Huygens' principle and anomalously small radiation tails

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Outline:

- Introduction and motivation
- Late-time tails of linear and nonlinear waves
- Anomalous tails and huygensian systems

Myron Mathisson and Huygens' principle

A wave equation is said to satisfy Huygens' principle if:

- the solution at a point P depends only on the initial data at the intersection of the past light cone of P with the Cauchy hypersurface or, equivalently,
- the solution vanishes at all points which cannot be reached from the initial data by a null geodesic (there are no tails).

Hadamard's conjecture: The only huygensian linear second-order hyperbolic equation of the form

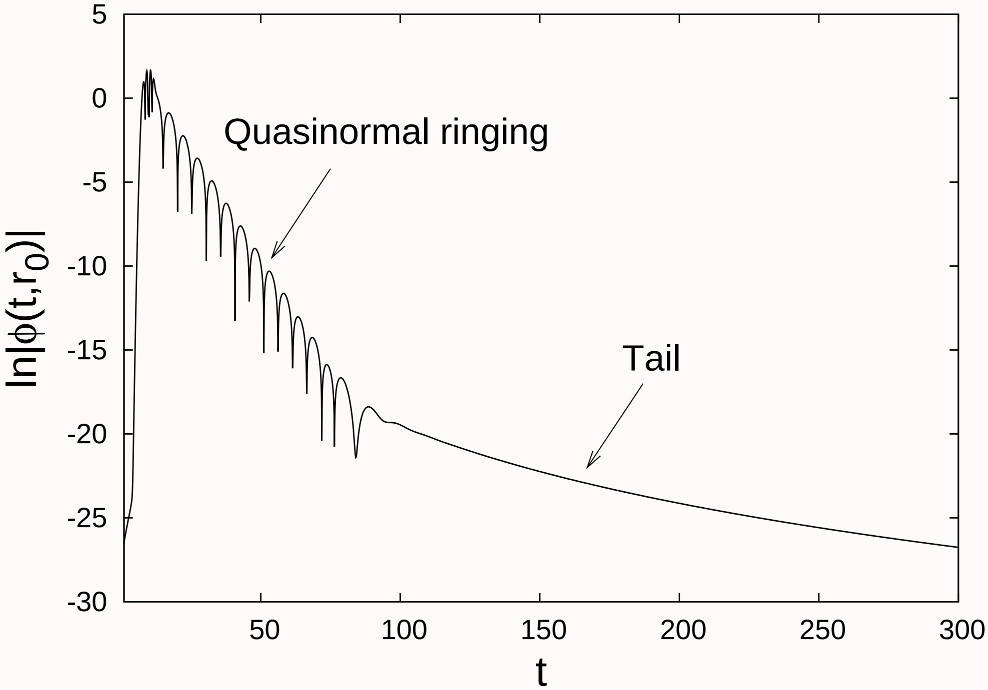
$$g^{\mu\nu}(x)\nabla_\mu \nabla_\nu \phi + A^\mu(x)\nabla_\mu \phi + B(x)\phi = 0$$

is the ordinary wave equation $\Box\phi = 0$ in even dimensions ≥ 4 (modulo trivial transformations).

Mathisson (1939) proved this conjecture in the case of four dimensional Minkowski spacetime.

Our motivation

- Goal: understanding of relaxation to equilibrium for nonlinear wave equations
- Mechanism of relaxation: dissipation by dispersion
  (only for spatially unbounded domains)
- Example: formation of the skyrmion



Model and assumptions

$$\Box\phi + V(x)\phi + N(t, x, \phi, \nabla\phi) = 0\,, \quad \Box = \partial_t^2 - \Delta\,, \quad (t,x) \in R^{1+d}$$

- Spherically symmetric smooth initial data with compact support
- Spatial dimension $d = \text{odd} \geq 3$
- Why $d > 3$?

(i) $$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right)\phi = \frac{1}{r^l}\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r + \frac{l(l+1)}{r^2}\right)(r^l\phi)$$
where $l = (d-3)/2$.

(ii) Some geometric wave equations in $3+1$ are equivalent to scalar wave equations in $d+1$ for $d > 3$. Example: Wave maps in $3+1$ dimensions.

