Higher–order binding corrections to the Lamb shift *

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Abstract

In this work a new analytical method for calculating the one–loop self–energy correction to the Lamb shift is presented in detail. The technique relies on division into the low and the high energy parts. The low energy part is calculated using the multipole expansion and the high energy part is calculated by expanding the Dirac–Coulomb propagator in powers of the Coulomb field. The obtained results are in agreement with those previously known, but are more accurate. A new theoretical value of the Lamb shift is also given.

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Part I
Motivation and results

1. Introduction

The Lamb shift is one of the oldest checks of quantum electrodynamics and is still an area of active research and increasing accuracy. The disagreement of hydrogen energy levels, obtained from the Dirac equation and from the experiment, created the impulse for the construction of the modern quantum electrodynamics. In the framework of this theory, one can perturbatively determine the radiative corrections to any physical quantity. Such corrections are well defined, finite in each order of the coupling constant, provide they are expressed in terms of the physical mass and charge of an electron. In the case of the Dirac hydrogen atom, the states $2S_{1/2}$ and $2P_{1/2}$ have the same energy. Quantum electrodynamics removes the degeneracy and splits these levels. The magnitude of this splitting was first calculated by H. A. Bethe [1] and agreed with the first experiment of Lamb [2]. This result was recognized as a success of quantum electrodynamics and confirmed the correctness of the perturbation theory.

At present, the goal is to check to what accuracy quantum electrodynamics describes reality, or in other words, at what accuracy its results are in agreement with experiment (an interesting overview is presented in [3]). It is worth reflecting at this point here, what will happen in the case of finding such disagreement. Will it be an impulse to create a new theory, and should this new theory, be (in the general meaning) the quantum theory? There is not much freedom to build a new renormalizable quantum field theory, and there is still an open problem of the relation between the measured object and the measuring device, usually called the measurement problem. I think, a better understanding of this problem, together with the discovery of a disagreement between theoretical and experimental results will be the corner stone for the construction of a new theory.

It is an interesting feature, that all current high–precision tests are related to the energy or to some other quantity that is connected with the energy. From the theoretical point of view, there are not too many gauge independent quantities, for which the radiative correction can be calculated. Matrix elements of S–operator are infrared divergent [4]. A finite quantity is the inclusive cross–section, but it depends on an additional parameter which is interpreted as a resolution of the measuring device. The other quantity, the electron charge density (the F_1 formfactor) in a bound state is well defined. One can calculate the radiative correction to it [5], but it can not be measured directly. It would be interesting to study the radiative correction to the temporal behaviour of any physical quantity, for example in the spontaneous emission [6]. But it is difficult in this case to perform the relevant calculations without drastic (dipole, two level) approximations.

The only quantities that have been measured and calculated with appropriate accuracy are the corrections to energy levels (the Lamb shift, *hfs* splitting), the anomalous magnetic moment, and the decay rate of metastable states. Of all the quantities mentioned, only the decay rate of orthopositronium indicates a disagreement between the theory [7] and the experiment [8],[9],

$$\begin{aligned}
\lambda^1_{expt} &= 7.0516(13)\,\mu s^{-1} \\
\lambda^2_{expt} &= 7.0482(16)\,\mu s^{-1} \\
\lambda_{theo} &= m\alpha^6\,\frac{2(\pi^2-9)}{9\pi}\left[1-10.282(3)\left(\frac{\alpha}{\pi}\right)-\frac{1}{3}\alpha^2\ln\left(\frac{1}{\alpha}\right)\right]+\frac{m\alpha^8}{\pi^3}X =
\end{aligned}$$

$$= \quad 7.038\,236(10)\,\mu s^{-1} + \frac{m\alpha^8}{\pi^3} X$$

To investigate this discrepancy, one needs to calculate the correction of order α^2 i.e. X. It is related to the evaluation of both one– and two–loop diagrams, which has not yet been done. For the Lamb shift, there is an agreement of the old theoretical [10] and the experimental results [11],[12] down to several kHz,

$$\begin{aligned} L^1_{expt}(2S - 2P_{1/2}) &= 1\,057\,845(9)\,\mathrm{kHz} \\ L^2_{expt}(2S - 2P_{1/2}) &= 1\,057\,851(2)\,\mathrm{kHz} \qquad (1.1) \\ L_{theo}(2S - 2P_{1/2}) &= \begin{cases} 1\,057\,855(11)\,\mathrm{kHz} & (\text{if } r_p = 0.805(11)\text{ fm [13]}), \\ 1\,057\,873(11)\,\mathrm{kHz} & (\text{if } r_p = 0.862(12)\text{ fm [14]}), \end{cases} \end{aligned}$$

if the smaller proton radius r_p is chosen.

The main contributions to the theoretical error are:

- the experimental uncertainty in the proton radius, giving about 3.5 kHz,
- the error coming from numerical integration of one–loop self–energy for 2S state, which gives about 8 kHz.

The necessity of a higher precision evaluation of the one–loop self–energy was indicated in many places [15], [16], [17], and this problem is solved in this work. There is an additional uncertainty related to unknown binding corrections to the two–loop self–energy $\alpha^2(Z\alpha)^5$ that is expected to give several kHz for 2S state. This is an important, and not yet calculated, correction.

The principal source of the experimental error is the natural line–width of $2P_{1/2}$ state, $\nu \approx 100$MHz, and further reduction of this error seems unlikely. The only possibility to test quantum electrodynamics for the bound state with a higher precision is the measurement of the energy of states with a much smaller natural line–width. The (2S–1S) transition is of particular interest, because its line–width is extremely small, 1.3 Hz. A direct comparison of (2S–1S) and (4S–2S) transition frequencies in the following combination

$$\Delta = (E_{4S} - E_{2S}) - \frac{1}{4}(E_{2S} - E_{1S}),$$

has the advantage that the main Bohr contribution to energy $E_n = -m\alpha^2/(2n^2)$, cancels out and thus there is no need to invoke an absolute optical frequency standard. There was a significant motivation of this work, the fact, that at the same time a high precision experiment was being performed [16],[18],[19], and the Lamb shift of a beat frequency defined by

$$\Delta_L = \frac{1}{4}\, L_{1S} - \frac{5}{4}\, L_{2S} + L_{4S} \qquad (1.2)$$

has been measured

$$(\Delta_L)_{expt} = 868.613(25)\ \mathrm{MHz}, \qquad (1.3)$$

and also a new experiment with the improved about 10 times accuracy is planned out [20]. This result is in agreement with the old theoretical value 868,64(2) MHz [16] assuming the smaller proton radius. The main goal of this work was the elimination of a large theoretical error in Δ_L, coming mainly from numerical integration of the one–loop self–energy for 1S state (about 15 kHz).

2. View into one–loop self–energy calculations

There are several contributions to the Lamb shift.

- one–loop self–energy
- vacuum polarization
- pure recoil correction
- radiative recoil correction
- two–loop correction
- finite nuclear size

A comprehensive review of calculations of these contributions is given in [15] (plus a recently calculated pure recoil correction in order $(Z\alpha)^6 m^2/M_{proton}$ in [10]). The dominating part of the Lamb shift is the one–loop self–energy. An explicit form for this contribution can be simply derived from the expression for the fermion propagator. The Dirac–Coulomb propagator S_F

$$\left(E\gamma^0 - \vec{p}\cdot\vec{\gamma} + \frac{\alpha}{r} - m\right) S_F(\vec{r},\vec{r}\,',E) = \delta^3(\vec{r}-\vec{r}\,'), \tag{2.1}$$

has poles (cuts) at all bound (continuum) eigenstates of the Dirac–Coulomb Hamiltonian.

$$S_F(\vec{r},\vec{r}\,',E) = \sum_n \frac{\psi_n(\vec{r})\overline{\psi}_n(\vec{r}\,')}{E-E_n}. \tag{2.2}$$

With the radiative corrections included, the renormalized fermion propagator takes the form

$$S_R(E) = \frac{1}{\not{p} - m + \frac{\alpha}{r} - \Sigma_R(E)}\,, \tag{2.3}$$

where Σ_R is the sum of renormalized one–particle irreducible diagrams. The perturbed energy levels are given by the positions of the poles of the fermion propagator, as a function of E. Taking the expectation value in a given unperturbed state ψ (by $\overline{\psi}$ I denote the Dirac adjoined spinor $\psi^*\gamma^0$).

$$\langle\overline{\psi}|\gamma^0 S_R(E)\gamma^0|\psi\rangle = \frac{Z_2}{E-E_\psi} \text{ for } E \approx E_\psi. \tag{2.4}$$

Expanding both sides in powers of the coupling constant one obtains for the energy shift in the second order the expression

$$E^{(2)} = \langle\overline{\psi}|\Sigma_R^{(2)}(E_0)|\psi\rangle, \tag{2.5}$$

and for the energy in the fourth order

$$\begin{aligned} E^{(4)} &= \langle\overline{\psi}|\Sigma_R^{(4)}(E_0)|\psi\rangle + \langle\overline{\psi}|\Sigma_R^{(2)}(E_0)|\psi\rangle\langle\overline{\psi}|\frac{d\Sigma_R^{(2)}(E_0)}{dE_0}|\psi\rangle \\ &+ \langle\overline{\psi}|\Sigma_R^{(2)}(E_0)S_F'(E_0)\Sigma_R^{(2)}(E_0)|\psi\rangle, \end{aligned} \tag{2.6}$$

where $S_F'(E_0)$ denotes the so called reduced Green function (the ground state is removed). In this work I will be exclusively concerned with the one–loop electron self–energy $\Delta E = E^{(2)}$, whose explicit form is the following [15]

$$\Delta E = \frac{e^2}{(2\pi)^4 i}\int d^4k\, D_{\mu\nu}(k)\langle\overline{\psi}|\gamma^\mu \frac{1}{\not{p} - \not{k} + \frac{\alpha}{r}\gamma^0 - m}\gamma^\nu|\psi\rangle - \langle\delta m\rangle, \tag{2.7}$$

where $p^0 = E$. In the Feynman gauge the photon propagator $D_{\mu\nu}$ has a simple form

$$D_{\mu\nu} = \frac{g_{\mu\nu}}{k^2}, \tag{2.8}$$

and in the Coulomb gauge

$$D_{\mu\nu} = \begin{cases} -\dfrac{1}{\vec{k}^2} & \mu = \nu = 0 \\ 0 & \mu = 0, \nu = i \text{ or } \mu = i, \nu = 0 \\ -\dfrac{1}{k^2}\left(\delta_{ij} - \dfrac{k_i k_j}{\vec{k}^2}\right) & \mu = i, \nu = j \end{cases}$$

There is no standard method of the evaluation of ΔE. The main difficulty is related to the nonperturbative treatment of the binding Coulomb field. All terms in the expansion of the Coulomb propagator in powers of the Coulomb field give the same order in $Z\alpha$, namely $\alpha(Z\alpha)^4$ (the zero– and single–vertex terms after mass subtraction give the divergent $\alpha(Z\alpha)^2$ contribution, which cancels out in the sum). The lowest order Lamb shift calculation involves the evaluation of the Bethe logarithm, which comes from the integral over the nonrelativistic Coulomb Green function. In a more precise treatment, the relativistic Coulomb Green function must be used, but its closed form is not known. The explicit form, the partial wave decomposition in terms of Whittaker functions, was used in the numerical calculation. For small Z values, one can expand ΔE in powers of $Z\alpha$. This expansion involves a constant and a logarithmic factor $\ln(Z\alpha)^{-2}$

$$\Delta E = \frac{m\alpha(Z\alpha)^4}{\pi n^3} F\,, \tag{2.9}$$

$$\begin{aligned} F = \Big\{ & A_{40} + A_{41}\ln(Z\alpha)^{-2} + (Z\alpha)A_{50} \\ & + (Z\alpha)^2\Big[A_{62}\ln^2(Z\alpha)^{-2} + A_{61}\ln(Z\alpha)^{-2} + A_{60} + o(Z\alpha)\Big]\Big\}\,. \end{aligned} \tag{2.10}$$

In this work I present a general scheme for the evaluation of radiative corrections, which allows for the nonperturbative treatment of the binding field. I apply this method to the one–loop self–energy contribution to simplify and improve the calculation of all the coefficients for the states 1S and 2S. The constant A_{40} includes the Bethe logarithm [21],[22], and thus involves all powers of the binding field. The constants: $A_{41}, A_{50}, A_{61}, A_{62}$ have been directly evaluated [23],[24],[25]. For their evaluation the full Green function is not needed. This fact is related to the nonanalyticity in $(Z\alpha)^2$. It was pointed out by Layzer [24], that all these terms arise from low orders in the expansion of S_F in powers of the Coulomb potential.

$$\begin{aligned} A_{40}(nS) &= \frac{10}{9} - \frac{4}{3}\ln k_0(nS) \\ \ln k_0(1S) &= 2.9841285557655(1) \\ \ln k_0(2S) &= 2.8117698931205(2) \\ A_{41}(nS) &= \frac{4}{3} \\ A_{50}(nS) &= 4\pi\left(\frac{139}{128} - \frac{1}{2}\ln 2\right) \\ A_{61}(1S) &= \frac{28}{3}\ln 2 - \frac{21}{20} \\ A_{61}(2S) &= \frac{16}{3}\ln 2 + \frac{67}{30} \\ A_{62}(nS) &= -1 \end{aligned} \tag{2.11}$$

The constant A_{60} is the most difficult to evaluate. It was a significant further motivation for this work, that the $(Z\alpha)^2$ correction has been studied for a long time without a satisfactory result.

