the pole-dipole approximation we find that

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} + \frac{\delta}{\mathrm{d}s}\left(n^{(\alpha}v^{\beta)}\right)$$

and $$m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)} + v^{\gamma}n^{(\alpha}v^{\beta)}$$

where $p^{\alpha} = Mv^{\alpha} + v_{\beta}\dfrac{\delta S^{\alpha\beta}}{\mathrm{d}s}$ and $n^{\alpha} = S^{\alpha\beta}v_{\beta}$.

For a given world line L, these express $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ in terms of a scalar mass M and antisymmetric spin tensor $S^{\alpha\beta}$ that satisfy the Mathisson-Papapetrou equations above. They take the simpler form

$$\frac{\delta p^{\alpha}}{\mathrm{d}s} = \tfrac{1}{2} v^{\beta} S^{\gamma\delta} R^{\alpha}_{\bullet\beta\gamma\delta}\,, \quad \frac{\delta S^{\alpha\beta}}{\mathrm{d}s} = 2 p^{[\alpha} v^{\beta]}$$

in terms of p^{α} and $S^{\alpha\beta}$ from which it follows that p^{α} has the above form with M now defined by $M = p^{\alpha} v_{\alpha}$.

Mathisson identified $n^{\alpha} \equiv S^{\alpha\beta} v_{\beta}$ as the static dipole moment of the body and put $n^{\alpha} = 0$ to characterise L as the centre-of-mass line of the body. However, this does not determine L uniquely. Tulczyjew (1959) identified p^{α} as momentum and adopted

$$p_{\beta} S^{\alpha\beta} = 0$$

as the mass centre condition. He showed that in special relativity this determines L uniquely and he adopted it in curved spacetime “owing to the lack of another definition”.

The terms involving n^{α} in the expressions for $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ do not contribute to Mathisson's moment-defining equation as they combine into a total derivative that integrates to zero. So these moments could instead be taken as

$$m^{\alpha\beta} = p^{(\alpha}v^{\beta)} \text{ and } m^{\gamma\alpha\beta} = S^{\gamma(\alpha}v^{\beta)},$$

but these do not satisfy the orthogonality condition (*b*). This is a first sign that (*b*) is not the most appropriate condition to ensure uniqueness of the moments.

Problems with the Mathisson approach are

- there is an underlying assumption that infinite series converge and functions are analytic to whatever extent is needed in the work, and
- the variational equation can only be solved if it is first truncated, so that constructed quantities such as the momentum p^{α} and spin $S^{\alpha\beta}$ are only defined within a particular approximation.

The aims of Mathisson’s new mechanics must be:

1. To remove all assumptions of convergence and analyticity from the statement of the moment defining equation and its associated existence and uniqueness proof.

2. To solve the variational equation exactly, and only then to truncate the result to provide equations of motion to any desired order of approximation.

If these are achieved, they should lead to exact expressions for momentum p^{α} and spin $S^{\alpha\beta}$. A third aim can then be added:

3. To show that the Tulczyjew condition

$$p_{\beta}S^{\alpha\beta}=0$$

determines L uniquely and that this L has properties which enable it to be identified naturally as the centre of mass line in general relativity.

We shall see in the next lecture that these can all be achieved.

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Part 2: Realisation

Recall Mathisson's moment defining equation

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = \int_L \sum {}^{1}\!/_{n!}\, m^{\delta\cdots\gamma\alpha\beta}(s)\nabla_{\delta\cdots\gamma}\varphi_{\alpha\beta}(z(s))\mathrm{d}s$$

Here and throughout, *n* is the number of indices in the set marked with dots, in this case $\delta\cdots\gamma$. Consider flat spacetime and define the Fourier transform $\tilde{\varphi}_{\alpha\beta}$ by

$$\tilde{\varphi}_{\alpha\beta}(k) = \int \varphi_{\alpha\beta}(x)\exp(\mathrm{i}k\cdot x)\mathrm{D}x$$

where $k\cdot x \equiv k_\alpha x^\alpha$. Then

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = {}^{1}\!/_{(2\pi)^4}\int_L \mathrm{d}s \sum \int \mathrm{D}k\, {}^{(-\mathrm{i})^n}\!/_{n!}$$
$$\times\ k_\delta\cdots k_\gamma m^{\delta\cdots\gamma\alpha\beta}(s)\tilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}k\cdot x)$$