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi + \frac{\sin(2\psi)}{r^2} = 0$$ after substitution $\psi = r\phi$ becomes

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\phi + \frac{4}{3}\phi^3 + \text{higher order terms} = 0$$

Tools (elementary)

- Solution of $\Box\phi = 0$ with smooth radial data ($d = 2l+3$):

$$\phi(t,r) = \frac{1}{r^{2l+1}} \sum_{k=0}^{l} \frac{2^{k-l}(2l-k)!}{k!(l-k)!} r^k \left(a^{(k)}(t-r) - (-1)^k a^{(k)}(t+r) \right) \quad (1)$$

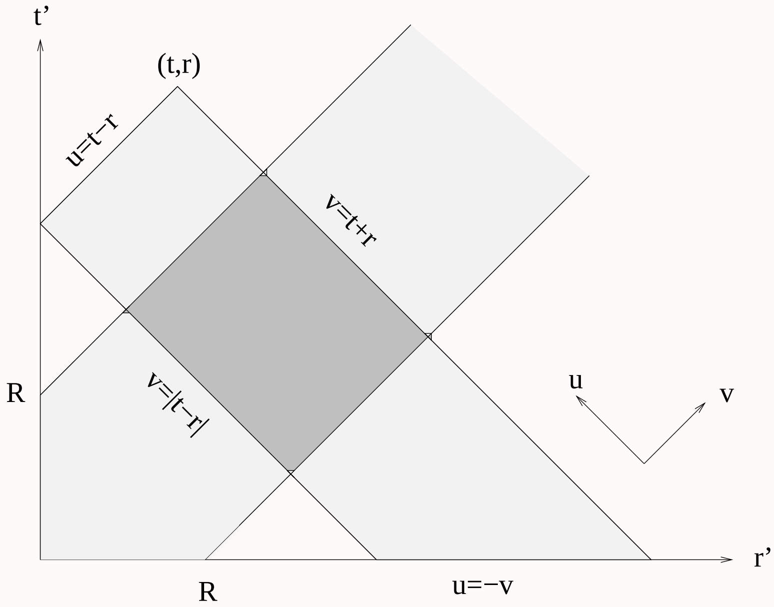
- Solution of $\Box\phi = F(t,r)$ with zero data (Duhamel formula):

$$\phi(t,r) = \frac{1}{2r^{l+1}} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} K(t,r;\tau,\rho) F(\tau,\rho) d\rho \,,$$

where $K(t,r;\tau,\rho) = \rho^{l+1} P_l(\mu)$ and $\mu = \dfrac{r^2+\rho^2-(t-\tau)^2}{2r\rho}$.
In terms of $u = \tau - \rho$ and $v = \tau + \rho$ this becomes

$$\phi(t,r) = \frac{1}{4r^{l+1}} \int_{|t-r|}^{t+r} dv \int_{-v}^{t-r} K(t,r;u,v) F(u,v) du \quad (2)$$

Simplification due to Huygens



$$F(u, v) = 0 \text{ for } |u| > R$$

$$\phi(t, r) = \frac{1}{4r^{l+1}} \int_{-\infty}^{\infty} du \int_{t-r}^{t+r} K(t, r; u, v) F(u, v) dv$$

Linear tails

$$\Box\phi + \lambda V\phi = 0\,, \quad (\phi(0,r), \partial_t\phi(0,r)) = (f(r), g(r))$$

Perturbation series:

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \ldots$$

Iteration:

$$\Box\phi_0 = 0, \quad \Box\phi_1 = -V\phi_0, \quad \Box\phi_2 = -V\phi_1\,, \text{etc.}$$

Assuming that $V(r) \sim r^{-\alpha}$ ($\alpha > 2$) for $r \to \infty$ we get (at fixed r and $t \to \infty$)

$$\phi_1(t,r) = \frac{C(l,\alpha)}{t^{\alpha+2l}}\left[A + \mathcal{O}\left(\frac{1}{t}\right)\right]$$

$$C(l,\alpha) = -\frac{2^{\alpha+2l-1}}{(2l+1)!!}\left(\frac{\alpha-3}{2}\right)^{\underline{l}}\left(\frac{\alpha}{2}\right)^{\overline{l}} \quad \text{and} \quad A = \int\limits_{-\infty}^{+\infty} a(u)\,du$$

