Equations of motion in the gauge gravity models

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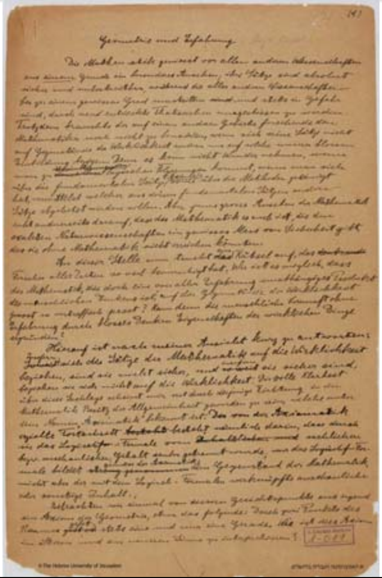
Outline

Introduction

- Crash-course in differential geometry:
- Spacetime = 4-dimensional smooth manifold
- Coframe $\vartheta^\alpha = h_i^\alpha dx^i$; dual frame field e_α (observer)
- Metric $\mathbf{g}$: $g_{\alpha\beta} = \mathbf{g}(e_\alpha, e_\beta)$ (lengths and angles)
- Linear connection: 1-form $\Gamma_\alpha{}^\beta$ (parallel transport)
- Notation: $\alpha, \beta, \dots = 0, 1, 2, 3$ anholonomic components; $i, j, \dots = 0, 1, 2, 3$ coordinate components
- η - volume 4-form; $\eta_\alpha = e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta)$, $\eta_{\alpha\beta\gamma} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$, and $\eta_{\alpha\beta\gamma\delta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma \wedge \vartheta_\delta)$

- Spacetime metric and linear connection structures are a priori *independent*.
- *Riemannian constraints:*
- $Dg_{\alpha\beta} = 0$ (metricity)
- $D\vartheta^\alpha = 0$ (no torsion)
- Then connection is expressed in terms of the metric:
- $$\Gamma_\alpha{}^\beta = \widetilde{\Gamma}_\alpha{}^\beta = \left\{{}_\alpha{}^\beta{}_\gamma\right\} \vartheta^\gamma$$
- $\left\{{}_\alpha{}^\beta{}_\gamma\right\} = \frac{1}{2} g^{\beta\delta} \left(\partial_\alpha g_{\delta\gamma} + \partial_\gamma g_{\delta\alpha} - \partial_\delta g_{\alpha\gamma}\right)$ Christoffel symbols.
- Einstein's GR is based on the Riemannian geometry.



“...The question whether this continuum has a Euclidean, Riemannian, or any other structure is a question of physics proper which must be answered by experience, and not a question of a convention to be chosen on grounds of mere expediency.”

A. Einstein, Geometrie und Erfahrung, Sitzungsber. preuss. Akad. Wiss. **1** (1921) 123-130.

- Non-Riemannian geometries in physics (examples)
- no gravity
  - Dislocations, Cosserat continuum
  - Continua with microstructure (Mindlin elastic bodies)
  - Thermodynamics (theory of equilibrium)
  - Physics of plasma (drift dynamics)
- gravity and spacetime geometry
  - Einstein-Cartan theory (minimal generalization of GR)
  - Ashtekar approach, first-order gravity
  - Supergravity
  - String theory (effective action)
  - Conformal symmetry (Penrose 1982)

- How can the non-Riemannian geometry of spacetime be detected?
- $\Longrightarrow$ dynamics of matter (particles, fluids, bodies)
- There are several widely spread erroneous claims and results about the motion in non-Riemannian spacetimes
- Point particle in GR moves along geodesics
- False $\#1$: Point particle in non-Riemannian geometries moves along autoparallel (H.Kleinert, e.g.)
- Compare: *neutral particle do not feel Lorentz force!*
- False $\#2$: Macroscopic collective dynamics of usual matter can feel non-Riemannian geometry (Y.Mao, M.Tegmark, A. Guth, S. Cabi, arXiv:gr-qc/0608121 on Gravity Probe B)
- False $\#3$: Equivalence principle violates gauge symmetry