Despite the complexity of the Dirac–Coulomb propagator, Mohr was able to make a direct numerical evaluation of ΔE [27],[28]. Due to the slow convergence in angular momentum for small Z, Mohr performed calculation only for $Z = 10$ and greater. For Z=1 the result can be obtained by extrapolation i.e. one fits the function

$$G_{SE}(Z\alpha) = A_{60} + Z\alpha\left[A_{71}\ln(Z\alpha)^{-2} + A_{70}\right]$$

or a similar one to the subtracted Mohr data. The extrapolation does not seem to be a particularly reliable method, but it is an interesting connection of analytical and numerical methods to obtain the value of G_{SE} at Z =1. One finds [29], [15], [17] for the coefficient G_{SE}

$$G_{SE}(1S) = -31.0(1.7)\,, \quad G_{SE}(2S) = -32.1(1.6)\,. \tag{2.12}$$

This result was not consistent with the earlier calculation performed by Erickson [30]

$$A_{60}(1S) = -25.79(67)\,,$$

but consistent with a later calculation of Sapirstein [31]

$$A_{60}(1S) = -33.2(1.2)\,.$$

My result is

$$A_{60}(1S) = -30.92890(1)\ \ [32],\ \ A_{60}(2S) = -31.84047(1)\,. \tag{2.13}$$

The last digit in (2.13) represents an error in frequency units below 1Hz. At the same time, as a check of my method I have calculated analytically all other coefficients and obtained full agreement with previous results.

What problems appear in the evaluation of ΔE? In the analytical calculation, although the initial expression looks simple, many terms appear, making this method very laborious. There is a problem in organizing the calculation in order to treat these terms in a unified way. There is a freedom in choosing the gauge: in the Feynman gauge the renormalization is performed simply, but the spurious terms of order $\alpha(Z\alpha)^2$ appear, which cancel out in the final result cause a significant decrease of the accuracy of a direct numerical calculation. In the Coulomb gauge, the spurious order terms cancel out at a lower level of the calculation, but the expression is more complicated and not suited for analytical calculation. This gauge was used by Sapirstein in the numerical determination of A_{60}. There is no representation of the Dirac–Coulomb propagator that well describes, simultaneously, both scales of energy: the nonrelativistic scale, measured by the Rydberg constant, and the mass energy scale. The multipole wave decomposition is suited for the low energy scale, but in the high energy region all angular momenta contribute. In contrast, the expansion in binding potential is appropriate for the high energy scale, but in the nonrelativistic region all powers in this expansion contribute.

3. ϵ–Method

The ϵ–method covers the problems presented in previous section and almost all calculations are performed with the computer symbolic program [37], using simple transformation rules. The leading order term A_{40} has not been analytically integrated, thus one can not expect that an analytical calculation of the term A_{60} should be possible. What has been possible, though, is an analytical integration of the dominant part, and a numerical evaluation with the desired precision of the reminder. It so happens, that the small numerical part of A_{60} is similar to the Bethe logarithm form and can therefore be easily integrated out. It is quite common, to make a mistake in the analytical calculation. I

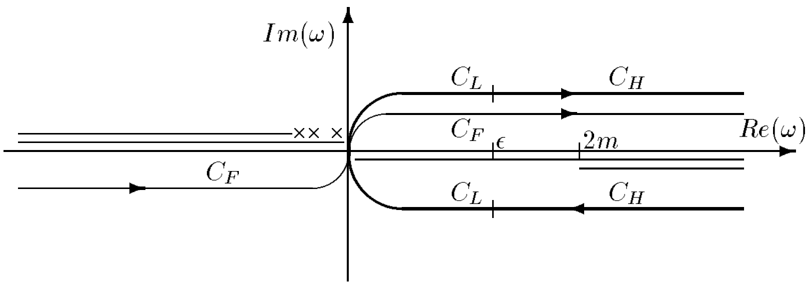

Figure 1: The contour of ω integration used in the ϵ–method

have suppressed the probability of such errors by simultaneous evaluation of the other terms, A_{61} and A_{62}, together with a special trick that is explained later in this section.

The method of evaluation is the following. The one–loop self–energy is given by the expression (2.7), which I rewrite here.

$$\Delta E = \frac{e^2}{(2\pi)^4 i} \int d^4k\, D_{\mu\nu}(k) \langle\overline{\psi}|\gamma^\mu \frac{1}{\not p - \not k + \frac{\alpha}{r}\gamma^0 - m}\gamma^\nu|\psi\rangle - \langle\delta m\rangle, \tag{3.1}$$

In contrast to the previous analytical method I first calculate the matrix element

$$\langle\overline{\psi}|\gamma^\mu(\not p - \not k + \frac{\alpha}{r}\gamma^0 - m)^{-1}\gamma^\nu|\psi\rangle,$$

that I denote by P. Next I integrate with respect to the photon momentum k. The final ω integration is a crucial point of this calculation. I treat the photon energy ω, as a complex variable. There are branch cuts coming from the photon and the Dirac propagator in the complex plane. The contour C_F of the ω integration is specified in Fig. 1; it passes through the branch point at $\omega = 0$, below all the poles and the branch cut on the left side, and above the cut on the right side. After removing the mass counter–term, one obtains a convergent but not absolutely convergent integral. Its value depends on the integration prescription. An example of the correct one is the regularization of the photon propagator,

$$\frac{1}{k^2} \to \frac{1}{k^2} - \frac{1}{(k^2 - M^2)} \tag{3.2}$$

by introducing the large mass M. After the integration, the limit $M \to \infty$ is taken. Other regularizations are also possible, but the Ward identity must be fulfilled. The dangerous ultraviolet terms in (3.1), are integrated independently, and thus for the remainder I put $M = \infty$. Changing now the contour C_F to C, I divide it into two parts: the low energy part

$$C_L = \{\omega \in \mathbf{C}; Re(\omega) < \epsilon\},$$

and the high energy part

$$C_H = \{\omega \in \mathbf{C}; Re(\omega) > \epsilon\}.$$

For the energy shift one has

$$\Delta E(\alpha) = E_L(\alpha, \epsilon) + E_H(\alpha, \epsilon).$$

After the expansion in α up to the order $(Z\alpha)^6$, one expands in ϵ up to the order ϵ^0. The terms divergent in ϵ cancel out in the sum $E_L + E_H$ because the whole term does not depend on ϵ. This fact is implied by the lemma, that for any functions:f, g, h of $x \in \mathbf{C}$ satisfying $f = g + h$, the asymptotic expansion of f is the sum of the asymptotic expansions of g and h, provided that the asymptotic expansions of g and h exist. This division of the contour is very helpful in the evaluation of ΔE. In these two parts I use different gauges. For the E_H part, the most appropriate is the Feynman gauge, and for the E_L part the Coulomb gauge. It is shown that, after the expansion in ϵ, both E_L and E_H are independent of gauge. It is sufficient to consider only E_L, because the sum $E_L + E_H$ does not depend on gauge at all. Any gauge transformation $D_{\mu\nu} \to D'_{\mu\nu}$ has the form

$$D'_{\mu\nu} = D_{\mu\nu} + \frac{1}{2k^2}(f_\mu k_\nu + f_\nu k_\mu) \tag{3.3}$$

where f is any nonsingular function of k. For example, for the transformation from the Feynman to the Coulomb gauge the function f is

$$f_\mu = \begin{cases} -\dfrac{k^0}{\vec{k}^2} & \mu = 0 \\ -\dfrac{k^i}{\vec{k}^2} & \mu = i \end{cases}$$

The dependence on f in E_L

$$\begin{aligned} E_L &= \frac{e^2}{(2\pi)^4 i}\int_{C_L} d^4k\, \frac{1}{2k^2}(f_\mu k_\nu + f_\nu k_\mu)\langle\overline{\psi}|\gamma^\mu \frac{1}{\not p - \not k + \frac{\alpha}{r}\gamma^0 - m}\gamma^\nu|\psi\rangle \\ &= \frac{e^2}{(2\pi)^4 i}\int_{C_L} d^4k\, \frac{1}{k^2}\langle\overline{\psi}|\, \not k \frac{1}{\not p - \not k + \frac{\alpha}{r}\gamma^0 - m}\, \not f|\psi\rangle \\ &= \frac{e^2}{(2\pi)^4 i}\int_{C_L} d^4k\, \frac{1}{k^2} f_\mu\langle\overline{\psi}|\gamma^\mu|\psi\rangle \end{aligned} \tag{3.4}$$

vanishes in the limit $\epsilon = 0$, and thus E_L does not depend on the gauge. For the E_H part ϵ plays a role of an infrared regulator. One can expand the electron propagator into powers of the Coulomb field and retain only the terms up to the third power. Higher vertex terms give higher order (in α) contributions, namely $(Z\alpha)^7$. All integrals with the electron momentum can be expressed in terms of the few basic integrals given in appendix B. The most important one is the following:

$$\begin{aligned} J = {} & \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}\,' \\ & \cdot\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\,\frac{1}{X^2+\vec{p}^2}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{X^2+\vec{p}\,'^2}\,\frac{1}{(\vec{p}\,'+\vec{k}')^2+\mu^2} \\ = {} & \frac{L_2(x_1)-L_2(x_2)}{b^{1/2}}, \end{aligned} \tag{3.5}$$

where L_2 is a dilogarithm [38], and x_1, x_2 and b are defined in appendix B. After the integration with respect to the electron momentum and the expansion in α (it is an important fact, that the expansion in α before the ω and k integration is allowed), one deals with an expression depending on ω and k, that is a combination of rational, logarithmic and dilogarithmic functions. Thus, the remaining integrals can be simply done.

The evaluation of the low energy part E_L is performed in the Coulomb gauge. Only the spatial components of the photon propagator contribute, because the time component is analytic inside the C_L contour. After performing the ω

integral, E_L becomes

$$E_L = e^2 \int_{|\vec{k}|<\,\epsilon} \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) \langle\overline{\psi}|\gamma^i \frac{1}{\not p - \not k + \frac{\alpha}{r}\gamma^0 - m}\gamma^j|\psi\rangle = \tag{3.6}$$

$$e^2 \int_{|\vec{k}|<\,\epsilon} \frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) \langle\overline{\psi}|\gamma^i e^{i\vec{k}\cdot\vec{r}} \frac{1}{\not p - \omega\gamma^0 + \frac{\alpha}{r}\gamma^0 - m}\gamma^j e^{-i\vec{k}\cdot\vec{r}}|\psi\rangle \tag{3.7}$$

The direct evaluation of the matrix element P is difficult to perform. The photon energy $\omega = |\vec{k}|$ is limited by ϵ, so there is no problem with ultraviolet divergences. Because, at the end, one expands in ϵ around zero, one can assume $\epsilon/m \ll 1$, but one can not assume that ϵ/m is smaller than α, because the expansion in α is performed first. After rationalizing the electron denominator in (3.6), it becomes

$$\frac{1}{(E - \omega + \frac{\alpha}{r})^2 - \vec{p}^2 - m^2 + i\,\alpha\frac{\vec{\alpha}\cdot\vec{r}}{r^2} + [2\vec{k}\cdot\vec{p} - \vec{k}^2]}\,. \tag{3.8}$$

Because $\epsilon/m \ll 1$, the inequality $\vec{k}^2 \ll E\,\omega$ holds. If $\vec{p}$ is of the order $m\alpha$, then $2\,\vec{k}\cdot\vec{p} \ll 2\,E\,\omega$. For large $\vec{p}$ comparable with the electron mass, $2\,\vec{k}\cdot\vec{p} \ll \vec{p}^2$. The conclusion is that one can expand in the last term in the bracket in (3.8). This expansion is equivalent to the expansion of the term

$$e^{i\,\vec{k}\cdot\vec{r}} = 1 + (i\,\vec{k}\cdot\vec{r}) + \frac{1}{2}(i\,\vec{k}\cdot\vec{r})^2 + \ ...$$

in (3.7). I name the first terms in this expansion dipole, quadrupole, etc. It is an important fact that after the expansion in ϵ, only the dipole and quadrupole terms contribute up to $\alpha(Z\alpha)^6$ order. It can be understood in the following way. Each term E_n of this expansion, transformed to atomic units $k \to \alpha^2 k, p \to \alpha p, r \to r/\alpha$, takes the form

$$E_n = \int_0^{\frac{\epsilon}{\alpha^2}} (\alpha^2 k)(\alpha^2 k)^n f_n(\alpha^2, k)\ \alpha^2\,dk \tag{3.9}$$