If we exchange the summation and *k*-space integration we get a form which no longer requires $\varphi_{\alpha\beta}$ to be analytic:

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = M^{\alpha\beta}[\Phi_{\alpha\beta}]$$

where

$$M^{\alpha\beta}[\Phi_{\alpha\beta}] = \frac{1}{(2\pi)^4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{m}^{\alpha\beta}(s,k)\tilde{\Phi}_{\alpha\beta}(z(s),k)$$

with $\tilde{m}^{\alpha\beta}(s,k) = \sum \frac{(-\mathrm{i})^n}{n!} k_\delta \cdots k_\gamma m^{\delta\cdots\gamma\alpha\beta}(s)$

and $\tilde{\Phi}_{\alpha\beta}(z,k) = \tilde{\varphi}_{\alpha\beta}(k)\exp(-\mathrm{i}\,k\cdot z)$.

Note that $\tilde{\Phi}_{\alpha\beta}(z,k)$ is simply the Fourier transform of $\varphi_{\alpha\beta}$ about z as origin.

If this is to be meaningful in curved spacetime we need

- k_α to be a vector at $z(s)$ on L
- $\tilde{m}^{\alpha\beta}(s,k)$ to be a tensor field on the tangent space $T_z(M)$ at z to the spacetime manifold M
- $\tilde{\Phi}_{\alpha\beta}(z,k)$ also to be such a tensor field, and so be the Fourier transform on the flat manifold $T_z(M)$ of another such tensor field $\Phi_{\alpha\beta}(z,X)$.

To complete this new form of the moment defining equation we only need to specify how $\Phi_{\alpha\beta}(z,X)$ is constructed from $\varphi_{\alpha\beta}$. For this we use the exponential map $\mathrm{Exp}_z : T_z(M) \to M$ and its derivative, which determine a map from tensor fields on $T_z(M)$ to tensor fields on *M*. By letting *z* vary, this gives a map Exp_A from fields such as $\Phi_{\alpha\beta}(z,X)$, which have tensor character at *z*, to two-point functions $\varphi_{\alpha\beta}(z,x)$ which are a tensor at *x* but a scalar at *z*. If *M* is flat then this is given simply by

$$\varphi_{\alpha\beta}(z,x) = \Phi_{\alpha\beta}(z,X) \text{ where } X^\alpha = x^\alpha - z^\alpha .$$

The map Exp_A has an inverse Exp^A that may be applied to an ordinary tensor field $\varphi_{\alpha\beta}(x)$ by treating it as a two-point field that is independent of *z*. We complete the moment defining equation by taking

$$\Phi_{\alpha\beta} = \mathrm{Exp}^\mathrm{A}\, \varphi_{\alpha\beta} .$$

Keep the symmetry and orthogonality conditions

(a) $$m^{\gamma\cdots\delta\alpha\beta} = m^{(\gamma\cdots\delta)(\alpha\beta)} \text{ and}$$

(b) $$v_\gamma m^{\gamma\cdots\delta\alpha\beta} = 0$$

and use a Minkowskian coordinate system on $T_z(M)$ in which the four-velocity v^α of L has $v^\alpha \neq 0$ only for $\alpha = 4$. Then $\tilde{m}^{\alpha\beta}(s,k)$ is independent of k_4. We can now perform the k_4 integration in

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = \frac{1}{(2\pi)^4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{m}^{\alpha\beta}(s,k)\tilde{\Phi}_{\alpha\beta}(z(s),k)$$

to show that for each s, the s-integrand on the right depends on $\Phi_{\alpha\beta}(z(s),X)$ only through its value on the hyperplane $X^4 = 0$, and hence depends on $\varphi_{\alpha\beta}(x)$ only through its value on the hypersurface $\Sigma(s)$ formed by all geodesics through $z(s)$ orthogonal to v^α.

By splitting the left hand integral similarly into one over $\Sigma(s)$ followed by one over s, we can equate the s-integrands. It is then straightforward to determine first $\tilde{m}^{\alpha\beta}(s,k)$ and then the moments $m^{\delta\cdots\gamma\alpha\beta}$ uniquely as integrals of $T^{\alpha\beta}$ over $\Sigma(s)$.

