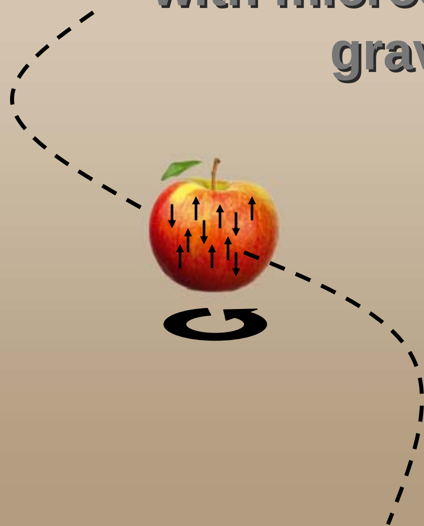
The motion of test bodies with microstructure in gauge gravity models



Dirk Puetzfeld
(Inst. of Theoretical Astrophysics - University of Oslo)

„Myron Mathisson: his life, work, and influence on current research“

Stefan Banach Center, Warsaw

17-20 October 2007

Outline

- Metric-affine gravity
- Conservation laws
- Propagation equations
- Special cases
- Open problems

Timeline test-particle EOM I

1923	1927	1931	1937	1938	1939
Weyl	Einstein & Grommer Lanczos	Mathisson	Mathisson Robertson	EIH	Fock

1940	1941	1949	1951	1953	1955
Papapetrou	Lanczos	Infeld & Schild	Papapetrou	Papapetrou Goldberg	Meister & Papapetrou

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Timeline test-particle EOM II

1957	1959	1960	1962	1964
Infeld	Kerr Fock Tulczyjew	Infeld & Plebanski Kerr Synge	Goldberg Havas & Goldberg Tulczyjew & Tulczyjew	Taub Dixon Havas

1969	1970	1971	1973	1979	1980	1997
Madore	Dixon	Hehl	Liebscher	Dixon	Yasskin & Stoeger Bailey & Israel	Ne'eman & Hehl

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Metric-affine gravity (MAG)

Potentials	Field strengths	Gauge currents	Excitations	Matter currents
$g_{\alpha\beta}$	Nonmetricity $Q_{\alpha\beta}$	$m^{\alpha\beta}$	$M^{\alpha\beta}$	$\sigma^{\alpha\beta}$
ϑ^{α}	Torsion T^{α}	E_{α}	H_{α}	Σ_{α}
$\Gamma_{\alpha}{}^{\beta}$	Curvature $R_{\alpha}{}^{\beta}$	$E^{\alpha}{}_{\beta}$	$H^{\alpha}{}_{\beta}$	$\Delta^{\alpha}{}_{\beta}$

$$L = V_{\mathsf{MAG}}(g_{\alpha\beta}, \vartheta^{\alpha}, Q_{\alpha\beta}, T^{\alpha}, R_{\alpha}{}^{\beta}) + L_{\mathsf{mat}}(g_{\alpha\beta}, \vartheta^{\alpha}, \psi, D\psi)$$

Field equations

$$\frac{\delta L_{\mathsf{mat}}}{\delta \psi} = 0$$

$$\mathbf{0} \qquad DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}$$

$$\mathbf{I} \qquad DH_{\alpha} - E_{\alpha} = \Sigma_{\alpha}$$

$$\mathbf{II} \qquad DH^{\alpha}{}_{\beta} - E^{\alpha}{}_{\beta} = \Delta^{\alpha}{}_{\beta}$$