Gauge approach to gravity

- History:
- Utiyama (1956), Kibble (1961), Sciama (1962)
- Reviews:
    - F.W. Hehl et al, *Rev. Mod. Phys.* **48** (1976) 393
    - F.W. Hehl et al, *Phys. Rep.* **258** (1995) 1
    - A. Trautman, in: *One hundred years after the birth of A. Einstein*, (NY, Plenum, 1980) vol. 1, p. 287
    - A. Trautman, in: *Encyclopedia of Mathematical Physics*, (Elsevier, Oxford, 2006) vol. 2, p. 189
- Yang-Mills type framework:
- Global symmetry $\Longrightarrow$ conserved current J_I (3-form)
- Local symmetry $\Longrightarrow$ gauge field A^I (1-form potential)
- $$\Longrightarrow \quad J_I \wedge A^I \quad \text{interaction}$$

- General scheme follows Yang-Mills for *internal symmetry* group G with N parameters ε^I. Invariance of action under *global* symmetry

$$\text{Noether theorem} \quad \Longrightarrow J_I, \qquad I = 1, \dots, N.$$

- Local symmetry with $\varepsilon^I(x)$ demands A^I with

$$\delta A^I = -D\varepsilon^I. \quad \text{Then} \quad DJ_I = 0.$$

- Gravity: *spacetime symmetry* group G. Lorentz group $\longrightarrow$ Poincaré group $\longrightarrow$ general affine group $GA(4, R)$.

Gravitational gauge potentials of metric-affine gravity (MAG)

$$\begin{array}{cc} \vartheta^\alpha & (\text{translations } \varepsilon^\alpha) \\ \Gamma_\alpha{}^\beta & (\text{linear group } \varepsilon_\alpha{}^\beta) \\ g_{\alpha\beta} & (\text{special status}) \end{array}$$

- Gauge field strength (recall: $F^I =$"DA^I"$= dA^I + [A\wedge A]^I$)

Gravitational gauge field strengths

$$
\begin{aligned}
T^\alpha &= D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha\wedge\vartheta^\beta && \text{(torsion)}\\
R_\alpha{}^\beta &= \text{``}D\Gamma_\alpha{}^\beta\text{''} = d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta\wedge\Gamma_\alpha{}^\gamma && \text{(curvature)}\\
Q_{\alpha\beta} &= -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha{}^\gamma\, g_{\gamma\beta} + \Gamma_\beta{}^\gamma\, g_{\alpha\gamma} && \text{(nonmetricity)}
\end{aligned}
$$

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- Sources arise as derivatives (recall: $J_I = \delta L_{\rm mat}/\delta A^I$)

Currents in metric-affine gravity

$$
\begin{aligned}
\Sigma_\alpha &= \frac{\delta L_{\rm mat}}{\delta\vartheta^\alpha} && \text{(canonical energy}-\text{momentum)}\\
\Delta^\alpha{}_\beta &= \frac{\delta L_{\rm mat}}{\delta\Gamma_\alpha{}^\beta} && \text{(intrinsic hypermomentum)}\\
\sigma^{\alpha\beta} &= 2\,\frac{\delta L_{\rm mat}}{\delta g_{\alpha\beta}} && \text{(metrical energy}-\text{momentum)}
\end{aligned}
$$

- In GR, only mass (energy-momentum) was the source of the gravitational field
- in MAG, in addition to the energy-momentum, there are sources that correspond to the microstructure of matter

“Fine structure” of hypermomentum

$$\Delta^{\alpha}{}_{\beta} = \tau^{\alpha}{}_{\beta} + \tfrac{1}{4}\,\Delta\,\delta^{\alpha}_{\beta} + \not\Delta^{\alpha}{}_{\beta}$$