The α^2 from the denominator in (3.7) is cancelled by two α's that come from the lower part of the electron wave function. One can put $\alpha^2 = 0$ in the first argument of f, which is related to the nonrelativitic approximation. Integrating over k one obtains

$$E_n = \alpha^{(4+n)} g\Big(\frac{\epsilon}{\alpha^2}\Big)\,. \tag{3.10}$$

When $\alpha^2 k = \epsilon$, E_n has two powers of α coming only from the wave function, thus in the expansion of E_n in α one has

$$\begin{aligned} E_0 &= \alpha^4\Big(a_0\Big(\frac{\epsilon}{\alpha^2}\Big) + b_0 + c_0\ln\Big(\frac{\epsilon}{\alpha^2}\Big) + d_0\Big(\frac{\alpha^2}{\epsilon}\Big) + ...\Big)\,, \\ E_2 &= \alpha^6\Big(a_2\Big(\frac{\epsilon}{\alpha^2}\Big)^2 + b_2\Big(\frac{\epsilon}{\alpha^2}\Big) + c_2 + d_2\ln\Big(\frac{\epsilon}{\alpha^2}\Big) + ...\Big)\,. \end{aligned} \tag{3.11}$$

After one goes with ϵ to zero, the terms in α in denominator vanish. Thus, up to $(Z\alpha)^6$ order, it is sufficient to consider only the nonrelativistic dipole E_d, the relativistic (correction) dipole E_{rd}, and the nonrelativistic quadrupole E_q. For these terms the matrix element P is calculated analytically. Next almost all terms, except some in A_{60}, are integrated analytically. The remainder is represented as a one dimensional integral over the infinite sum of hypergeometric functions and is integrated numerically. The dependence on ϵ in $E_L + E_H$ should cancel out, and is an important check of the obtained result. In this calculation many terms appear. Because they have a simple structure, they can be handled using a computer symbolic program. In fact, apart from the evaluation of P, symbolic procedures were written. for the entire evaluation the symbolic procedures were written.

4. Results and conclusions

There are two results of this work. The first, is the presentation of a analytical method for bound state problems of quantum electrodynamics. This method allows one to divide this calculation into two parts: a nonrelativistic and a relativistic one. In each part the appropriate expansion is performed, what leads to a significant simplification and the improvement of the calculation. The symbolic program improves the calculation, but also eliminates some errors that can appear in such a long calculation. Also, a canceling of ϵ–singularities tests the final result. There are widely different methods applied to the high– and the low– energy parts, and both of them depend on the same parameter ϵ. Almost any change made in the high or low energy part will cause the appearance of an ϵ–singularity in the final result. Of course, another problem appears, that symbolic programs are not free from errors. The main difficulty that arises in symbolic calculation on the computer is the appearance of many terms, even for potentially simple expressions. Thus one should pay particular attention to the rearranging of large expressions.

The second result of this work is a direct evaluation of the A_{60} coefficients for 1S and 2S states, that have been known with an insufficient accuracy.

$$A_{60}(1S) = -30.92890(1)\,, \tag{4.1}$$

$$A_{60}(2S) = -31.84047(1)\,. \tag{4.2}$$

I would like to stress at this point, that a large analytical calculation can not be fully reliable, so it is worth checking all the results by an improved numerical method.

Using these values for A_{60} Drake has calculated [43], applying his method of extrapolation [17], higher order coefficients in the expansion of the self–energy into the powers of α (2.10)

$$A_{71}(1S) = 6.92(7),\;\; A_{70}(1S) = 23.11(25),\;\; G_{SE}(1S) = -30.268(13), \tag{4.3}$$

$$A_{71}(2S) = 7.04(7),\;\; A_{70}(2S) = 25.38(29),\;\; G_{SE}(2S) = -31.149(13). \tag{4.4}$$

Using these results one obtains for the new theoretical value of the *classic* Lamb shift,

$$L_{theo}(2S - 2P_{1/2}) = \begin{cases} 1057860(5)\,\mathrm{kHz} & r_p = 0.805(11)\,\mathrm{fm} \\ 1057879(5)\,\mathrm{kHz} & r_p = 0.862(12)\,\mathrm{fm} \end{cases} \tag{4.5}$$

and for the Lamb shift of the beat frequency Δ_L defined in (1.2)

$$(\Delta_L)_{theo} = \begin{cases} 868641(4)\mathrm{kHz} & r_p = 0.805(11)\,\mathrm{fm} \\ 868657(4)\mathrm{kHz} & r_p = 0.862(12)\,\mathrm{fm} \end{cases} \tag{4.6}$$

The self–energy of 2P state ($G_{SE} = -0.8(3)$) gives for the $L(2S - 2P_{1/2})$ an error of 1.5 kHz. This error can be eliminated by a direct evaluation of $A_{60}(2P)$, alike for the 1S and 2S states. The uncertainty in the proton radius gives for $L(2S - 2P_{1/2})$ error of 3.5 kHz and for Δ_L error of 3 kHz. The experimental results were presented in (1.1) and (1.3). The comparison of current theory and experiment is limited by a discrepancy in the proton radius. There are two measurements that differ by about 5 times the quoted errors. The smaller proton radius gives results which are closer to the experimental values. A further more accurate measurement of r_p is very desirable to solve the experimental discrepancy and to diminish the theoretical error. As I wrote in the Introduction there is an additional uncertainty coming from unknown binding correction to the two–loop contribution of order $\alpha^2(Z\alpha)^5$. This contribution could cause a small difference between theoretical and experimental results

$$\begin{aligned} (\Delta_L)_{theo} - (\Delta_L)_{expt} &= 28(4)(25)\ \mathrm{kHz}\,, \\ L_{theo}(2S-2P_{1/2}) - L_{expt2}(2S-2P_{1/2}) &= 9(5)(2)\ \mathrm{kHz}\,. \end{aligned}$$

The evaluation of this correction is difficult and thus interesting. Because this correction is nonanalytic in $Z\alpha$, only the zero–, single– and double–vertex diagrams conmtributes, but in fact many terms have to be considered. In the leading order the two–loop contribution has been calculated analytically in [44], using the scattering approximation. The binding corrections consist of: two–photon loop, one–photon and one–fermion, and two–fermion loop diagrams. The second type diagrams can be represented as an expectation value of the one–loop correction to the current, multipled by one–loop correction to the Coulomb potential,

$$\delta E = \langle\phi|\delta j^0\,\delta V|\phi\rangle\,.$$

In fact a problem of radiative correction to the charge density has been solved [5]. The most difficult diagram is that with two crossed photon loops. After experience with the one loop problem, I think that the best way to calculate this diagram is at first to separate the desired order terms and next perform an integration with respect to photon and electron momenta. The nonanalyticity in α could simplify this calculation. A numerical integration of this two loop diagram has not been yet performed, and is also desirable, because the comparison of the analytical and numerical results is a strong test for both methods of the calculation.

The most interesting, but also most difficult, I think, is the evaluation of the ortho–positronium decay rate. In this order a nonperturbative treatment of the Coulomb kernel is necessary, thus the separation into the low and high energy part would be helpful. There is still a barrier in the large number of terms, that contribute in this order. The partial solution, I think, is the writing of a general symbolic procedure for the evaluation of integrals with the photon and electron propagators. Evaluation of the binding correction to the Lamb shift was possible because general integration formulae could be found. In this case, a more complicated integral with several momenta has to be done. The evaluation of a general integral with three momenta, derived in [45], suggests that this is possible.

There is a large open field for the calculation of the radiative correction in the area of atomic optics, where the ϵ–method can be applied directly. There have been no systematic evaluation of the radiative corrections to the Rayleigh scattering, the Stark effect, or to the life time of 2P state in H-atom. A recent measurement by Drake *et. al.* [46] gives for the latter:

$$\tau = 0.99717(75)\cdot 10^{-10}\mathrm{s} \tag{4.7}$$

The radiative corrections are about α^{-3} smaller, so they are still below the limits of experimental observability.

Part II

The calculation

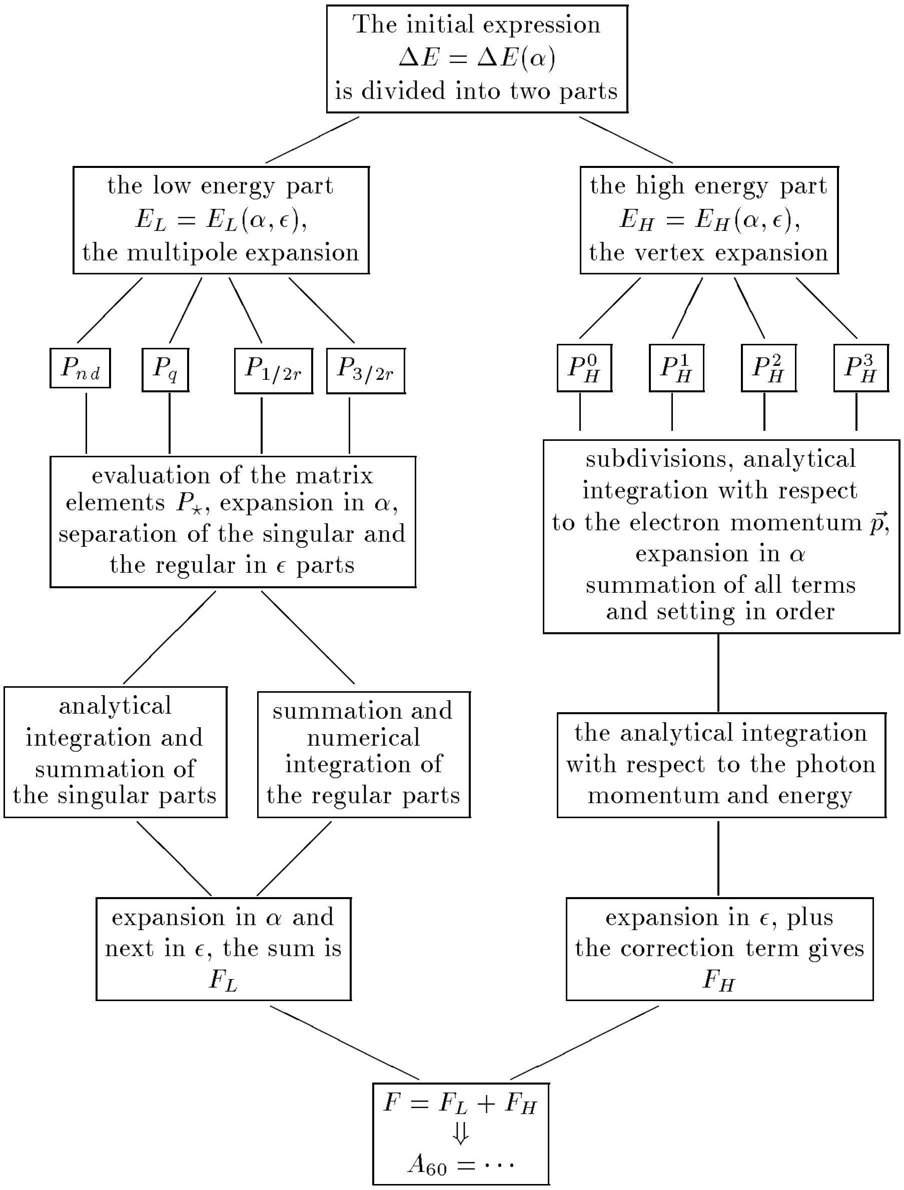

Figure 2: Evaluation scheme

5. 1S high–energy part

The energy shift ΔE is divided into two parts

$$\Delta E = E_H + E_L\ .$$

The high–energy part

$$E_H = \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\, \frac{1}{\omega^2-k^2}\,\langle\psi|\gamma^\mu \frac{1}{\not p - \not k + \frac{\alpha}{r}\gamma^0 - m}\gamma^\nu|\psi\rangle - \langle\delta m\rangle, \tag{5.1}$$

is integrated along the C_H contour. Because the denominator in the Dirac–Coulomb propagator is now limited from below by ϵ one can expand the propagator in the powers of Coulomb field. Let E_H^i denotes the i–th term in this expansion. I use in this section a three–dimensional notation (except for $\not k$), and $\vec{k}^2 = k^2$ for any vector k.