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

A more general connection

$$\Gamma_{bc}{}^{a} = \frac{1}{2} g^{ak} \left(\partial_b g_{ck} + \partial_\gamma g_{bk} - \partial_k g_{bc}\right)$$
$$+ \; S_{bc}{}^{a} - S_{c}{}^{a}{}_{b} + S^{a}{}_{bc}$$
$$+ \; \frac{1}{2}\left(Q_{bc}{}^{a} + Q_{c}{}^{a}{}_{b} - Q^{a}{}_{bc}\right)$$

$$S^{a}{}_{bc} := \Gamma_{[bc]}{}^{a}$$

Torsion

$$Q_{abc} := -\nabla_a g_{bc}$$

Nonmetricity

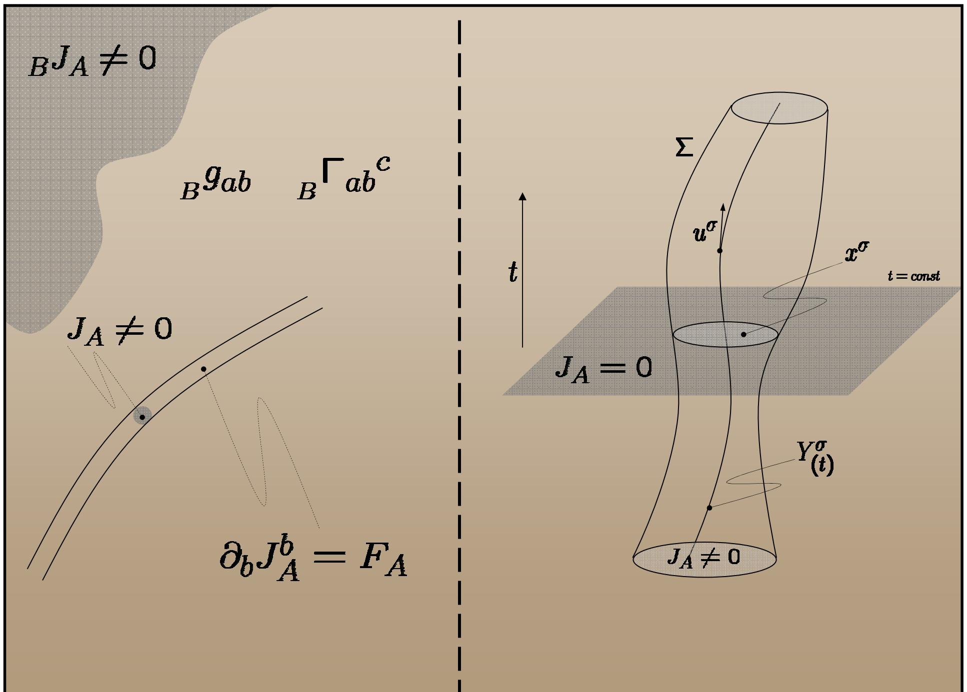
Conservation laws

$$\overset{\{\}}{\nabla}_j \left(T_i^{\ j} - N_{ikl}\,\Delta^{klj}\right) = \left(\overset{\{\}}{R}_{ijkl} - \overset{\{\}}{\nabla}_i N_{jkl}\right)\Delta^{klj}$$

Energy-momentum conservation

$$\overset{\{\}}{\nabla}_j \Delta^{klj} - N_{ij}{}^{k}\Delta^{jli} + N^{jli}\Delta^{k}{}_{ij} + T^{lk} - t^{kl} = 0$$

Hypermomentum conservation



Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

$$\frac{d}{dt}\int\left(\prod_{j=1}^{n}\delta x^{b_j}\right)J_A{}^0=\sum_{i=1}^{n}\rho^{b_i}{}_a\int\left(\prod_{j=1,j\neq i}^{n}\delta x^{b_j}\right)J_A{}^a+\int\left(\prod_{j=1}^{n}\delta x^{b_j}\right)J_A{}^a{}_{,a}$$

$$\begin{aligned}\delta x^a &= x^a - Y^a\\ \rho^b{}_a &= \delta x^b{}_{,a} = \delta^b_a - v^b\delta^0_a = \delta^b_a - \delta^b_0\delta^0_a = \delta^b_\alpha\delta^\alpha_a\end{aligned}$$

$$\frac{d}{dt}\int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\widetilde{T}^{i0}=\sum_{\beta=1}^{n}\left[\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\widetilde{T}^{ib_\beta}-v^{b_\beta}\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\widetilde{T}^{i0}\right]$$
$$+\int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\left(R^i{}_{jkl}\widetilde{\Delta}^{klj}-N^i{}_{kl}\widetilde{T}^{lk}-\overset{\{\}}{\Gamma}{}_{kj}{}^i\widetilde{T}^{(kj)}+N^i{}_{kl}\widetilde{t}^{kl}\right)$$