We get the variational equation by taking

$$\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}\,.$$

It follows from this that ω_α satisfies

$$\frac{\delta^2}{\mathrm{d}u^2}\omega_\alpha + \omega_\beta \dot{x}^\gamma \dot{x}^\delta R^{\beta}_{\bullet\gamma\delta\alpha} = \dot{x}^\beta \dot{x}^\gamma \nabla_{\{\beta}\varphi_{\alpha\gamma\}}$$

along all affinely parametrised geodesics $x^\alpha(u)$, where $\dot{x}^\alpha = \mathrm{d}x^\alpha / \mathrm{d}u$ and $A_{\{\alpha\beta\gamma\}} \equiv A_{\alpha\beta\gamma} - A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta}$.

Given an arbitrary symmetric $\varphi_{\alpha\beta}(x)$ let $\lambda_\alpha(z,x)$ be the unique field that satisfies this equation for ω_α along all such geodesics through z and that has

$$\lambda_\alpha = 0 \text{ and } \nabla_\beta \lambda_\alpha = \varphi_{\beta\alpha} \text{ at } x = z\,.$$

When $\varphi_{\alpha\beta}$ is constructed as above from ω_α, define also $\xi_\alpha(z,x)$ such that

$$\omega_\alpha(x) = \lambda_\alpha(z,x) + \xi_\alpha(z,x)\,.$$

Put $$G_{\alpha\beta} = \mathrm{Exp}^{\mathrm{A}}\, g_{\alpha\beta}, \quad \Lambda^\alpha = \mathrm{Exp}^{\mathrm{A}}\, \lambda^\alpha,$$

$$\mathrm{M}_\alpha = \mathrm{Exp}^{\mathrm{A}}\, \lambda_\alpha \equiv G_{\alpha\beta}\Lambda^\beta \quad \text{and} \quad \Xi_{\alpha\beta} \equiv \mathrm{Exp}^{\mathrm{A}}\, \nabla_{(\alpha}\xi_{\beta)}.$$

When $\varphi_{\alpha\beta} = \nabla_{(\alpha}\omega_{\beta)}$ then these constructions give

$$\Phi_{\alpha\beta} + \tfrac{1}{2}\Lambda^{\gamma}\nabla_{*\{\alpha}G_{\gamma\beta\}} = \nabla_{*(\alpha}\mathrm{M}_{\beta)} + \Xi_{\alpha\beta}$$

where $\nabla_{*\alpha} \equiv \partial/\partial X^{\alpha}$, which is a covariant operation. Motivated by this, we now modify the moment defining equation, changing it to

$$\int T^{\alpha\beta}\varphi_{\alpha\beta}\mathrm{D}X = M^{\alpha\beta}[\Phi_{\alpha\beta} + \tfrac{1}{2}\Lambda^{\gamma}\nabla_{*\{\alpha}G_{\gamma\beta\}}]$$

for arbitrary $\varphi_{\alpha\beta}$, so that the variational equation becomes

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)} + \Xi_{\alpha\beta}] = 0.$$

From the definition of $M^{\alpha\beta}[\cdots]$ we also have

$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)}] = 1\big/(2\pi)^4\int \mathrm{d}s\int \mathrm{D}k\,\tilde{t}^{\alpha}\tilde{\mathrm{M}}_{\alpha}$$

where

$$\tilde{t}^{\alpha}(s,k) \equiv -\mathrm{i}k_{\beta}\tilde{m}^{\alpha\beta}(s,k) = \sum (-\mathrm{i})^{n}\big/n!\; k_{\gamma}\cdots k_{\beta}t^{\gamma\cdots\beta\alpha}$$

with $t^{\alpha} = 0$ and $t^{\delta\cdots\gamma\beta\alpha} = (n+1)m^{(\delta\cdots\gamma\beta)\alpha}$ for $n \geq 0$.

For $n \geq 2$ we can decompose $m^{\delta \cdots \gamma\beta\alpha}$ through the use of symmetry operations into two tensors, one of which is $t^{\delta \cdots \gamma\beta\alpha}$. The other can be taken as the appropriate one from the set

$$J^{\zeta \cdots \varepsilon\delta\gamma\beta\alpha} \equiv \tfrac{1}{2}(m^{\zeta \cdots \varepsilon\delta[\beta\alpha]\gamma} - m^{\zeta \cdots \varepsilon\gamma[\beta\alpha]\delta}) \quad \text{for } n \geq 0 .$$

For $n = 0$ and 1 the m's and t's are equivalent as either can be expressed in terms of the other.