5.1 Three–vertex part P_H^3

Let us start the evaluation from E_H^3. I recall the definition of P

$$\Delta E = \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3k\, \frac{1}{\omega^2-k^2}P$$

Thus I have

$$P_H^3 = \left(\prod_{i=1}^4 \int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)\overline{\psi}(\vec{p}_1)\gamma^\mu \frac{1}{(\not p_1 - \not k - m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_2-\vec{p}_1)^2}\,\gamma^0\,\frac{1}{(\not p_2 - \not k - m)}$$
$$\cdot\frac{(-4\pi\alpha)}{(\vec{p}_3-\vec{p}_2)^2}\,\gamma^0\frac{1}{(\not p_3 - \not k - m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_4-\vec{p}_3)^2}\,\gamma^0\,\frac{1}{(\not p_4 - \not k - m)}\gamma^\mu\psi(\vec{p}_4). \tag{5.2}$$

Because P_H^3 is of $(Z\alpha)^6$ order, one can put $p_0 \equiv E = m$, and make the non-relativistic approximation for the wave functions. After performing the algebra with gamma matrices, a polynomial in p_i appears in the numerator. I divide $P_H^3 = P_H^{31} + P_H^{32}$ into two parts. The first part P_H^{31} without p_i in the numerator, has the form

$$\begin{aligned} P_H^{31} &= 64\pi\lambda^5\Big\{-16m(m-\omega)\Big((m-\omega)^2+m^2+\vec{k}^2\Big) \\ &\quad +2\Big((m-\omega)^2+m^2+\vec{k}^2\Big)^2 + 8(m^2+\vec{k}^2)(m-\omega)^2\Big\}R, \end{aligned} \tag{5.3}$$

$$\begin{aligned} R &= \left(\prod_{i=1}^4\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)\frac{1}{((\vec{p}_4+\vec{k})^2+\lambda^2)^2}\,\frac{1}{X^2+\vec{p}_4^{\,2}}\,\frac{4\pi\alpha}{(\vec{p}_4-\vec{p}_3)^2}\,\frac{1}{X^2+\vec{p}_3^{\,2}} \\ &\quad\cdot\frac{4\pi\alpha}{(\vec{p}_3-\vec{p}_2)^2}\,\frac{1}{X^2+\vec{p}_2^{\,2}}\,\frac{4\pi\alpha}{(\vec{p}_2-\vec{p}_1)^2}\,\frac{1}{X^2+\vec{p}_1^{\,2}}\,\frac{1}{((\vec{p}_1+\vec{k})^2+\lambda^2)^2}, \end{aligned} \tag{5.4}$$

where $X^2 = 2m\,\omega - \omega^2,\ \lambda = m\,\alpha$. The R integral is calculated in the appendix B, with the result

$$R = -\frac{\alpha}{4\pi}\,\frac{1}{(2m)^2}\,\frac{1}{(X^2+\vec{k}^2)^4}\ln\left(\frac{4\lambda X}{X^2+\vec{k}^2}\right) + o(\alpha^2), \tag{5.5}$$

The second part P_H^{32} has the numerator, which vanishes for $p_i = 0$. It allows one to put wave function on mass shell.

$$\phi(r) = \sqrt{\frac{\lambda^3}{\pi}}e^{-\lambda r} \to \sqrt{\frac{\lambda^3}{\pi}}. \tag{5.6}$$

In the momentum representation

$$\phi(\vec{p}) \to \sqrt{\frac{\lambda^3}{\pi}}(2\pi)^3\delta^3(\vec{p}). \tag{5.7}$$

Due to the delta function in (5.7), there remain only two integrals in p_i in P_H^{32}.

$$P_H^{32} = -\frac{\lambda^3}{\pi}(4\pi\alpha)^3\frac{1}{(X^2+k^2)^2}\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}$$
$$\cdot\frac{1}{(\vec{p}+\vec{k})^2}\,\frac{1}{X^2+\vec{p}^{\,2}}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{X^2+\vec{p}\,'^2}\,\frac{1}{(\vec{p}\,'+\vec{k})^2}$$
$$\cdot 4\left[(\vec{p}\cdot\vec{p}\,'-\vec{k}^2)(m^2-\omega\,m-\frac{\omega^2}{2}-\frac{\vec{k}^2}{2})+(\vec{p}\cdot\vec{k}+\vec{p}\,'\cdot\vec{k}+2\,\vec{k}^2)(\omega^2-m^2)\right] \tag{5.8}$$

It is convenient to group p_i's in the numerator in such a way that the appropriate factors in the denominator cancel out

$$\vec{p}\cdot\vec{p}\,' = \frac{1}{2}\Big((\vec{p}^{\,2}+X^2)+(\vec{p}\,'^2+X^2)-(\vec{p}-\vec{p}\,')^2\Big)-X^2\,, \tag{5.9}$$

$$\vec{p}\cdot\vec{k}+\vec{p}\,'\cdot\vec{k}+2\,\vec{k}^2 =$$
$$\frac{1}{2}\Big((\vec{p}+\vec{k})^2-(\vec{p}^{\,2}+X^2)\Big)+\frac{1}{2}\Big((\vec{p}\,'+\vec{k})^2-(\vec{p}\,'^2+\vec{X}^2)\Big)+(X^2+\vec{k}^2)\,. \tag{5.10}$$

The expression for P_H^{32} becomes

$$P_H^{32} = -4m^3\alpha^6\,\frac{1}{(X^2+\vec{k}^2)^2}\Big\{(J_2-J_1\,(X^2+\vec{k}^2))\,(2m^2-\omega\,m-\frac{3}{2}\omega^2-\frac{\vec{k}^2}{2})$$
$$-\frac{K_1^2}{2}\Big(m^2-\omega\,m-\frac{\omega^2}{2}-\frac{\vec{k}^2}{2}\Big)+J_3\,(\omega^2-m^2)\Big\}. \tag{5.11}$$

Because the integral for separate terms will be divergent, I introduce an infrared cutoff λ, that is present in J and K integrals. This cutoff must of course cancel out in the sum (5.11). The definitions of J_i and K_i integrals and all others together with its evaluation are given in the appendix B. I mention only that all J_i integrals can be evaluated from the one basic integral J given in (3.5), by passing with parameters to some limits and expanding in λ. I use the notation

$$J_i = J_i(X),\quad K_i = K_i(X),\quad G_i = G_i(X)\,.$$

The final expression for P_H^{32} becomes

$$P_H^{32} = -4m^3\alpha^6\frac{1}{(X^2+k^2)^2}\left\{\left[\frac{4}{X^2+k^2}\Big(1+\ln\Big(\frac{4\,\lambda\,X}{X^2+k^2}\Big)\Big)\right.\right.$$
$$\left.-\frac{2}{i\,k}\Big(\frac{1}{X+i\,k}\ln\Big(\frac{X+i\,k}{2\,\lambda}\Big)-\frac{1}{X-i\,k}\ln\Big(\frac{X-ik}{2\,\lambda}\Big)\Big)\right]$$
$$\cdot\Big(2m^2-\omega\,m-\frac{3}{2}\omega^2-\frac{k^2}{2}\Big)-\frac{1}{2k^2}\ln^2\Big(\frac{X-i\,k}{X+i\,k}\Big)\Big(m^2-\omega\,m-\frac{\omega^2}{2}-\frac{k^2}{2}\Big)$$
$$\left.+\frac{1}{i\,k\,X}\Big(L_2\Big(-\frac{X-i\,k}{X-i\,k}\Big)-L_2\Big(-\frac{X+i\,k}{X-i\,k}\Big)\Big)(\omega^2-m^2)\right\}. \tag{5.12}$$

5.2 Double–vertex part P_H^2

In the second order in the Coulomb field, one has for the matrix element P

$$P_H^2 = \left(\prod_{i=1}^{3}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)\overline{\psi}(\vec{p}_1)\gamma^\mu\frac{1}{(\not{p}_1-\not{k}-m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_2-\vec{p}_1)^2}\,\gamma^0$$
$$\cdot\frac{1}{(\not{p}_2-\not{k}-m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_3-\vec{p}_2)^2}\,\gamma^0\,\frac{1}{(\not{p}_3-\not{k}-m)}\,\gamma^\mu\psi(\vec{p}_3). \tag{5.13}$$

I separate out the leading part

$$P_H^2 = P_H^{2L} + P_H^{2R} \tag{5.14}$$

$$\begin{aligned}
P_H^{2L} = & \left(\prod_{i=1}^{3}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)\overline{\psi}(\vec{p}_1)\gamma^\mu\,\frac{1}{(E\gamma_0 - \not k - m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_2-\vec{p}_1)^2}\gamma^0 \\
& \cdot\frac{1}{(E\gamma_0 - \not k - m)}\,\frac{(-4\pi\alpha)}{(\vec{p}_3-\vec{p}_2)^2}\gamma^0\frac{1}{(E\gamma_0 - \not k - m)}\,\gamma^\mu\psi(\vec{p}_3) \\
& = -\frac{1}{(X^2+k^2)^3}\Big[\Big((4m^3 + 12m(E-\omega)^2 + 4m\,k^2 - 6m^2(E-\omega) \\
& - 2(E-\omega)^3 - 6(E-\omega)k^2\Big)\langle\psi_u|(\frac{\alpha}{r})^2|\psi_u\rangle - \Big(4m^3 + 12m(E-\omega)^2 \\
& + 4m\,k^2 + 6m^2(E-\omega) + 2(E-\omega)^3 + 6(E-\omega)k^2\Big)\langle\psi_d|(\frac{\alpha}{r})^2|\psi_d\rangle\Big]\,,
\end{aligned} \tag{5.15}$$

where ψ_u and ψ_d are the upper and the lower component of the relativistic wave function. The expectation values of $(\frac{\alpha}{r})^2$, and of other operators are given in table 1 in appendix B, but for clarity I rewrite the two that appeared in (5.15).

$$\langle\psi_u|\left(\frac{\alpha}{r}\right)^2|\psi_u\rangle = 2m^2\alpha^4(1+\frac{5}{4}\alpha^2) + O(\alpha^8)\,, \tag{5.16}$$

$$\langle\psi_d|\left(\frac{\alpha}{r}\right)^2|\psi_d\rangle = \frac{1}{2}m^2\alpha^6 + O(\alpha^8)\,. \tag{5.17}$$

The remaining part P_H^{2R} is calculated, by replacing the wave function ψ by the Pauli–Schrödinger wave function $\psi \to (\phi, \vec{\sigma}\cdot\vec{p}/(2m)\phi)$, and putting $E = m$.

$$\begin{aligned}
P_H^{2R} = & -\left(\prod_{i=1}^{3}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)(4\pi\alpha)^2\frac{\phi(p_3)}{(\vec{p}_3-\vec{p}_2)^2}\,\frac{\phi(p_1)}{(\vec{p}_1-\vec{p}_2)^2}\Bigg\{ \\
& \Big[4m^3 + 12m(m-\omega)^2 + 4m\,k^2 - 6m^2(m-\omega) - 2(m-\omega)^3 - 6(m-\omega)k^2\Big] \\
& \cdot\left(\frac{1}{X^2+(\vec{p}_1-\vec{k})^2}\,\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\,\frac{1}{X^2+(\vec{p}_3-\vec{k})^2} - \frac{1}{(X^2+k^2)^3}\right) \\
& + \Big[\vec{k}\cdot(\vec{p}_1+\vec{p}_3)\Big(3m - 4\omega - \frac{3}{m}(m-\omega)^2 - \frac{k^2}{m}\Big) - 4\vec{k}\cdot\vec{p}_2(m+\omega) \\
& + \vec{p}_1\cdot\vec{p}_3\Big(-4m + 2\omega - 2\frac{(m-\omega)^2}{m} + 2\frac{k^2}{m}\Big) \\
& + \vec{p}_2\cdot(\vec{p}_1+\vec{p}_3)\Big(m + 2\omega + \frac{(m-\omega)^2}{m} - \frac{k^2}{m}\Big) \\
& + (\vec{p}_1^{\,2}+\vec{p}_3^{\,2})\Big(m + \frac{(m-\omega)^2}{m} + \frac{k^2}{m}\Big) - \frac{(\vec{k}\cdot\vec{p}_2)}{m}(\vec{p}_1^{\,2}+\vec{p}_3^{\,2}) + 2\frac{(\vec{k}\cdot\vec{p}_2)}{m}(\vec{p}_1+\vec{p}_3)\cdot\vec{k}\Big] \\
& \cdot\frac{1}{X^2+(\vec{p}_1-\vec{k})^2}\,\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\,\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\Bigg\}\,.
\end{aligned} \tag{5.18}$$