$$\frac{d}{dt}\int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\widetilde{\Delta}^{kl0}=\sum_{\beta=1}^{n}\left[\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\widetilde{\Delta}^{klb_\beta}-v^{b_\beta}\int\left(\prod_{\alpha=1,\alpha\neq\beta}^{n}\delta x^{b_\alpha}\right)\widetilde{\Delta}^{kl0}\right]$$
$$+\int\left(\prod_{\alpha=1}^{n}\delta x^{b_\alpha}\right)\left(N_{mj}{}^k\widetilde{\Delta}^{jlm}-\overset{\{\}}{\Gamma}{}_{mj}{}^k\widetilde{\Delta}^{mlj}-\overset{\{\}}{\Gamma}{}_{mj}{}^l\widetilde{\Delta}^{k(mj)}-N^{jlm}\widetilde{\Delta}^k{}_{mj}-\widetilde{T}^{lk}+\widetilde{t}^{kl}\right)$$

Integrated conservation laws

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

$$
\begin{aligned}
\overline{\Delta}^{b_1\cdots b_n ijk} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{\Delta}^{ijk} \\
\overline{T}^{b_1\cdots b_n ij} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{T}^{ij} \\
\overline{t}^{b_1\cdots b_n ij} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{t}^{ij}
\end{aligned}
$$

$$
\frac{d}{dt}\overline{T}^{b_1\cdots b_n i0} = \sum_{\beta=1}^{n}\left(\overline{T}^{b_1\cdots \breve{b}_\beta \cdots b_n i b_\beta} - v^{b_\beta}\overline{T}^{b_1\cdots \breve{b}_\beta \cdots b_n i0}\right)
+ \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right)\left(R^{i}{}_{jkl}\widetilde{\Delta}^{klj} - N^{i}{}_{kl}\widetilde{T}^{lk} - \overset{\{\}}{\Gamma}{}_{kj}{}^{i}\widetilde{T}^{(kj)} + N^{i}{}_{kl}\widetilde{t}^{kl}\right)
$$

$$
\frac{d}{dt}\overline{\Delta}^{b_1\cdots b_n kl0} = \sum_{\beta=1}^{n}\left(\overline{\Delta}^{b_1\cdots \breve{b}_\beta \cdots b_n kl b_\beta} - v^{b_\beta}\overline{\Delta}^{b_1\cdots \breve{b}_\beta \cdots b_n kl0}\right)
+ \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right)\left(N_{mj}{}^{k}\widetilde{\Delta}^{jlm} - \overset{\{\}}{\Gamma}{}_{mj}{}^{k}\widetilde{\Delta}^{mlj} - \overset{\{\}}{\Gamma}{}_{mj}{}^{l}\widetilde{\Delta}^{k(mj)} - N^{jlm}\widetilde{\Delta}^{k}{}_{mj} - \widetilde{T}^{lk} + \widetilde{t}^{kl}\right)
$$

General form of the integrated conservation laws in MAG

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

$$
\begin{aligned}
\underline{\Delta}^{b_1\cdots b_n i}{}_{j}{}^{k} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{\Delta}^{i}{}_{j}{}^{k}, \\
\underline{T}^{b_1\cdots b_n}{}_{i}{}^{j} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{T}_{i}{}^{j}, \\
\underline{t}^{b_1\cdots b_n i}{}_{j} &:= \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right) \widetilde{t}^{i}{}_{j}.
\end{aligned}
$$

$$
\frac{d}{dt}\underline{T}^{b_1\cdots b_n}{}_{i}{}^{0} = \sum_{\beta=1}^{n}\left(\underline{T}^{b_1\cdots \breve{b}_\beta \cdots b_n}{}_{i}{}^{b_\beta} - v^{b_\beta}\,\underline{T}^{b_1\cdots \breve{b}_\beta \cdots b_n}{}_{i}{}^{0}\right)
+ \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right)\left(R_{ijk}{}^{l}\widetilde{\Delta}^{k}{}_{l}{}^{j} + \Gamma_{ij}{}^{k}\widetilde{T}_{k}{}^{j} + N_{ij}{}^{k}\widetilde{t}^{j}{}_{k}\right)
$$