Now reconsider the orthogonality condition

$$(b) \qquad v_\gamma m^{\gamma \cdots \delta\alpha\beta} = 0 \ \text{ for } n \geq 1.$$

First generalise it by replacing the four-velocity v^α by an arbitrary timelike unit vector field n^α along L. The set of these for $n \geq 2$ then decomposes by symmetry operations into the two sets

$$(b1) \qquad n_\zeta J^{\zeta \cdots \varepsilon\delta\gamma\beta\alpha} = 0 \quad \text{for } n \geq 1,$$

$$(b2) \qquad n_\delta m^{\delta(\gamma \cdots \beta)\alpha} = 0 \quad \text{for } n \geq 2.$$

We revise the moment definitions further by replacing ($b2$) by the set

$$(b3) \qquad n_\delta t^{\delta\gamma \cdots \beta\alpha} \equiv n_\delta m^{(\delta\gamma \cdots \beta)\alpha} = 0 \quad \text{for } n \geq 2.$$

We omit any replacement of (*b*) for $n = 1$ and see later what freedom this leaves us. It is (*b3*) that enables the variational equation to be solved exactly. Let us split the variational equation into two parts:

- those contributions to
$$M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)}] \equiv \frac{1}{(2\pi)^4}\int \mathrm{d}s \int \mathrm{D}k\, \tilde{t}^{\alpha}\tilde{\mathrm{M}}_{\alpha}$$
arising from the *t*'s constrained by (*b3*), and
- the term $M^{\alpha\beta}[\Xi_{\alpha\beta}]$ together with the contributions to $M^{\alpha\beta}[\nabla_{*(\alpha}\mathrm{M}_{\beta)}]$ from the remaining *t*'s.

Then we find that

- as a consequence of (*b3*) the *s*-integrand in the first part depends on ω_α only through its value on $\Sigma(s)$, the hypersurface formed by all geodesics through *z*(*s*) orthogonal to n^α, and
- the second part depends on ω_α only through the values on *L* of it and its first two derivatives.

These properties lead to a result that each of the two parts must vanish separately. The vanishing of the first part shows that its *s*-integrand itself must vanish. This in turn requires that the *t*'s constrained by (*b3*) themselves all vanish, so that

$$t^{\gamma\cdots\beta\alpha} = 0 \quad \text{for } n \geq 3.$$

The vanishing of the second part brings us back to familiar territory known from treating the pole-dipole approximation. We find that $m^{\alpha\beta}$ and $m^{\gamma\alpha\beta}$ can be expressed in terms of a vector p^{α} and a tensor $S^{\alpha\beta}$, which at this time is not necessarily symmetric, by

$$m^{\gamma\beta\alpha} = \tfrac{1}{2}(S^{[\gamma\beta]}v^{\alpha} + S^{[\gamma\alpha]}v^{\beta} + S^{(\beta\alpha)}v^{\gamma}),$$

$$m^{\beta\alpha} = p^{(\beta}v^{\alpha)} + \tfrac{1}{2}\frac{\delta}{\mathrm{d}s}S^{(\beta\alpha)}.$$

The value of $S^{(\alpha\beta)}$ is arbitrary, since if we substitute the above expressions for $m^{\beta\alpha}$ and $m^{\gamma\beta\alpha}$ into the moment defining equation then the two contributions from $S^{(\alpha\beta)}$ combine into a total *s*-derivative that integrates to zero.

The arbitrariness of $S^{(\alpha\beta)}$ is the freedom we gain by omitting any constraint corresponding to the $n = 1$ case of the orthogonality condition (*b*), i.e. to $n_\gamma m^{\gamma\beta\alpha} = 0$. We choose $S^{(\alpha\beta)} = 0$, in which case

$$m^{\alpha\beta} = p^{(\alpha} v^{\beta)} \text{ and } m^{\alpha\beta\gamma} = S^{\alpha(\beta} v^{\gamma)}.$$