I have omitted from this expression the terms $\vec{p}_1\,\vec{p}_3\,\vec{p}_i$, because they give higher order contributions. The direct evaluation of p_i integrals is difficult, therefore I proceed with their simplification. Let $f(\vec{p}_1,\vec{p}_2,\vec{p}_3)$ denote the terms in a braces in the above formula. Since the lowest order has been separated out,

$$f(0,0,0) = 0. \tag{5.19}$$

First, one notices that if some term in f contains $\vec{p}_1\vec{p}_3$, it gives a $(Z\alpha)^6$ contribution, and each additional power of $\vec{p}_i$ gives a higher order contribution. Therefore, expanding f

$$
\begin{aligned}
f(\vec{p}_1,\vec{p}_2,\vec{p}_3) &= f(0,\vec{p}_2,0) + f_{,3}(0,\vec{p}_2,0)\vec{p}_3 + f_3^R(0,\vec{p}_2,\vec{p}_3)\vec{p}_3\vec{p}_3 \\
&\quad + f_{,1}(0,\vec{p}_2,0)\vec{p}_1 + f_{,13}(0,\vec{p}_2,0)\vec{p}_1\vec{p}_3 + f_1^R(\vec{p}_1,\vec{p}_2,0)\vec{p}_1\vec{p}_1 \\
&= \underbrace{f(0,\vec{p}_2,\vec{p}_3)}_{I} + \underbrace{f(\vec{p}_1,\vec{p}_2,0)}_{I} + \underbrace{f_{,13}(0,0,0)\vec{p}_1\vec{p}_3}_{II} \underbrace{-f(0,\vec{p}_2,0)}_{III}\,. \qquad (5.20)
\end{aligned}
$$

one can reduce the momentum integration. Thus, P_H^{2R} is naturally divided

$$
P_H^{2R} = 2P_2^I + P_2^{II} + P_2^{III}. \qquad (5.21)
$$

For P_2^{III} I have the expression

$$
\begin{aligned}
P_2^{III} = &\left(\prod_{i=1}^{3}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)(4\pi\alpha)^2\frac{\phi(p_3)}{(\vec{p}_3-\vec{p}_2)^2}\,\frac{\phi(p_1)}{(\vec{p}_1-\vec{p}_2)^2}\,\frac{1}{(X^2+k^2)^2}\Bigg\{ \\
&\left[4m^3+12m(m-\omega)^2+4m\,k^2-6m^2(m-\omega)-2(m-\omega)^3-6(m-\omega)k^2\right] \\
&\cdot\left(\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}-\frac{1}{X^2+k^2}\right)-4\vec{k}\cdot\vec{p}_2\frac{(m+\omega)}{X^2+(\vec{p}_2-\vec{k})^2}\Bigg\}. \qquad (5.22)
\end{aligned}
$$

Using the integration formula

$$
\int d^3\vec{q}\,\frac{1}{(\vec{q}^{\,2}+\lambda^2)^2}\,\frac{1}{(\vec{q}-\vec{Q})^2+\Delta^2}=\frac{\pi^2}{\lambda}\,\frac{1}{(\Delta+\lambda)^2+\vec{Q}^2}\,, \qquad (5.23)
$$

one can perform the momentum integration and P_2^{III} becomes

$$
\begin{aligned}
P_2^{III} = &-2m^3\alpha^5\frac{1}{(X^2+k^2)^2}\Bigg[\Big(4m^3+12m(m-\omega)^2+4m\,k^2-6m^2(m-\omega) \\
&-2(m-\omega)^3-6(m-\omega)k^2\Big)\left(-K_2+\frac{1}{X^2+k^2}K_5\right)+2(m+\omega)K_6\Bigg]\,, \qquad (5.24)
\end{aligned}
$$

where all K_i integrals are given in appendix B.

Using (5.20) and (5.21) one obtains for P_2^{II}

$$
\begin{aligned}
P_2^{II} = &-\langle\phi|\vec{p}\Big(\frac{\alpha}{r}\Big)^2\vec{p}\,|\phi\rangle\Bigg\{\Big[4m^3+12m(m-\omega)^2+4m\,k^2-6m^2(m-\omega) \\
&-2(m-\omega)^3-6(m-\omega)k^2\Big]\,\frac{4}{3}\,\frac{k^2}{(X^2+k^2)^5}+\left[3m-4\omega-3\frac{(m-\omega)^2}{m}-\frac{k^2}{m}\right] \\
&\cdot\frac{4}{3}\frac{k^2}{(X^2+k^2)^4}+\left[-4m+2\omega+2\frac{(m-\omega)^2}{m}+2\frac{k^2}{m}\right]\frac{1}{(X^2+k^2)^3}\Bigg\}\,, \qquad (5.25)
\end{aligned}
$$

where

$$
\langle\phi|\vec{p}\Big(\frac{\alpha}{r}\Big)^2\vec{p}\,|\phi\rangle = 2m^4\alpha^6\,. \qquad (5.26)
$$

P_2^I is more difficult to evaluate, because many terms give contributions in the desired order.

$$
\begin{aligned}
P_2^I = &-\left(\prod_{i=1}^{3}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)(4\pi\alpha)^2\frac{\phi(p_3)}{(\vec{p}_3-\vec{p}_2)^2}\frac{\phi(p_1)}{(\vec{p}_1-\vec{p}_2)^2}\Bigg\{ \\
&\left[4m^3+12m(m-\omega)^2+4m\,k^2-6m^2(m-\omega)-2(m-\omega)^3-6(m-\omega)k^2\right]
\end{aligned}
$$

$$
\cdot\frac{1}{X^2+k^2}\left(\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}-\frac{1}{(X^2+k^2)^2}\right)
$$
$$
+\Big[\vec{k}\cdot\vec{p}_3\Big(3m-4\omega-\frac{3}{m}(m-\omega)^2-\frac{k^2}{m}\Big)+\vec{p}_2\cdot\vec{p}_3\Big(m+2\omega+\frac{(m-\omega)^2}{m}-\frac{k^2}{m}\Big)
$$
$$
-4\vec{k}\cdot\vec{p}_2(m+\omega)+\vec{p}_3^{\,2}\Big(m+\frac{(m-\omega)^2}{m}+\frac{k^2}{m}\Big)-\frac{\vec{k}\cdot\vec{p}_2}{m}\vec{p}_3^{\,2}+(\vec{k}\cdot\vec{p}_2)\,(\vec{k}\cdot\vec{p}_3)\frac{2}{m}\Big]
$$
$$
\cdot\frac{1}{X^2+k^2}\;\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\;\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\Big\}. \tag{5.27}
$$

Using the formula (5.23), the integral over p_1 is performed. The integration with respect to p_2 and p_3 is performed separately for each term using J and K integrals.

$$
P_2^I=-2m^4\alpha^6\frac{1}{X^2+k^2}\Big\{
$$
$$
L_1\Big[4m^3+12m(m-\omega)^2+4m\,k^2-6m^2(m-\omega)-2(m-\omega)^3-6(m-\omega)k^2\Big]
$$
$$
+L_2\Big(3m-4\omega-\frac{3}{m}(m-\omega)^2-\frac{k^2}{m}\Big)-L_3\;4(m+\omega)
$$
$$
+L_4\Big(m+2\omega+\frac{(m-\omega)^2}{m}-\frac{k^2}{m}\Big)+L_5\Big(m+\frac{(m-\omega)^2}{m}+\frac{k^2}{m}\Big)
$$
$$
-L_6\frac{1}{m}+L_7\frac{2}{m}\Big\}, \tag{5.28}
$$

where all L_i are defined and calculated below by removing p_i from the numerator.

$$
L_1=\frac{1}{\pi^4}\int d^3\vec{p}_2\,d^3\vec{p}_3\,\frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\left(\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\right.
$$
$$
\left.-\frac{1}{(X^2+k^2)^2}\right)\frac{1}{(\vec{p}_2-\vec{p}_3)^2}\frac{1}{(\vec{p}_3^{\,2}+\lambda^2)^2}
$$
$$
=J_5-\frac{1}{(X^2+k^2)^2}J_4=\frac{-2X}{\lambda(X^2+k^2)^3}+\frac{4(X^2-k^2)}{(X^2+k^2)^4}\left(1+\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\right) \tag{5.29}
$$

$$
L_2=\frac{1}{\pi^4}\int d^3\vec{p}_2\,d^3\vec{p}_3
$$
$$
\vec{k}\cdot\vec{p}_3\;\frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\;\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\;\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\;\frac{1}{(\vec{p}_2-\vec{p}_3)^2}\;\frac{1}{(\vec{p}_3^{\,2}+\lambda^2)^2}
$$
$$
=J_6=\frac{-4k^2}{(X^2+k^2)^3}\left(\frac{1}{2}+\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\right) \tag{5.30}
$$

$$
L_3=\frac{1}{\pi^4}\int d^3\vec{p}_2\,d^3\vec{p}_3
$$
$$
\vec{k}\cdot\vec{p}_2\;\frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\;\frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\;\frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\;\frac{1}{(\vec{p}_2-\vec{p}_3)^2}\;\frac{1}{(\vec{p}_3^{\,2}+\lambda^2)^2}
$$
$$
=\frac{1}{2}\Big(J_7-J_9+(X^2+k^2-\lambda^2)J_5\Big)
$$

$$
\begin{aligned}
&= \frac{-X}{\lambda(X^2+k^2)^2} + \frac{i}{2k\lambda(X^2+k^2)} \ln\Big(\frac{X-i\,k}{X+i\,k}\Big) \\
&- \frac{32}{9}\frac{1}{(X^2+k^2)^2} + \frac{38}{9}\frac{X^2}{(X^2+k^2)^3} - \frac{1}{(X^2+k^2)^2}\ln\Big(\frac{X^2}{X^2+k^2}\Big) \\
&+ \frac{i}{3k}\Big[\frac{1}{(X+i\,k)^3}\ln\Big(\frac{X+i\,k}{2\lambda}\Big) - \frac{1}{(X-i\,k)^3}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\Big] \\
&+ \Big[\frac{-2}{(X^2+k^2)^2} + \frac{4X^2}{(X^2+k^2)^3}\Big]\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big) \qquad (5.31)
\end{aligned}
$$

$$
\begin{aligned}
L_4 &= \frac{1}{\pi^4}\int d^3\vec{p}_2\, d^3\vec{p}_3 \\
&\vec{p}_2\cdot\vec{p}_3\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\ \frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\ \frac{1}{(\vec{p}_2-\vec{p}_3)^2}\ \frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\ \frac{1}{(\vec{p}_3^{\,2}+\lambda^2)^2} \\
&= \frac{1}{2}(J_7+J_1-K_1K_2) - \lambda^2 J_5 \\
&= \frac{-1}{(X^2+k^2)^2}\Big(1+\ln\Big(\frac{4X^2}{X^2+k^2}\Big) + 2\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big) - \frac{i\,X}{k}\ln\Big(\frac{X-i\,k}{X+i\,k}\Big)\Big) \\
&\qquad (5.32)
\end{aligned}
$$

$$
\begin{aligned}
L_5 &= \frac{1}{\pi^4}\int d^3\vec{p}_2\, d^3\vec{p}_3 \\
&\vec{p}_3^{\,2}\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\ \frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\ \frac{1}{(\vec{p}_2-\vec{p}_3)^2}\ \frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)^2} \\
&= J_1 - \lambda^2 J_5 = \frac{-1}{(X^2+k^2)^2}\Big(1+4\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\Big) \qquad (5.33)
\end{aligned}
$$

$$
\begin{aligned}
L_6 &= \frac{1}{\pi^4}\int d^3\vec{p}_2\, d^3\vec{p}_3 \\
&(\vec{k}\cdot\vec{p}_2)\,\vec{p}_3^{\,2}\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\ \frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\ \frac{1}{(\vec{p}_2-\vec{p}_3)^2}\ \frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)^2} \\
&= \frac{1}{2}\Big(J_3 - J_2 + \lambda^2 J_9 + (X^2+k^2-\lambda^2)(J_1-\lambda^2 J_5)\Big) \\
&= \frac{1}{2i\,kX}\Big(L_2\Big(-\frac{X-i\,k}{X+i\,k}\Big) - L_2\Big(\frac{X+i\,k}{X-i\,k}\Big)\Big) - \frac{1}{X^2+k^2}\Big(1+\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\Big) \\
&+ \frac{1}{i\,k}\Big(\frac{1}{X+i\,k}\ln\Big(\frac{X+i\,k}{2\lambda}\Big) - \frac{1}{X-i\,k}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\Big) \qquad (5.34)
\end{aligned}
$$