$$
\frac{d}{dt}\underline{\Delta}^{b_1\cdots b_n k}{}_{l}{}^{0} = \sum_{\beta=1}^{n}\left(\underline{\Delta}^{b_1\cdots \breve{b}_\beta \cdots b_n k}{}_{l}{}^{b_\beta} - v^{b_\beta}\,\underline{\Delta}^{b_1\cdots \breve{b}_\beta \cdots b_n k}{}_{l}{}^{0}\right)
+ \int \left(\prod_{\alpha=1}^{n} \delta x^{b_\alpha}\right)\left(\Gamma_{jl}{}^{m}\widetilde{\Delta}^{k}{}_{m}{}^{j} - \Gamma_{mj}{}^{k}\widetilde{\Delta}^{j}{}_{l}{}^{m} - \widetilde{T}_{l}{}^{k} + \widetilde{t}^{k}{}_{l}\right)(1)
$$

Alternative form of the integrated conservation laws in MAG

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

$$\overline{\Delta}^{ijk}, \overline{T}^{ij}, \overline{T}^{ijk}, \overline{t}^{ij}, \overline{t}^{ijk} \neq 0$$

$$
\begin{aligned}
R^i{}_{jkl}\big|_x &= R^i{}_{jkl}\big|_Y + \delta x^a\, R^i{}_{jkl,a}\big|_Y + \ldots \\
\overset{\{\}}{\Gamma}_{ij}{}^k\Big|_x &= \overset{\{\}}{\Gamma}_{ij}{}^k\Big|_Y + \delta x^a\, \overset{\{\}}{\Gamma}_{ij}{}^k{}_{,a}\Big|_Y + \ldots \\
N_{ij}{}^k\big|_x &= N_{ij}{}^k\big|_Y + \delta x^a\, N_{ij}{}^k{}_{,a}\big|_Y + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\overline{T}^{i0} &= R^i{}_{jkl}\overline{\Delta}^{klj} - N^i{}_{kl}\overline{T}^{lk} - N^i{}_{kl,a}\overline{T}^{alk} - \overset{\{\}}{\Gamma}_{kj}{}^i\overline{T}^{(kj)} - \overset{\{\}}{\Gamma}_{kj}{}^i{}_{,a}\overline{T}^{a(kj)} \\
&\quad + N^i{}_{kl}\overline{t}^{kl} + N^i{}_{kl,a}\overline{t}^{akl} \\
\frac{d}{dt}\overline{T}^{ai0} &= \overline{T}^{ia} - v^a\overline{T}^{i0} - N^i{}_{kl}\overline{T}^{alk} - \overset{\{\}}{\Gamma}_{kj}{}^i\overline{T}^{a(kj)} + N^i{}_{kl}\overline{t}^{akl} \\
0 &= \overline{T}^{bia} + \overline{T}^{aib} - v^a\overline{T}^{bi0} - v^b\overline{T}^{ai0} \\
\frac{d}{dt}\overline{\Delta}^{kl0} &= N_{mj}{}^k\overline{\Delta}^{jlm} - \overset{\{\}}{\Gamma}_{mj}{}^k\overline{\Delta}^{mlj} - \overset{\{\}}{\Gamma}_{mj}{}^l\overline{\Delta}^{k(mj)} - N^{jlm}\overline{\Delta}^k{}_{mj} - \overline{T}^{lk} + \overline{t}^{kl} \\
0 &= \overline{\Delta}^{kla} - v^a\overline{\Delta}^{kl0} - \overline{T}^{alk} + \overline{t}^{akl}
\end{aligned}
$$