This was first derived by Ching et al. (1995). Notation:

$$x^{\underline{0}} := 1, \qquad x^{\underline{k}} := x\cdot(x-1)\cdot\cdots\cdot(x-(k-1)), \quad k>0\,,$$
$$x^{\overline{0}} := 1, \qquad x^{\overline{k}} := x\cdot(x+1)\cdot\cdots\cdot(x+(k-1)), \quad k>0\,.$$

How good is the first order approximation?

- All higher-order terms $\phi_n(t,r)$ decay in the same manner as $\phi_1(t,r)$
- Convergence of the perturbation series? (proved for $d=3$, Szpak)
- The perturbation series is asymptotic to the solution (for any $d \geq 3$)

$$\phi(t,r) - \lambda\phi_1(t,r) \sim \mathcal{O}(\lambda^2) t^{-(\alpha+2l)} \tag{3}$$

Numerical evidence.

Example:

$$V = \lambda \frac{\tanh^{\alpha+2} r}{r^\alpha}$$

$$\lambda = 0.1$$

$\alpha = 4$		Theory	Numerics
$d=3$	Exponent	4	4.00002
$(l=0)$	Amplitude	-0.3545	-0.3320
$d=5$	Exponent	6	5.9999
$(l=1)$	Amplitude	-0.2363	-0.2318
$d=7$	Exponent	8	7.9999
$(l=2)$	Amplitude	0.1418	0.1404

Nonlinear tails

$$\Box\phi - \phi^p = 0\,, \quad (\phi(0,r), \partial_t\phi(0,r)) = (\varepsilon f(r), \varepsilon g(r))$$

Perturbation series:

$$\phi = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \dots$$

Iteration:

$$\Box\phi_0 = 0, \quad \Box\phi_p = \phi_0^p \text{ , etc.}$$

At fixed r and $t \to \infty$

$$\phi_p(t,r) = \frac{\tilde{C}(l,p)}{t^{(l+1)p-1}} \left[\tilde{A} + \mathcal{O}\left(\frac{1}{t}\right)\right]$$

$$\tilde{C}(l,p) = (-1)^l \frac{2^{(l+1)(p+1)-1}}{(2l+1)!!}[(l+1)(p-1)-2]^l \quad \text{and} \quad \tilde{A} = \int_{-\infty}^{+\infty} [a^{(l)}(u)]^p \, du$$

The perturbation series is convergent for $d = 3\,(l = 0)$ (Szpak) and, at least, asymptotic for $d > 3\,(l > 0)$.

Competition between linear and nonlinear tails

$$\text{linear tail} \sim t^{-(\alpha+2l)} \quad vs. \quad \text{nonlinear tail} \sim t^{1-(l+1)p}$$

Thus

$$\phi(t,r) \sim t^{-\gamma}\,, \qquad \gamma = \min\{\alpha + 2l, (l+1)p - 1\}$$

Example: relaxation of the skyrmion

$$(w_F \dot{F})^{\cdot} - (w_F F')' + \sin(2F) + \alpha^2 \sin(2F)\left(\frac{\sin^2 F}{r^2} + F'^2 - \dot{F}^2\right) = 0\,,$$

where $w_F = r^2 + 2\alpha^2 \sin^2 F$. Asymptotically $F(t,r) \to S(r)$ =skyrmion.
The perturbation $\phi(t,r) = r{w_S}^{-1/2}[F(t,r) - S(r)]$ satisfies

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\phi + V\phi + \frac{4}{3}\phi^3 + \text{higher order terms} = 0\,, \quad V(r) \sim r^{-6}\,.$$

We have $d = 5\,(l = 1)$, $\alpha = 6$, $p = 3$, thus

$$\text{linear tail} \sim t^{-8}, \quad \text{nonlinear tail} \sim t^{-5}$$

Anomalous tails

- Linear case ($V(r) = \lambda r^{-\alpha}$ for $r > R$)

$$C(l,\alpha) \propto \left(\frac{\alpha-3}{2}\right)^{\underline{l}} = 0 \quad \text{if} \quad \alpha = \text{odd integer} \leq 2l+1$$