$$
\begin{aligned}
L_7 &= \frac{1}{\pi^4}\int d^3\vec{p}_2\, d^3\vec{p}_3\ (\vec{k}\cdot\vec{p}_2)\ (\vec{k}\cdot\vec{p}_3) \\
&\cdot\frac{1}{(\vec{p}_2^{\,2}+\lambda^2)}\ \frac{1}{X^2+(\vec{p}_2-\vec{k})^2}\ \frac{1}{(\vec{p}_2-\vec{p}_3)^2}\ \frac{1}{X^2+(\vec{p}_3-\vec{k})^2}\ \frac{1}{(\vec{p}_2^{\,2}+\lambda^2)^2} \\
&= \frac{1}{4}\Big(J_3 - J_{10} - J_2 + \frac{1}{\lambda}K_2 + (X^2+k^2-\lambda^2)(J_7-J_9-J_8) \\
&+ (X^2+k^2)J_1 + (X^2+k^2-\lambda^2)^2 J_5\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-16}{9}\frac{1}{X^2+k^2} + \frac{10}{9}\frac{X^2}{(X^2+k^2)^2} - \frac{1}{2(X^2+k^2)}\ln\Big(\frac{4X^2}{X^2+k^2}\Big) \\
&+ \frac{1}{4i\,k\,X}\Big(L_2\Big(-\frac{X-i\,k}{X+i\,k}\Big) - L_2\Big(-\frac{X+i\,k}{X-i\,k}\Big)\Big) \\
&+ \frac{1}{2i\,k}\Big(\frac{1}{X+i\,k}\ln\Big(\frac{X+i\,k}{2\lambda}\Big) - \frac{1}{X-i\,k}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\Big) \\
&- \frac{1}{6i\,k}(X^2+k^2)\Big(\frac{1}{(X+i\,k)^3}\ln\Big(\frac{X+i\,k}{2\lambda}\Big) - \frac{1}{(X-i\,k)^3}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\Big) \\
&- \frac{-2\,k^2}{(X^2+k^2)^2}\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\,.
\end{aligned} \tag{5.35}
$$

Having calculated all L_i, P_2^{II} and thus P_H^2 is determined.

5.3 Single vertex part P_H^1

Passing to the single vertex part, one has the expression

$$
\begin{aligned}
P_H^1 &= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\overline{\psi}(\vec{p})\,\gamma^\mu\,\frac{1}{(\not p - \not k - m)}\,\frac{(-4\pi\alpha)}{(\vec{p}-\vec{p}\,')^2}\,\gamma_0\,\frac{1}{(\not p' - \not k - m)}\,\gamma_\mu\,\psi(\vec{p}\,') \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\overline{\psi}(\vec{p})\,\Gamma_0(p,p')\,V(\vec{p}-\vec{p}\,')\,\psi(\vec{p}\,')\,.
\end{aligned} \tag{5.36}
$$

This expression, integrated over k, is ultraviolet divergent. To eliminate this divergence I separate out the term related to the Z_2 renormalization constant, which cancels out with the term in the zero–vertex part.

$$
\begin{aligned}
\gamma_0 Z_2^{(2)} &\equiv \Gamma_0(p,p')_{p=p'=(m,0)} \\
&= \gamma_0\frac{1}{(\omega^2-k^2-2m\,\omega)^2}\Big(4m^2-2\omega^2-2k^2-4m\,\omega\Big)\,,
\end{aligned} \tag{5.37}
$$

$$
\Gamma_0'(p,p') = \Gamma_0(p,p') - Z_2^{(2)}\gamma_0\,. \tag{5.38}
$$

I have for the modified matrix element P_H^1 $(Y^2 = m^2-(E-\omega)^2)$

$$
\begin{aligned}
P_H^1 = -\int &\frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\overline{\psi}(\vec{p})\frac{4\pi\alpha}{\vec{q}^{\,2}}\Big[\\
&\left.\frac{4(E-\omega)(m+\not k)}{(Y^2+(\vec{p}-\vec{k})^2)\,(Y^2+(\vec{p}\,'-\vec{k})^2)} - \frac{4\gamma_0(m^2-\omega^2)}{(X^2+k^2)^2}\quad\right\}1 \\
&\left.+\frac{\gamma_0\,\vec{q}^{\,2}}{(Y^2+(\vec{p}-\vec{k})^2)\,(Y^2+(\vec{p}\,'-\vec{k})^2)}\quad\right\}2 \\
&\left.-\frac{\gamma_0}{(Y^2+(\vec{p}-\vec{k})^2)} - \frac{\gamma_0}{(Y^2+(\vec{p}\,'-\vec{k})^2)} + \frac{2\gamma_0}{X^2+k^2}\quad\right\}3 \\
&\left.-\frac{2(E-\omega)[(\not p - m)+(\not p' - m)]}{(Y^2+(\vec{p}-\vec{k})^2)\,(Y^2+(\vec{p}\,'-\vec{k})^2)}\quad\right\}4 \\
&\left.-\frac{2\,i\,\gamma_0(\vec{p}\,'-\vec{k})\times(\vec{p}-\vec{k})\cdot\vec{\sigma}}{(Y^2+(\vec{p}-\vec{k})^2)\,(Y^2+(\vec{p}\,'-\vec{k})^2)}\Big]\psi(\vec{p})\quad\right\}5 \\
= \;& P_{11}+P_{12}+P_{13}+P_{14}+P_{15}\,.
\end{aligned} \tag{5.39}
$$

All P_{1i} are defined as parts of the expression (5.39) indicated by the corresponding number. To evaluate P_{11} a further subdivision is necessary. I replace the wave function ψ by the sum of the Schrödinger–Pauli wave function ψ_n and the relativistic correction ψ_r, and separate the upper ψ_u and lower ψ_d component (all these functions are given in appendix A). So that,

$$
\begin{aligned}
P_{11} = -\int & \frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{4\pi\alpha}{\vec{q}^{\,2}}\Bigg[\\
& \left.\overline{\psi}_n(\vec{p})\left(\frac{4(E-\omega)(m+\not{k})}{(Y^2+(\vec{p}-\vec{k})^2)(Y^2+(\vec{p}\,'-\vec{k})^2)} - \frac{4\gamma_0(m^2-\omega^2)}{(X^2+k^2)^2}\right)\psi_n(\vec{p}\,')\right\} a \\
& \left.-2\psi_{nd}(\vec{p})\left(\frac{4\left((m-\omega)^2+m^2-\omega^2\right)}{(X^2+k^2)^2}\right)\psi_{rd}(\vec{p}\,')\right\} b \\
& \left.-2\overline{\psi}_n(\vec{p})\left(\frac{4(m-\omega)(\vec{k}\cdot\vec{\gamma})(2\vec{p}\cdot\vec{k}+2\vec{p}\,'\cdot\vec{k})}{(X^2+k^2)^3}\right)\psi_r(\vec{p}\,')\right\} cd \\
& \left.-2\psi_{nu}(\vec{p})\frac{4(m-\omega)(m+\omega)}{(X^2+k^2)^3}\left(\vec{p}^{\,2}-\frac{(2\,\vec{p}\cdot\vec{k})^2}{X^2+k^2}\right)\psi_{ru}(\vec{p}\,')\right\} e \\
& \left.+2\psi_{nu}(\vec{p})\frac{4(m-\omega)(m+\omega)}{(X^2+k^2)^4}(2\,\vec{p}\cdot\vec{k})(2\,\vec{p}\,'\cdot\vec{k})\psi_{ru}(\vec{p}\,')\right\} f \\
& \left.+2\psi_{nu}(\vec{p})\left(\frac{4(E-\omega)(m+\omega)}{(Y^2+k^2)(Y^2+(\vec{p}\,'-\vec{k})^2} - \frac{4(m^2-\omega^2)}{(X^2+k^2)^2}\right)\psi_{ru}(\vec{p}\,')\Bigg]\right\} g \\
= {} & P_{11a}+P_{11b}+P_{11cd}+P_{11e}+P_{11f}+P_{11g}\,.
\end{aligned} \tag{5.40}
$$

where all $P_{11\star}$ are calculated below. The general scheme is the same as before; I remove p_i from numerator and find the appropriate integral from appendix B, so that,

$$
\begin{aligned}
P_{11a} = & -16m^5\alpha^6(E-\omega)\Bigg[\left(m+\omega-\lambda^2\frac{(\omega-m)}{4m^2}-\frac{Y^2+k^2-\lambda^2}{2m}\right)J_{11}(Y) \\
& -\frac{\omega-m}{8m^2}K_2^2(Y)+\frac{\omega-3m}{4m^2}J_5(Y)+\frac{1}{2m}J_{12}(Y)\Bigg] \\
& -\frac{4(m^2-\omega^2)}{X^2+k^2}(m\alpha^2+\frac{1}{4}m\alpha^4)
\end{aligned} \tag{5.41}
$$

$$
P_{11b} = \frac{16m(m-\omega)}{(X^2+k^2)^2}\langle\psi_{nd}|\frac{\alpha}{r}|\psi_{rd}\rangle = \frac{3}{2}\alpha^6\,\frac{m^2(m-\omega)}{(X^2+k^2)^2} \tag{5.42}
$$

$$
P_{11cd} = -\frac{16}{3}\frac{(\omega-m)}{(X^2+k^2)^3}k^2\langle\overline{\psi}_n|\vec{p}\cdot\vec{\gamma}\,\frac{\alpha}{r}+\frac{\alpha}{r}\,\vec{p}\cdot\vec{\gamma}|\psi_r\rangle = 16\alpha^6\,\frac{k^2m^2(m-\omega)}{(X^2+k^2)^3} \tag{5.43}
$$

$$
\begin{aligned}
P_{11e} &= -8\frac{m^2-\omega^2}{(X^2+k^2)^2}\left(\frac{4}{3}\frac{k^2}{(X^2+k^2)^2}-\frac{1}{X^2+k^2}\right)\langle\psi_{nu}|\vec{p}^{\,2}\frac{\alpha}{r}|\psi_{ru}\rangle \\
&= -19\alpha^6\,\frac{m^3(m^2-\omega^2)}{(X^2+k^2)^3}\left(\frac{4}{3}\frac{k^2}{X^2+k^2}-1\right)
\end{aligned} \tag{5.44}
$$

$$
P_{11f} = -\frac{32}{3}\frac{(m^2-\omega^2)k^2}{(X^2+k^2)^4}\langle\psi_{nu}|\vec{p}\,\frac{\alpha}{r}\,\vec{p}|\psi_{ru}\rangle = -12\alpha^6\,\frac{m^3k^2(m^2-\omega^2)}{(X^2+k^2)^4} \tag{5.45}
$$

$$P_{11g} = -8\frac{(E-\omega)(m+\omega)}{Y^2+k^2}\langle\psi_{nu}|\Big(\frac{\alpha}{r}\Big)\frac{1}{Y^2+(\vec{p}\,'-\vec{k})^2}|\psi_{ru}\rangle$$
$$+8\frac{(m^2-\omega^2)}{(X^2+k^2)^2}\langle\psi_{nu}|\frac{\alpha}{r}|\psi_{ru}\rangle$$
$$= -4\alpha^6\,\frac{m^3(E-\omega)(m+\omega)}{Y^2+k^2}\Big(\lambda K_3(Y)+G_1(Y)\Big)+\alpha^4\,\frac{m(m^2-\omega^2)}{(X^2+k^2)^2}. \tag{5.46}$$

In the evaluation of P_{11g} I used ψ_{ru} in the momentum representation (A29). From the second term in (A29) a G type integral arises in (5.46), which is calculated in appendix B. At this point P_{11} is calculated.

For P_{12} I have

$$P_{12} = -4\pi\alpha\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\frac{1}{[Y^2+(\vec{p}-\vec{k})^2][Y^2+(\vec{p}\,'-\vec{k})^2]}$$
$$\Big(\psi_{nu}(\vec{p})\psi_{nu}(\vec{p}\,')+2\psi_{nu}(\vec{p})\psi_{ru}(\vec{p}\,')+\psi_{nd}(\vec{p})\psi_{nd}(\vec{p}\,')\Big)$$
$$= -4\alpha\lambda^5\Big[K_2^2(Y)+\frac{\alpha}{4m}K_2(Y)\Big(\frac{1}{Y^2+k^2}+G_2(Y)\Big)+\frac{1}{16m^2k^2}K_6^2(Y)\Big]. \tag{5.47}$$

The next term is P_{13}

$$P_{13} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{d^3\vec{p}\,'}{(2\pi)^3}$$
$$\overline{\psi}(\vec{p})\gamma_0\Big(\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\,\frac{1}{Y^2+(\vec{p}\,'-\vec{k})^2}-\frac{2}{X^2+k^2}\Big)\psi(\vec{p}\,')\frac{4\pi\alpha}{\vec{q}^{\,2}} \tag{5.48}$$

This contribution is exactly canceled by the similar term in P_H^0, so I do not calculate it.