Pole-dipole propagation equations

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

$$\underline{\Delta}^{ijk}, \underline{T}^{ij}, \underline{T}^{ijk}, \underline{t}^{ij}, \underline{t}^{ijk} \neq 0$$

$$
\begin{aligned}
R_{ijk}{}^l\big|_x &= R_{ijk}{}^l\big|_Y + \delta x^a\, R_{ijk}{}^l{}_{,a}\big|_Y + \ldots \\
\Gamma_{ij}{}^k\big|_x &= \Gamma_{ij}{}^k\big|_Y + \delta x^a\, \Gamma_{ij}{}^k{}_{,a}\big|_Y + \ldots \\
N_{ij}{}^k\big|_x &= N_{ij}{}^k\big|_Y + \delta x^a\, N_{ij}{}^k{}_{,a}\big|_Y + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\underline{T}_i{}^0 &= R_{ijk}{}^l\underline{\Delta}^k{}_l{}^j + \Gamma_{ij}{}^k\underline{T}_k{}^j + \Gamma_{ij}{}^k{}_{,a}\underline{T}^a{}_k{}^j + N_{ij}{}^k\underline{t}^j{}_k + N_{ij}{}^k{}_{,a}\underline{t}^{aj}{}_k \\
\frac{d}{dt}\underline{T}^a{}_i{}^0 &= \underline{T}_i{}^a - v^a\underline{T}_i{}^0 + \Gamma_{ij}{}^k\underline{T}^a{}_k{}^j + N_{ij}{}^k\underline{t}^{aj}{}_k \\
0 &= \underline{T}^b{}_i{}^a + \underline{T}^a{}_i{}^b - v^a\underline{T}^b{}_i{}^0 - v^b\underline{T}^a{}_i{}^0 \\
\frac{d}{dt}\underline{\Delta}^k{}_l{}^0 &= \Gamma_{jl}{}^m\underline{\Delta}^k{}_m{}^j - \Gamma_{mj}{}^k\underline{\Delta}^j{}_l{}^m - \underline{T}_l{}^k + \underline{t}^k{}_l \\
0 &= \underline{\Delta}^k{}_l{}^a - v^a\underline{\Delta}^k{}_l{}^0 - \underline{T}^a{}_l{}^k + \underline{t}^{ak}{}_l
\end{aligned}
$$

Alternative form of the pole-dipole propagation equations

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Rewritten propagation equations

$$
\begin{aligned}
\underline{P}_i &:= \underline{T}_i{}^0 \\
\underline{L}^k{}_l &:= \underline{T}^k{}_l{}^0 \\
\underline{Y}^k{}_l &:= \underline{\Delta}^k{}_l{}^0 \\
\mathcal{P}_i &:= \underline{T}_i{}^0 - N_{ik}{}^l \underline{Y}^k{}_l - \overset{\{\}}{\Gamma}_{ik}{}^l \underline{L}^k{}_l \\
\overset{(c)}{\underline{\Delta}}{}^k{}_l{}^m &:= \underline{\Delta}^k{}_l{}^m - v^m \underline{\Delta}^k{}_l{}^0 \\
\overset{(c)}{\underline{T}}{}^k{}_l{}^m &:= \underline{T}^k{}_l{}^m - v^m \underline{T}^k{}_l{}^0 \\
\nabla_v \underline{Y}^i{}_k &:= d/dt\, \underline{Y}^i{}_k + v^m \Gamma_{mj}{}^i \underline{Y}^j{}_k - v^m \Gamma_{mk}{}^j \underline{Y}^i{}_j
\end{aligned}
$$

Rewritten propagation equations

$$
\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= \left(\overset{\{\}}{R}_{ijk}{}^l - \overset{\{\}}{\nabla}_i N_{jk}{}^l\right) \underline{\Delta}^k{}_l{}^j + \overset{\{\}}{R}_{ijk}{}^l \overset{(c)}{\underline{T}}{}^k{}_l{}^j \\
\underline{T}_k{}^i &= v^i \underline{P}_k + \frac{d}{dt} \underline{L}^i{}_k - \overset{\{\}}{\Gamma}_{kj}{}^l \underline{T}^i{}_l{}^j + N_{kj}{}^l \overset{(c)}{\underline{\Delta}}{}^j{}_l{}^i \\
\overset{(c)}{\underline{T}}{}^{(a}{}_i{}^{b)} &= 0 \\
\nabla_v \underline{Y}^i{}_k &= -\underline{T}_k{}^i + \underline{t}^i{}_k - \Gamma_{jl}{}^i \overset{(c)}{\underline{\Delta}}{}^l{}_k{}^j + \Gamma_{jk}{}^l \overset{(c)}{\underline{\Delta}}{}^i{}_l{}^j \\
\overset{(c)}{\underline{\Delta}}{}^k{}_l{}^a &= \underline{T}^a{}_l{}^k - \underline{t}^{ak}{}_l
\end{aligned}
$$