$\tau_{\alpha\beta} = -\tau_{\beta\alpha}$	spin
$\Delta = \Delta^{\alpha}{}_{\alpha}$	dilation current
$\not\Delta_{\alpha\beta} = \not\Delta_{\beta\alpha}\ (\not\Delta^{\alpha}{}_{\alpha} = 0)$	shear hypermomentum

- This decomposition corresponds to the Lorentz (rotation) group, dilation (isotropic volume scaling), and anisotropic shear deformation with fixed volume

- Noether: Symmetry $\Longrightarrow$ conservation law (recall: $DJ_I = 0$)

Diffeomorphism invariance: energy-momentum conservation

$$D\Sigma_\alpha = (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \Delta^\beta{}_\gamma - \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})\, \sigma^{\beta\gamma}$$

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- Additionally

Linear group transformations: hypermomentum conservation

$$D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - \sigma^\alpha{}_\beta = 0$$

- In gravity, equations of motion need not be postulated separately, they can be derived from the conservation laws

Matter in metric-affine approach

- Matter in metric-affine gravity is characterized by *mass* (energy-momentum) and, in addition, by *microstructural properties* (spin, dilation, and shear “charges”).
- Specific examples:
    - Medium with defects, Cosserat continuum
    - Elastic medium with microstructure (Mindlin continuum)
    - Hadronic matter (composite objects describing excitation bands of nuclei), see Y. Dothan, M. Gell-Mann, and Y. Ne’eman, *Phys. Lett.* **17** (1965) 148
- Types of matter to study:
    - Fundamental matter fields (e.g., multispinors of Ne’eman and Šijački)
    - Point particles
    - Continuous media (fluids)
    - Test bodies (multipole approach)

Elastic test particle

- [Spinning particle: W. Kopczyński, *Phys. Rev.* **D34** (1986) 352]
- We describe test particle with microstructure as a physical point with attached frame θ^i_a, $a = 0, 1, 2, 3$
- Frame is not orthonormal (g_{ij} spacetime metric)

$$g_{ab} = g_{ij}\theta^i_a\theta^j_b \qquad \text{“internal metric”}$$

- With the evolution parameter t, motion is described by

$$x^i(t), \qquad \theta^i_a(t), \qquad g_{ab}(t)$$

- θ^a_i is inverse, $\theta^a_i\theta^i_b = \delta^a_b$

- Action $A = \int_{t_0}^{t_1} L\,dt$. The Lagrangian is a function of

$$x^i, \quad v^i = \frac{dx^i}{dt}, \quad \theta^i_a, \quad \dot{\theta}^i_a, \quad g_{ab}$$

- Here dot denotes covariant derivative along the trajectory

$$\dot{\theta}^i_a = v^k \nabla_k \theta^i_a = \frac{d\theta^i_a}{dt} + v^k \Gamma_{kj}{}^i \theta^j_a$$

- We demand invariance under (i) diffeomorphisms, (ii) global general linear transformation

$$\theta^i_a \longrightarrow \Lambda_a{}^b \theta^i_b, \quad g_{ab} \longrightarrow \Lambda_a{}^c \Lambda_b{}^d g_{cd}$$

- Introduce the generalized velocity deformation tensor

$$W_j{}^i := \theta^a_j \dot{\theta}^i_a$$

- *Invariant* Lagrangian is then unspecified function

$$L = L(v^i,\ W_i{}^j,\ g_{ij})$$

- Define the momentum, hypermomentum, and symmetric stress as

Physical characteristics of a particle

$$P_i := \frac{\delta L}{\delta v^i}, \qquad J^i{}_j := \frac{\delta L}{\delta W_i{}^j}, \qquad I^{ij} := 2\frac{\delta L}{\delta g_{ij}}$$