Using the Dirac equation, the expression for P_{14} becomes,

$$P_{14} = -4(E-\omega)\langle\overline{\psi}|\gamma_0\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\Big(\frac{\alpha}{r}\Big)\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\Big(\frac{\alpha}{r}\Big)|\psi\rangle \tag{5.49}$$
$$= -4(E-\omega)\Big[\langle\psi_{nu}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\Big(\frac{\alpha}{r}\Big)\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\Big(\frac{\alpha}{r}\Big)|\psi_{nu}\rangle$$
$$+\frac{1}{(Y^2+k^2)^2}\langle\psi_{nd}|\Big(\frac{\alpha}{r}\Big)^2|\psi_{nd}\rangle+\frac{2}{(Y^2+k^2)^2}\langle\psi_{ru}|\Big(\frac{\alpha}{r}\Big)^2|\psi_{nu}\rangle\Big]$$
$$= -4(E-\omega)\Big[2m^4\alpha^6J_5(Y)+3m^2\alpha^6\frac{1}{(X^2+k^2)^2}\Big]. \tag{5.50}$$

The last single vertex part P_{15} has many terms to evaluate.

$$P_{15} = -\frac{2m^3\alpha^6}{\pi^4}\int d^3\vec{p}\,d^3\vec{p}\,'$$
$$\frac{\vec{p}^{\,2}\vec{p}\,'^{\,2}-(\vec{p}\cdot\vec{p}\,')^2+(\vec{p}\cdot\vec{k})(\vec{p}\cdot\vec{p}\,'-\vec{p}\,'^{\,2})+(\vec{p}\,'\cdot\vec{k})(\vec{p}\cdot\vec{p}\,'-\vec{p}^{\,2})}{(p^2+\lambda^2)^2(X^2+(\vec{p}-\vec{k})^2)\vec{q}^{\,2}(X^2+(\vec{p}\,'-\vec{k})^2)(p'^2+\lambda^2)^2}. \tag{5.51}$$

Eliminating $\vec{p}$ and $\vec{p}\,'$ from the numerator I obtain,

$$P_{15} = -2m^3\alpha^6\Big[\frac{1}{2}K_1K_2-\frac{\lambda^2}{2}K_2^2+\frac{1}{8k^2}K_6^2-\frac{1}{2}K_2K_6+\frac{1}{2}J_8-\frac{1}{2}J_9\Big]$$

$$
\begin{aligned}
&= a^6\, m^3 \left[\frac{32\,X^2}{9\,(X^2+k^2)^3} + \frac{1}{9\,(X^2+k^2)^2} - \frac{X^2}{k^2\,(X^2+k^2)^2}\right.\\
&+ \frac{i\,X}{k^3\,(X^2+k^2)}\,\ln\Big(\frac{X-i\,k}{X+i\,k}\Big) + \frac{1}{4k^4}\ln^2\Big(\frac{X-i\,k}{X+i\,k}\Big)\\
&\left. + \frac{2\,i}{3}\,\left(\frac{1}{k(X-i\,k)^3}\,\ln\Big(\frac{X-i\,k}{2\,\lambda}\Big) - \frac{1}{k(X+i\,k)^3}\,\ln\Big(\frac{X+i\,k}{2\,\lambda}\Big)\right)\right]. \qquad (5.52)
\end{aligned}
$$

At this point the single vertex contribution to matrix element P is calculated.

5.4 Zero–vertex part P_H^0

The zero vertex contribution to energy is

$$
\begin{aligned}
E_H^0 &= \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\,\frac{1}{\omega^2-k^2}\,\langle\overline{\psi}|\gamma^\mu\,\frac{1}{\not{p}-\not{k}-m}\gamma_\mu|\psi\rangle - \delta m\langle\overline{\psi}|\psi\rangle\\
&= \langle\overline{\psi}|\Sigma(p)|\psi\rangle - \delta m\langle\overline{\psi}|\psi\rangle\,. \qquad (5.53)
\end{aligned}
$$

The free electron self–energy can be rewritten as follows

$$
\begin{aligned}
\Sigma(p) = \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\,\frac{1}{\omega^2-k^2}&\\
\frac{2}{(E-\omega)^2-(\vec{p}-\vec{k})^2-m^2}&\Big[m+\not{k}-(\not{p}-m)\Big]\,. \qquad (5.54)
\end{aligned}
$$

I separate out the mass renormalization term and e_1 term.

$$
\Sigma(p) = \delta m + e_1(\not{p}-m) + O(\not{p}-m)^2\,, \qquad (5.55)
$$

$$
\delta m = \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\,\frac{1}{\omega^2-k^2}\,\frac{2(m+\omega)}{k^2-2m\,\omega}\,, \qquad (5.56)
$$

$$
\begin{aligned}
e_1 = \frac{\partial\Sigma}{\partial E}\Big|_{\not{p}\,=\,m} &= -\frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\,\frac{1}{\omega^2-k^2}\\
&\cdot\left[\frac{4(m^2-\omega^2)}{(\omega^2-k^2-2m\,\omega)^2} + \frac{2}{\omega^2-k^2-2m\,\omega}\right] = -Z_2^{(2)}\,. \qquad (5.57)
\end{aligned}
$$

Due to the Ward identity the sum of a infinite terms: with e_1 from the zero vertex and with $Z_2^{(2)}$ from the single vertex is zero,

$$
\langle\overline{\psi}|e_1(\not{p}-m) + Z_2^{(2)} A^0\gamma^0|\psi\rangle = -Z_2^{(2)}\langle\overline{\psi}|\;\not{p}-m-A_0\gamma_0|\psi\rangle = 0\,, \qquad (5.58)
$$

therefore

$$
\Sigma(p) - e_1(\not{p}-m) - \delta m = \frac{e^2}{(2\pi)^4 i}\int_{C_H} d\omega \int d^3\vec{k}\frac{1}{\omega^2-k^2}\;P_H^0
$$

$$
\begin{aligned}
&P_H^0 =\\
&\left[\frac{2(m+\not{k})}{(E-\omega)^2-(\vec{p}-\vec{k})^2-m^2} - \frac{2(m+\omega)}{\omega^2-k^2-2\omega m} + \frac{4(m^2-\omega^2)}{(\omega^2-k^2-2m\,\omega)^2}(\not{p}-m)\right.\\
&\left. + \left\{-\frac{2(\not{p}-m)}{(E-\omega)^2-(\vec{p}-\vec{k})^2-m^2} + \frac{2(\not{p}-m)}{\omega^2-k^2-2m\,\omega}\right\}\right]. \qquad (5.59)
\end{aligned}
$$

The term in braces cancels out with P_{13}, so finally I have to calculate

$$
\begin{aligned}
P_H'^{\,0} = \langle\overline{\psi}|&\frac{2(m+\not{k})}{(E-\omega)^2-(\vec{p}-\vec{k})^2-m^2}\\
&-\frac{2(m+\omega)}{\omega^2-k^2-2m\,\omega} + \frac{4(m^2-\omega^2)}{(\omega^2-k^2-2m\,\omega)^2}(\not{p}-m)|\psi\rangle\,. \qquad (5.60)
\end{aligned}
$$

To calculate this expression I make the following decomposition that is valid up to the order $(Z\alpha)^6$.

$$
\begin{aligned}
&{P'_H}^0 = \\
&-2(m+\omega)\langle\psi_{nu}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}-\frac{1}{Y^2+k^2}+\frac{p^2}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)|\psi_{nu}\rangle \\
&-4(m+\omega)\langle\psi_{nu}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}-\frac{1}{Y^2+k^2}+\frac{p^2}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)|\psi_{ru}\rangle \\
&+2(m-\omega)\langle\psi_{nd}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}-\frac{1}{Y^2+k^2}+\frac{p^2}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)|\psi_{nd}\rangle \\
&+\left(\langle\psi_{nu}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2}-\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\vec{p}\cdot\vec{\sigma}|\psi_{nd}\rangle+c.c.\right) \\
&+\left(\langle\psi_{ru}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2}-\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\vec{p}\cdot\vec{\sigma}|\psi_{nd}\rangle+c.c.\right) \\
&+\left(\langle\psi_{nu}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2}-\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\vec{p}\cdot\vec{\sigma}|\psi_{rd}\rangle+c.c.\right) \\
&-2(m+\omega)\langle\psi_u|\frac{1}{Y^2+k^2}-\frac{p^2}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)|\psi_u\rangle \\
&+2(m-\omega)\langle\psi_d|\frac{1}{Y^2+k^2}-\frac{p^2}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)|\psi_d\rangle \\
&+\left(\langle\psi_u|\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\vec{p}\cdot\vec{\sigma}|\psi_d\rangle+c.c.\right) \\
&+\langle\overline{\psi}|\frac{2(m+\omega)}{X^2+k^2}|\psi\rangle+\langle\overline{\psi}|\frac{4(m^2-\omega^2)}{(X^2+k^2)^2}(\not{p}-m)|\psi\rangle \\
&=\sum_{i=1}^{22}Q_i\,.
\end{aligned}
\tag{5.61}
$$

where all Q_i are defined and calculated below.

$$Q_1=-2(m+\omega)\langle\psi_{nu}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}|\psi_{nu}\rangle=-2(m+\omega)\,8\lambda^5\,K_4(Y) \tag{5.62}$$

$$Q_2=\frac{2(m+\omega)}{Y^2+k^2}\langle\psi_{nu}|\psi_{nu}\rangle=\frac{2(m+\omega)}{Y^2+k^2} \tag{5.63}$$

$$
\begin{aligned}
Q_3&=-\frac{2(m+\omega)}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)\langle\psi_{nu}|p^2|\psi_{nu}\rangle \\
&=-\frac{2(m+\omega)}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)\,\lambda^2
\end{aligned}
\tag{5.64}
$$

$$
\begin{aligned}
Q_4&=-4(m+\omega)\langle\psi_{nu}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}|\psi_{ru}\rangle \\
&=-4\alpha^6m^4(m+\omega)\,\Big(\lambda K_4(Y)+G_3(Y)\Big)
\end{aligned}
\tag{5.65}
$$

$$Q_5=\frac{4(m+\omega)}{Y^2+k^2}\langle\psi_{nu}|\psi_{ru}\rangle=\frac{4(m+\omega)}{Y^2+k^2}\Big(-\frac{\alpha^2}{8}\Big) \tag{5.66}$$

$$Q_6=-\frac{4(m+\omega)}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)\langle\psi_{nu}|p^2|\psi_{ru}\rangle$$

$$= -\frac{4(m+\omega)}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)\frac{3}{8}m^2\alpha^4 \tag{5.67}$$

$$Q_7 = 2(m-\omega)\langle\psi_{nd}|\frac{1}{Y^2+(\vec{p}-\vec{k})^2}|\psi_{nd}\rangle$$

$$= \frac{(m-\omega)}{m^2}\,4\lambda^5\left(K_3(Y)-\lambda^2 K_4(Y)\right) \tag{5.68}$$

$$Q_8 = -\frac{2(m-\omega)}{Y^2+k^2}\langle\psi_{nd}|\psi_{nd}\rangle = -\frac{2(m-\omega)}{Y^2+k^2}\,\frac{\alpha^2}{4} \tag{5.69}$$

$$Q_9 = \frac{2(m-\omega)}{(Y^2+k^2)^2}\Big(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\Big)\langle\psi_{nd}|p^2|\psi_{nd}\rangle$$

$$= \frac{5}{2}\,\alpha^4\,\frac{m^2(m-\omega)}{(Y^2+k^2)^2}\left(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\right) \tag{5.70}$$

$$Q_{10} = \langle\psi_{nu}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2}|\psi_{nd}\rangle + c.c. = \frac{8\lambda^5}{m}K_7(Y) \tag{5.71}$$

$$Q_{11} = -\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\langle\psi_{nu}|\vec{p}\cdot\vec{\sigma}|\psi_{nd}\rangle + c.c. = -\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\,\frac{\lambda^2}{m} \tag{5.72}$$

$$Q_{12} = \langle\psi_{ru}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2}|\psi_{nd}\rangle + c.c. = m^3\alpha^6\Big(\lambda K_7(Y)+G_4(Y)\Big) \tag{5.73}$$

$$Q_{13} = -\frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\langle\psi_{ru}|\vec{p}\cdot\vec{\sigma}|\psi_{nd}\rangle + c.c. = -\frac{1}{2}m\alpha^4\frac{k^2}{(Y^2+k^2)^2} \tag{5.74}$$