Detecting post-Riemannian structures

$$
\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= \left(\overset{\{\}}{R}_{ijk}{}^{l} - \overset{\{\}}{\nabla}_i N_{jk}{}^{l}\right) \underline{\Delta}^{k}{}_{l}{}^{j} + \overset{\{\}}{R}_{ijk}{}^{l} \overset{(c)}{\underline{T}}{}^{k}{}_{l}{}^{j} \\
\underline{T}_k{}^{i} &= v^i \underline{P}_k + \frac{d}{dt} \underline{L}^{i}{}_{k} - \overset{\{\}}{\Gamma}_{kj}{}^{l} \underline{T}^{i}{}_{l}{}^{j} + N_{kj}{}^{l} \overset{(c)}{\underline{\Delta}}{}^{j}{}_{l}{}^{i} \\
\overset{(c)}{\underline{T}}{}^{(a}{}_{i}{}^{b)} &= 0 \\
\nabla_v \underline{Y}^{i}{}_{k} &= -\underline{T}_k{}^{i} + \underline{t}^{i}{}_{k} - \Gamma_{jl}{}^{i} \overset{(c)}{\underline{\Delta}}{}^{l}{}_{k}{}^{j} + \Gamma_{jk}{}^{l} \overset{(c)}{\underline{\Delta}}{}^{i}{}_{l}{}^{j} \\
\overset{(c)}{\underline{\Delta}}{}^{k}{}_{l}{}^{a} &= \underline{T}^{a}{}_{l}{}^{k} - \underline{t}^{ak}{}_{l}
\end{aligned}
$$

$$
\mathcal{P}_i := \underline{T}_i{}^{0} - N_{ik}{}^{l} \underline{Y}^{k}{}_{l} - \overset{\{\}}{\Gamma}_{ik}{}^{l} \underline{L}^{k}{}_{l}
$$

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Single-poles <u>without</u> hypermomentum

$$
\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= 0 \\
\underline{T}_k{}^{i} &= v^i \underline{P}_k \\
\underline{T}_k{}^{i} &= \underline{t}^{i}{}_{k} \\
\mathcal{P}_i &:= \underline{T}_i{}^{0}
\end{aligned}
$$

→ Geodesic equation based on $\overset{\{\}}{\Gamma}_{kj}{}^{l}$

In MAG <u>single-pole</u> test particles <u>without microstructure</u> move exactly in the same way as in GR!

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Weyl-Cartan pole-dipole case

$$Q_{\alpha\beta} = g_{\alpha\beta} Q_i dx^i \quad \rightarrow \quad N_{i\alpha}{}^{\beta} = -\frac{1}{2}\delta_{\alpha}^{\beta} Q_i + K_{i\alpha}{}^{\beta}$$