- Since L is scalar under diffeomorphisms, we find *identity*

Momentum, hypermomentum and stress are related

$$P_i v^j = I_i{}^j + W_i{}^k J^j{}_k - W_k{}^j J^k{}_i$$

Technical details: Variation

- For a function $f(x)$ its total variation $\delta f(x) = f'(x') - f(x)$ with $x' = x + \delta x$ reads $\delta f = \delta_{(s)} f + \delta x^\mu \partial_\mu f$ where $\delta_{(s)} f(x) = f'(x) - f(x)$ is the *substantial variation*.
- Then the total variation of the Lagrangian

$$\delta L = P_i(\delta_{(s)} + \delta_{(x)})v^i + J^i{}_j(\delta_{(s)} + \delta_{(x)})W_i{}^j + \frac{1}{2}I^{ij}(\delta_{(s)} + \delta_{(x)})g_{ij}$$

with $\delta_{(x)} f = \delta x^\mu \nabla_\mu f$ for any f.

- Obviously $\delta_{(s)} x^i = \delta x^i$ and then direct calculation yields

$$\delta_{(s)} v^i = 0, \qquad \delta_{(x)} v^i = (\dot{\delta x^i}) + T_{kl}{}^i \delta x^k v^l$$

$$\delta_{(s)} W_i{}^j = \theta_i^a(\dot{\delta_{(s)}\theta_a^i}) - W_k{}^j \theta_i^a \delta_{(s)} \theta_a^k + v^k \delta_{(s)} \Gamma_{ki}{}^j$$

$$\delta_{(x)} W_i{}^j = (\theta_i^a \dot{\delta_{(x)}\theta_a^i}) + W_i{}^k \theta_k^a \delta_{(x)} \theta_a^i - W_k{}^j \theta_i^a \delta_{(x)} \theta_a^k + R_{kli}{}^j \delta x^k v^l$$

Equations of motion of test particle

- For a motion on a time interval t_0, t_1 from initial state $\{x^i(t_0), \theta^i_a(t_0), g_{ab}(t_0)\}$ to final one $\{x^i(t_1), \theta^i_a(t_1), g_{ab}(t_1)\}$ on a *fixed MAG background geometry*, dynamics follows from the variational principle $\delta \int_{t_0}^{t_1} L dt = 0$ with

$$\begin{aligned}
\delta x(t_0) = \delta_{(s)}\theta(t_0) = \delta_{(s)}g_{ab}(t_0) &= 0,\\
\delta x(t_1) = \delta_{(s)}\theta(t_1) = \delta_{(s)}g_{ab}(t_1) &= 0,\\
\delta_{(s)}g_{ij} = 0, \quad \delta_{(s)}\Gamma_{ki}{}^{j} &= 0.
\end{aligned}$$

Equations of motion from variations of δx and $\delta\theta$

$$\dot{P}_i = T_{ij}{}^{k} v^j P_k + R_{ijk}{}^{l} v^j J^k{}_l - \frac{1}{2} Q_{ijk}(P^j v^k - W^{jl} J^k{}_l + W_l{}^k J^{jl})$$

$$\dot{J}^i{}_j = W_k{}^i J^k{}_j - W_j{}^k J^i{}_k$$

- Special case 1: When particle does not have microstructure, $J^i{}_j = 0$, equations of motion reduce to

$$v^j \widetilde{\nabla}_j P_i = 0.$$

- $\Longrightarrow$ trajectory is *Riemannian geodesics*, even in MAG.
- Special case 2: When the geometry of spacetime is Riemannian, the terms with torsion and nonmetricity vanish
- Then we recover usual *Mathisson-Papapetrou* equation

$$v^j \widetilde{\nabla}_j P_i = \widetilde{R}_{ijk}{}^l v^j \tau^k{}_l$$

Dynamical currents as gravitational sources

$$\Sigma_\alpha{}^i = v^i\, P_\alpha, \quad \Delta^\alpha{}_\beta{}^i = v^i\, J^\alpha{}_\beta, \quad \sigma^{\alpha\beta} = I^{\alpha\beta}$$