No tail in the first order! $\Leftrightarrow$ The system $\Box\phi_0 = 0$, $\Box\phi_1 = -V\phi_0$ is huygensian
Solving the second order equation $\Box\phi_2 = -V\phi_1$ we get

$$\phi(t,r) \approx \lambda^2 \phi_2(t,r) = \lambda^2 \frac{D(l,\alpha)}{t^{2(\alpha+l-1)}} \left[A + \mathcal{O}\left(\frac{1}{t}\right)\right]$$

- Nonlinear case ($\Box\phi = \phi^p$):

$$C(l,p) \propto [(l+1)(p-1)-2]^{\underline{l}} = 0 \quad \text{if} \quad p = 2 \quad \text{and} \quad l \geq 1$$

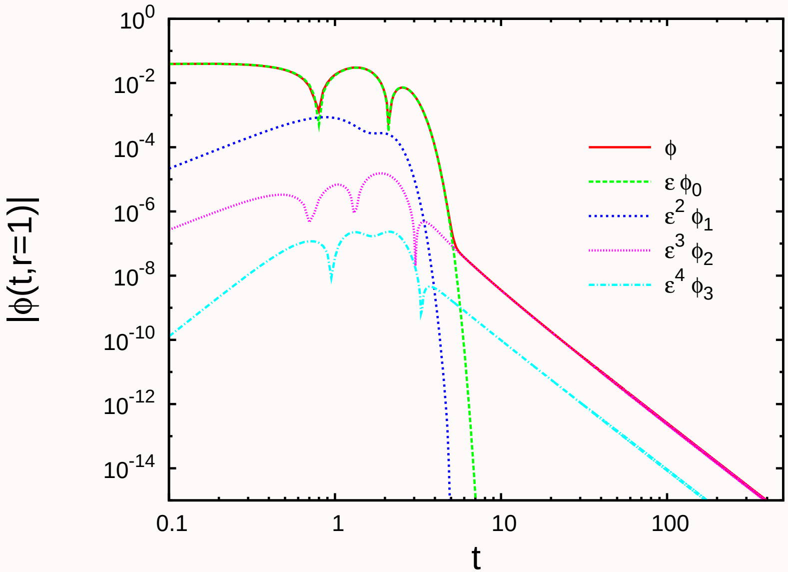
No tail in the first order!$\Leftrightarrow$ The system $\Box\phi_0 = 0$, $\Box\phi_1 = \phi_0^2$ is huygensian
Solving the second order equation $\Box\phi_2 = 2\phi_0\phi_1$ we get

$$\phi(t,r) \approx \varepsilon^3 \phi_2(t,r) \sim \varepsilon^3 \frac{c(l)}{t^{3l+1}}$$

Examples of anomalous tails

- Spherically-symmetric Yang-Mills in $3+1$ dimensions

$$(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r)\phi + 3\phi^2 + r^2\phi^3 = 0$$



Near timelike infinity

$$\phi(t,r) \approx \varepsilon^3 \phi_2(t,r) \sim \varepsilon^3\, c\, t^{-4}$$

$$c = -8 \int_{-\infty}^{+\infty} a(u) a'(u)^2 du$$

Tails on Schwarzschild background

- Thoroughly studied in $d = 3$. Price's law: $\phi(t, r) \sim t^{-3}$
- Almost unexplored for $d > 3$

$$\partial_t^2 \psi - \partial_x^2 \psi + U(x)\psi = 0, \qquad dr/dx = 1 - 1/r^{d-2}$$

For $x \longrightarrow \infty$

$$U(x) \sim \frac{l(l+1)}{x^2} + \frac{a}{x^d} + \frac{b}{x^{2d-2}}, \qquad l = j + \frac{d-3}{2}$$

Conjecture: For odd $d > 3$ the tail behaves as $t^{-\gamma}$, where

$$\gamma = \alpha + 2l = 2d - 2 + 2(j + \frac{d-3}{2}) = 2j + 3d - 5$$

In higher dimensions black holes become bald very fast