$$\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= \left(\overset{\{\}}{R}{}_{ijk}{}^{l} - \overset{\{\}}{\nabla}_i K_{jk}{}^{l}\right) \underline{\tau}^{k}{}_{l}{}^{j} + \overset{\{\}}{R}{}_{ijk}{}^{l} \overset{(c)}{\underline{T}}{}^{k}{}_{l}{}^{j} + \frac{1}{2}(\overset{\{\}}{\nabla}_i Q_j)\, \underline{Z}^j \\
\underline{T}_k{}^{i} &= v^i \underline{P}_k + \frac{d}{dt}\underline{L}^{i}{}_{k} - \overset{\{\}}{\Gamma}{}_{kj}{}^{l} \underline{T}^{i}{}_{l}{}^{j} + K_{kj}{}^{l} \overset{(c)}{\underline{\tau}}{}^{j}{}_{l}{}^{i} - \frac{1}{2} Q_k \overset{(c)}{\underline{Z}}{}^{i} \\
\overset{(c)}{\underline{T}}{}^{(a}{}_{i}{}^{b)} &= 0, \\
\overset{\{\}}{\nabla}_v \underline{Y}^{i}{}_{k} &= -\underline{T}_k{}^{i} + \underline{t}^{i}{}_{k} - \overset{\{\}}{\Gamma}{}_{jl}{}^{i} \overset{(c)}{\underline{\Delta}}{}^{l}{}_{k}{}^{j} + \overset{\{\}}{\Gamma}{}_{jk}{}^{l} \overset{(c)}{\underline{\Delta}}{}^{i}{}_{l}{}^{j} + K_{jl}{}^{i} \underline{\Delta}^{l}{}_{k}{}^{j} - K_{jk}{}^{l} \underline{\Delta}^{i}{}_{l}{}^{j} \\
\overset{(c)}{\underline{\Delta}}{}^{k}{}_{l}{}^{a} &= \underline{T}^{a}{}_{l}{}^{k} - \underline{t}^{ak}{}_{l}
\end{aligned}$$

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Weyl-Cartan single-pole case

$$Q_{\alpha\beta} = g_{\alpha\beta} Q_i dx^i \quad \rightarrow \quad N_{i\alpha}{}^{\beta} = -\frac{1}{2}\delta_{\alpha}^{\beta} Q_i + K_{i\alpha}{}^{\beta}$$

$$\begin{aligned}
\overset{\{\}}{\nabla}_v \underline{P}_i + K_{ij}{}^{k} v^j \underline{P}_k &= R_{ijk}{}^{l} v^j \underline{\tau}^{k}{}_{l} + \frac{1}{2} f_{ij} v^j \underline{Z} - \frac{1}{2} Q_i \frac{d\underline{Z}}{dt} \\
\underline{T}_k{}^{i} &= v^i \underline{P}_k \\
\nabla_v \underline{Y}^{i}{}_{k} &= -\underline{T}_k{}^{i} + \underline{t}^{i}{}_{k} \\
\overset{(c)}{\underline{\Delta}}{}^{k}{}_{l}{}^{a} &= 0
\end{aligned}$$

$$f_{ij} := \partial_i Q_j - \partial_j Q_i$$

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Riemann-Cartan pole-dipole case

$$Q_i = 0 \quad \rightarrow \quad N_{ijk} = K_{ijk} = \frac{1}{2}\left(S_{jki} + S_{ikj} + S_{jik}\right)$$

$$\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= \left(\overset{\{\}}{R}{}_{ijk}{}^l - \overset{\{\}}{\nabla}_i K_{jk}{}^l\right) \underline{\tau}^k{}_l{}^j + \overset{\{\}}{R}{}_{ijk}{}^l \,\overset{(c)}{\underline{T}}{}^k{}_l{}^j \\
\underline{T}_k{}^i &= v^i \underline{P}_k + \frac{d}{dt} \underline{L}^i{}_k - \overset{\{\}}{\Gamma}{}_{kj}{}^l \underline{T}^i{}_l{}^j + K_{kj}{}^l \,\overset{(c)}{\underline{\tau}}{}^j{}_l{}^i \\
\overset{(c)}{\underline{T}}{}^{(a}{}_i{}^{b)} &= 0, \\
\overset{\{\}}{\nabla}_v \underline{Y}^i{}_k &= -\underline{T}_k{}^i + \underline{t}^i{}_k - \overset{\{\}}{\Gamma}{}_{jl}{}^i \,\overset{(c)}{\underline{\Delta}}{}^l{}_k{}^j + \overset{\{\}}{\Gamma}{}_{jk}{}^l \,\overset{(c)}{\underline{\Delta}}{}^i{}_l{}^j + K_{jl}{}^i \underline{\Delta}^l{}_k{}^j - K_{jk}{}^l \underline{\Delta}^i{}_l{}^j \\
\overset{(c)}{\underline{\Delta}}{}^k{}_l{}^a &= \underline{T}^a{}_l{}^k - \underline{t}^{ak}{}_l
\end{aligned}$$