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Hyperfluid: continuous medium with microstructure

- Fluid model is constructed along the similar lines
- History: Weyssenhoff-Raabe (1947), Kopczyński (1985)
- Fluid's element is characterized by four-velocity u^α (*flow* 3-form $u = u^\alpha \eta_\alpha$) and *triad* b^α_A, $A = 1, 2, 3$ attached
- Together (u^α, b^α_A) form material frame $u^\alpha u_\alpha = 1, u_\alpha b^\alpha_A = 0.$
- Internal structure of fluid is described by: *particle density* ρ, *specific entropy* s, *Lin (identity) variable* X, *specific hypermomentum density* $\mu^A{}_B$
- Constraints imposed:

$$d(\rho u) = 0, \quad u \wedge ds = 0, \quad u \wedge dX = 0$$

(number of particles, identity, and entropy are conserved)

Motion of hyperfluid

- Fluid's Lagrangian

$$L = \epsilon(\rho, s, \mu^A{}_B)\,\eta - \frac{1}{2}\,\mu^A{}_B b^B_\alpha u \wedge D b^\alpha_A + \text{constraints}$$

- Internal energy density ϵ; second term is kinetic energy
- Variation wrt: field g, ϑ, Γ and matter $u, b, \rho, s, X, \mu^A{}_B$
- Define density $J^\alpha{}_\beta := \frac{1}{2}\,\mu^A{}_B b^B_\beta b^\alpha_A$ and pressure $p = \rho\frac{\partial\epsilon}{\partial\rho} - \epsilon$
- With $P_\alpha = \epsilon\, u_\alpha + 2u^\beta g_{\gamma[\alpha}\dot{J}^\gamma{}_{\beta]}$ and $h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ then

dynamical currents as gravitational sources

$$\begin{aligned}
\Sigma_\alpha &= u\,P_\alpha - p(\eta_\alpha - u u_\alpha)\\
\Delta^\alpha{}_\beta &= u\,J^\alpha{}_\beta\\
\sigma^{\alpha\beta} &= \eta(\epsilon u^\alpha u^\beta - p h^{\alpha\beta}) + 2u^\gamma u^{(\alpha} D\Delta^{\beta)}{}_\gamma
\end{aligned}$$

- The dynamics of the hypermomentum (derived from $\delta\mu$) is described by the equation

$$D\Delta^{\alpha}{}_{\beta} = u^{\alpha}u_{\gamma}D\Delta^{\gamma}{}_{\beta} + u_{\beta}u^{\gamma}D\Delta^{\alpha}{}_{\gamma}$$

- The hypermomentum density satisfies

$$J^{\alpha}{}_{\beta}\,u^{\beta} = 0, \qquad J^{\alpha}{}_{\beta}\,u_{\alpha} = 0$$

- This generalizes the Frenkel-Pirani condition
- Translational equation of motion (Euler equation) reads

$$D\Sigma_{\alpha} = (e_{\alpha}\rfloor T^{\beta}) \wedge \Sigma_{\beta} + (e_{\alpha}\rfloor R_{\beta}{}^{\gamma}) \wedge \Delta^{\beta}{}_{\gamma} - \frac{1}{2}(e_{\alpha}\rfloor Q_{\beta\gamma})\,\sigma^{\beta\gamma}$$

- Hyperfluid model generalizes phenomenological Weyssenhoff-Raabe model of spinning particles/fluids
- It reproduces the elastic particle results when the pressure vanishes $p = 0$ (collisionless gas)

Summary and Conclusions

- Following Einstein, geometrical structure of spacetime is a *physical question*, and not a matter of convention, it should be determined *experimentally*
- Metric-affine gravity provides framework to go beyond Riemannian geometry
- Matter with microstructure is a physical tool to detect non-Riemannian structures
- Other aspects (not touched in this talk):
    - Semiclassical derivation from fundamental quantum theory?
    - Extended bodies

Thanks !

Dziękue !