The remaining restrictions from the variational equation are then

$$\frac{\delta}{\mathrm{d}s} p^\alpha = \tfrac{1}{2} v^\beta S^{\gamma\delta} R^\alpha_{\bullet\beta\gamma\delta} + F^\alpha$$

and
$$\frac{\delta}{\mathrm{d}s} S^{\alpha\beta} = 2 p^{[\alpha} v^{\beta]} + L^{\alpha\beta}$$

which are the equations of the pole-dipole approximation with the addition of a force F^α and torque $L^{\alpha\beta}$ given by

$$F_\alpha = \frac{1}{(2\pi)^4} \int \tilde{m}^{\beta\gamma} \nabla_{\alpha *} \tilde{G}_{\beta\gamma} \mathrm{D}k,$$

$$L_{\alpha\beta} = \frac{2\mathrm{i}}{(2\pi)^4} \int \tilde{G}_{\gamma\delta[\alpha} \nabla_{*\beta]} \tilde{m}^{\gamma\delta} \mathrm{D}k.$$

Here $\nabla_{\alpha *}$ is a generalisation of the usual covariant derivative which has an extra term as a result of the dependence of the operand on position in the tangent space $T_z(M)$ and $\tilde{G}_{\alpha\beta\gamma}$ is the Fourier transform of

$$G_{\alpha\beta\gamma} \equiv \tfrac{1}{2}\nabla_{*\{\alpha} G_{\gamma\beta\}}.$$

These complete the exact solution of the variational equation but the more useful form is their multipole approximation

$$F_\alpha = \tfrac{1}{2}\sum_{n\geq 2} {}^{1}\!/_{n!}\, m^{\varepsilon\cdots\delta\gamma\beta}\, \nabla_\alpha g_{\gamma\beta,\varepsilon\cdots\delta}$$

and $$L^{\alpha\beta} = \sum_{n\geq 1} {}^{1}\!/_{n!}\, g^{\gamma[\alpha} m^{\beta]\varepsilon\cdots\delta\zeta\eta}\, g_{\{\gamma\eta,\zeta\}\varepsilon\cdots\delta}.$$

As infinite series these do not converge, but they may be truncated at any desired order. The quantities $g_{\alpha\beta,\gamma\cdots\delta}$ are the *Veblen extensions* of the metric tensor $g_{\alpha\beta}$, defined as the tensors that reduce to the partial derivatives of the metric tensor at the pole of a normal coordinate system.

The quadrupole and higher moments that occur in the force and torque are determined by the J's. which have the symmetry and orthogonality properties

$$J^{\alpha\cdots\beta\gamma\delta\varepsilon\zeta} = J^{(\alpha\cdots\beta)[\gamma\delta][\varepsilon\zeta]} \text{ and}$$

$$J^{\alpha\cdots\beta\gamma[\delta\varepsilon\zeta]} = 0 \quad \text{for } n \geq 0,$$

$$J^{\alpha\cdots[\beta\gamma\delta]\varepsilon\zeta} = 0 \quad \text{and} \quad n_\alpha J^{\alpha\cdots\beta\gamma\delta\varepsilon\zeta} = 0 \quad \text{for } n \geq 1.$$

They are otherwise arbitrary and in particular their dependence on s is unconstrained. The m's are determined from them by

$$m^{\alpha\cdots\beta\gamma\delta\varepsilon} = \frac{4(n-1)}{n+1} J^{(\alpha\cdots\beta|\delta|\gamma)\varepsilon} \quad \text{for } n \geq 1.$$

Explicit expressions can be obtained for all the moments as integrals over $\Sigma(s)$ and shown to satisfy all the above properties, so proving their existence and uniqueness. In particular p^α and $S^{\alpha\beta}$ have simple integral expressions which show that for a given $T^{\alpha\beta}$ they depend only on the point z and vector n^α (not, for example, also on v^α).

We are now in a position to characterise the centre of mass. For any point z we choose n^{α} to be parallel to p^{α}, as a solution of the implicit equation

$$p^{\alpha}(z,n) = M(z)n^{\alpha}$$

which serves also to define the mass M. With this choice of n^{α} we can consider p^{α} and $S^{\alpha\beta}$ as functions only of z. We then take L to be the set of points z that satisfy the Tulczyjew condition

$$p_{\beta}S^{\alpha\beta} = 0.$$

This is the world line of the centre of mass. It has been proved by Schattner (1978) that subject to mild restrictions on the strength of the gravitational field, these conditions do determine the field n^{α} and the line L uniquely.

This completes the realisation of the three aims of Mathisson’s “New Mechanics”.

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