Riemann-Cartan single-pole case

$$Q_i = 0 \quad \rightarrow \quad N_{ijk} = K_{ijk} = \frac{1}{2}\left(S_{jki} + S_{ikj} + S_{jik}\right)$$

$$\begin{aligned}
\dot{\underline{P}}_i &= S_{ij}{}^k u^j \underline{P}_k + R_{ijk}{}^l u^j \underline{\tau}^k{}_l \\
u^0 \underline{T}_k{}^i &= u^i \underline{P}_k \\
\dot{\underline{\tau}}_{ij} &= u_{[i} \underline{P}_{j]} \\
\dot{\underline{Y}}_{(ij)} &= u^0 \left(\underline{t}_{(ij)} - \underline{T}_{(ij)}\right)
\end{aligned}$$

Dirk Puetzfeld (U Oslo), “*The motion of test bodies with microstructure in gauge gravity models*”

Riemannian pole-dipole case

$$S_{ij}{}^k = 0, \quad Q_{ijk} = 0 \quad \rightarrow \quad N_{ijk} = 0$$

$$\begin{aligned}
\overset{\{\}}{\nabla}_v \mathcal{P}_i &= \overset{\{\}}{R}{}_{ijk}{}^l (\underline{\tau}^k{}_l{}^j + \underline{\overset{(c)}{T}}{}^k{}_l{}^j) \\
\underline{T}_k{}^i &= v^i \underline{P}_k + \frac{d}{dt}\underline{L}^i{}_k - \overset{\{\}}{\Gamma}{}_{kj}{}^l \underline{T}^i{}_l{}^j \\
\underline{\overset{(c)}{T}}{}^{(a}{}_i{}^{b)} &= 0, \\
\overset{\{\}}{\nabla}_v \underline{Y}^i{}_k &= -\underline{T}_k{}^i + \underline{t}^i{}_k - \overset{\{\}}{\Gamma}{}_{jl}{}^i \underline{\overset{(c)}{\Delta}}{}^l{}_k{}^j + \overset{\{\}}{\Gamma}{}_{jk}{}^l \underline{\overset{(c)}{\Delta}}{}^i{}_l{}^j \\
\underline{\overset{(c)}{\Delta}}{}^k{}_l{}^a &= \underline{T}^a{}_l{}^k - \underline{t}^{ak}{}_l
\end{aligned}$$

Mathisson-Papapetrou equations

$$S_{ij}{}^k = 0, \quad Q_{ijk} = 0, \quad \Delta^{ijk} = 0$$

$$\begin{aligned}
\frac{d}{dt}\overline{T}^{i0} &= -\overset{\{\}}{\Gamma}{}_{kj}{}^i \overline{T}^{(kj)} - \overset{\{\}}{\Gamma}{}_{kj}{}^i{}_{,a} \overline{T}^{a(kj)} \\
\frac{d}{dt}\overline{T}^{ai0} &= \overline{T}^{ia} - v^a \overline{T}^{i0} - \overset{\{\}}{\Gamma}{}_{kj}{}^i \overline{T}^{a(kj)} \\
v^a \overline{T}^{bi0} + v^b \overline{T}^{ai0} &= \overline{T}^{bia} + \overline{T}^{aib} \\
\overline{T}^{lk} &= \overline{t}^{kl} \\
\overline{T}^{alk} &= \overline{t}^{akl}
\end{aligned}$$

Dirk Puetzfeld (U Oslo), "*The motion of test bodies with microstructure in gauge gravity models*"

Conclusions

- Very general framework, contains nearly all known theories as special cases
- Only test particles with microstructure allow for a detection of non-Riemannian spacetime features

Open problems

- Invariant definition of moments
- Supplementary conditions
- Higher-orders
- Relation to other approximation schemes
- Numerical implementation