$$Q_{14}+Q_{15} = \langle\psi_{nu}|\frac{2\vec{k}\cdot\vec{\sigma}}{Y^2+(\vec{p}-\vec{k})^2} - \frac{4}{3}\frac{k^2\,\vec{p}\cdot\vec{\sigma}}{(Y^2+k^2)^2}|\psi_{rd}\rangle + c.c.$$

$$= 33m^3\alpha^6\left(\frac{2}{5}\frac{k^4}{(Y^2+k^2)^4} - \frac{1}{3}\frac{k^2}{(Y^2+k^2)^3}\right) \tag{5.75}$$

$$Q_{16} = -\frac{2(m+\omega)}{Y^2+k^2}\langle\psi_u|\psi_u\rangle = -\frac{2(m+\omega)}{Y^2+k^2}\left(\frac{1+\gamma}{2}\right) \tag{5.76}$$

$$Q_{17} = \frac{2(m+\omega)}{(Y^2+k^2)^2}\left(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\right)\langle\psi_u|p^2|\psi_u\rangle$$

$$= \frac{2(m+\omega)}{(Y^2+k^2)^2}\left(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\right)\lambda^2\left(1+\frac{3}{4}\alpha^2+\frac{15}{16}\alpha^4\right) \tag{5.77}$$

$$Q_{18} = \frac{2(m-\omega)}{Y^2+k^2}\langle\psi_d|\psi_d\rangle = \frac{2(m-\omega)}{Y^2+k^2}\,\frac{1-\gamma}{2} \tag{5.78}$$

$$Q_{19} = -\frac{2(m-\omega)}{(Y^2+k^2)^2}\left(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\right)\langle\psi_d|p^2|\psi_d\rangle$$

$$= -\frac{2(m-\omega)}{(Y^2+k^2)^2}\left(1-\frac{4}{3}\frac{k^2}{Y^2+k^2}\right)m^2\left(\frac{5}{4}\alpha^4+\frac{33}{16}\alpha^6\right) \tag{5.79}$$

$$Q_{20} = \frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\langle\psi_d|\vec{\sigma}\cdot\vec{p}|\psi_u\rangle + c.c. = \frac{4}{3}\frac{k^2}{(Y^2+k^2)^2}\frac{m\alpha^2}{\gamma} \tag{5.80}$$

$$Q_{21} = \frac{2(m+\omega)}{X^2+k^2}\,\langle\overline{\psi}|\psi\rangle = \frac{2(m+\omega)}{X^2+k^2}\,\gamma \tag{5.81}$$

$$Q_{22} = \frac{4(m^2-\omega^2)}{(X^2+k^2)^2}\,\langle\overline{\psi}|\not{p}-m|\psi\rangle = \frac{4(m^2-\omega^2)}{(X^2+k^2)^2}\left(-\frac{m\alpha^2}{\gamma}\right)\,. \tag{5.82}$$

At this point the integration with respect to electron momentum in the high energy part is done.

5.5 $\omega-$ and $k-$integration

Summing up all terms,

$$P_H = P_H^3 + P_H^2 + P_H^1 + P_H^0 \ , \tag{5.83}$$

expanding E and Y in α^2, I obtain the following expression ($\omega, X^2 = 2\omega - \omega^2, k$ are now dimensionless)

$$\mathcal{F}(\omega, k) \equiv m\alpha^{-4}\ P_H =$$

$$\alpha^{-2}\left[\frac{8\,X^2}{3\,(k^2+X^2)^3} + \frac{8\,w\,X^2}{3\,(k^2+X^2)^3} - \frac{8}{3\,(k^2+X^2)^2} - \frac{2\,w}{3\,(k^2+X^2)^2} + \frac{2\,w^2}{(k^2+X^2)^2}\right.$$

$$\left. - \frac{4\,X^2}{3\,(k^2+X^2)^2} + \frac{4}{3\,(k^2+X^2)} - \frac{w}{k^2+X^2}\right] + \left[\frac{-32\,X^4}{(k^2+X^2)^5} - \frac{32\,w\,X^4}{(k^2+X^2)^5} + \frac{160\,X^2}{3\,(k^2+X^2)^4}\right.$$

$$+ \frac{24\,w\,X^2}{(k^2+X^2)^4} - \frac{88\,w^2\,X^2}{3\,(k^2+X^2)^4} + \frac{16\,X^4}{(k^2+X^2)^4} - \frac{64}{3\,(k^2+X^2)^3} + \frac{16\,w}{(k^2+X^2)^3} - \frac{26\,w^2}{3\,(k^2+X^2)^3}$$

$$+ \frac{2\,w^3}{(k^2+X^2)^3} - \frac{32\,X^2}{(k^2+X^2)^3} + \frac{76\,w\,X^2}{3\,(k^2+X^2)^3} + \frac{4}{(k^2+X^2)^2} - \frac{77\,w}{6\,(k^2+X^2)^2} + \frac{3\,w^2}{2\,(k^2+X^2)^2}$$

$$\left. - \frac{2\,X^2}{3\,(k^2+X^2)^2} + \frac{2}{3\,(k^2+X^2)} - \frac{w}{4\,(k^2+X^2)}\right] + \alpha\left[\frac{512\,X^5}{3\,(k^2+X^2)^6} + \frac{512\,w\,X^5}{3\,(k^2+X^2)^6}\right.$$

$$- \frac{896\,X^3}{3\,(k^2+X^2)^5} - \frac{512\,w\,X^3}{3\,(k^2+X^2)^5} + \frac{128\,w^2\,X^3}{(k^2+X^2)^5} - \frac{256\,X^5}{3\,(k^2+X^2)^5} + \frac{128\,X}{(k^2+X^2)^4} - \frac{16\,w\,X}{(k^2+X^2)^4}$$

$$- \frac{40\,w^2\,X}{(k^2+X^2)^4} + \frac{8\,w^3\,X}{(k^2+X^2)^4} + \frac{520\,X^3}{3\,(k^2+X^2)^4} - \frac{72\,w\,X^3}{(k^2+X^2)^4} - \frac{56\,X}{(k^2+X^2)^3} + \frac{40\,w\,X}{(k^2+X^2)^3}$$

$$\left. + \left(\frac{4\,i}{k\,(k^2+X^2)^2} + \frac{4\,i\,w}{k\,(k^2+X^2)^2}\right)\ln\left(\frac{X-i\,k}{X+i\,k}\right)\right] + \alpha^2\left[\frac{-640\,X^6}{(k^2+X^2)^7} - \frac{640\,w\,X^6}{(k^2+X^2)^7}\right.$$

$$+ \frac{4736\,X^4}{3\,(k^2+X^2)^6} + \frac{800\,w\,X^4}{(k^2+X^2)^6} - \frac{2336\,w^2\,X^4}{3\,(k^2+X^2)^6} + \frac{320\,X^6}{(k^2+X^2)^6} - \frac{1280\,X^2}{(k^2+X^2)^5} + \frac{448\,w\,X^2}{3\,(k^2+X^2)^5}$$

$$+ \frac{2224\,w^2\,X^2}{3\,(k^2+X^2)^5} - \frac{176\,w^3\,X^2}{(k^2+X^2)^5} - \frac{22576\,X^4}{25\,(k^2+X^2)^5} + \frac{33472\,w\,X^4}{75\,(k^2+X^2)^5} + \frac{1024}{3\,(k^2+X^2)^4}$$

$$- \frac{880\,w}{3\,(k^2+X^2)^4} - \frac{80\,w^2}{3\,(k^2+X^2)^4} + \frac{46\,w^3}{(k^2+X^2)^4} - \frac{2\,w^4}{(k^2+X^2)^4} + \frac{55496\,X^2}{75\,(k^2+X^2)^4}$$

$$- \frac{153862\,w\,X^2}{225\,(k^2+X^2)^4} + \frac{602\,w^2\,X^2}{9\,(k^2+X^2)^4} + \frac{872\,X^4}{75\,(k^2+X^2)^4} - \frac{15608}{75\,(k^2+X^2)^3} + \frac{54226\,w}{225\,(k^2+X^2)^3}$$

$$- \frac{488\,w^2}{9\,(k^2+X^2)^3} - \frac{w^3}{(k^2+X^2)^3} - \frac{7987\,X^2}{225\,(k^2+X^2)^3} + \frac{40\,w\,X^2}{(k^2+X^2)^3} + \frac{2101}{225\,(k^2+X^2)^2}$$

$$- \frac{87\,w}{4\,(k^2+X^2)^2} + \frac{5\,w^2}{4\,(k^2+X^2)^2} - \frac{X^2}{2\,(k^2+X^2)^2} - \frac{2\,X^2}{k^2\,(k^2+X^2)^2} + \frac{1}{2\,(k^2+X^2)} - \frac{w}{8\,(k^2+X^2)}$$

$$\left(+\frac{8\,i}{5}\frac{1}{k\,(-i\,k+X)^5} - \frac{2\,i}{5}\frac{k}{(-i\,k+X)^5} + \frac{8\,i}{5}\frac{w}{k\,(-i\,k+X)^5} - \frac{2\,i}{5}\frac{X^2}{k\,(-i\,k+X)^5}\right.$$

$$+ \frac{4\,i}{3}\frac{1}{k\,(-i\,k+X)^3} + \frac{16\,i}{k\,(-i\,k+X)\,(k^2+X^2)^2} - \frac{4\,i\,k}{(-i\,k+X)\,(k^2+X^2)^2}$$

$$- \frac{8\,i\,w}{k\,(-i\,k+X)\,(k^2+X^2)^2} - \frac{12\,i\,w^2}{k\,(-i\,k+X)\,(k^2+X^2)^2} - \frac{8\,i}{k\,(-i\,k+X)^3\,(k^2+X^2)}$$

$$+ \frac{4\,i}{3}\frac{k}{(-i\,k+X)^3\,(k^2+X^2)} - \frac{16\,i}{3}\frac{w}{k\,(-i\,k+X)^3\,(k^2+X^2)} + \frac{8\,i}{3}\frac{w^2}{k\,(-i\,k+X)^3\,(k^2+X^2)}$$

$$\left. + \frac{4\,i}{3}\frac{X^2}{k\,(-i\,k+X)^3\,(k^2+X^2)} + \frac{2\,i}{k\,(-i\,k+X)\,(k^2+X^2)}\right)\ln\left(\frac{-i\,k+X}{2\,\alpha}\right)$$

$$
\begin{aligned}
&+\left(\frac{-8\,i}{5}\frac{1}{k\,(i\,k+X)^5}+\frac{2\,i}{5}\frac{k}{(i\,k+X)^5}-\frac{8\,i}{5}\frac{w}{k\,(i\,k+X)^5}+\frac{2\,i}{5}\frac{X^2}{k\,(i\,k+X)^5}\right.\\
&-\frac{4\,i}{3}\frac{1}{k\,(i\,k+X)^3}-\frac{16\,i}{k\,(i\,k+X)\,(k^2+X^2)^2}+\frac{4\,i\,k}{(i\,k+X)\,(k^2+X^2)^2}\\
&+\frac{8\,i\,w}{k\,(i\,k+X)\,(k^2+X^2)^2}+\frac{12\,i\,w^2}{k\,(i\,k+X)\,(k^2+X^2)^2}+\frac{8\,i}{k\,(i\,k+X)^3\,(k^2+X^2)}\\
&-\frac{4\,i}{3}\frac{k}{(i\,k+X)^3\,(k^2+X^2)}+\frac{16\,i}{3}\frac{w}{k\,(i\,k+X)^3\,(k^2+X^2)}-\frac{8\,i}{3}\frac{w^2}{k\,(i\,k+X)^3\,(k^2+X^2)}\\
&\left.-\frac{4\,i}{3}\frac{X^2}{k\,(i\,k+X)^3\,(k^2+X^2)}-\frac{2\,i}{k\,(i\,k+X)\,(k^2+X^2)}\right)\ln\left(\frac{i\,k+X}{2\,\alpha}\right)\\
&+\left(\frac{-8\,i\,X}{k\,(k^2+X^2)^3}+\frac{4\,i\,k\,X}{(k^2+X^2)^3}-\frac{4\,i\,w^2\,X}{k\,(k^2+X^2)^3}+\frac{2\,i\,X}{k^3\,(k^2+X^2)}\right)\ln\left(\frac{-i\,k+X}{i\,k+X}\right)\\
&+\left(\frac{1}{2\,k^4}+\frac{1}{(k^2+X^2)^2}-\frac{2}{k^2\,(k^2+X^2)^2}+\frac{2\,w}{k^2\,(k^2+X^2)^2}+\frac{w^2}{k^2\,(k^2+X^2)^2}\right)\ln^2\left(\frac{-i\,k+X}{i\,k+X}\right)\\
&+\left(\frac{1024\,X^4}{3\,(k^2+X^2)^6}-\frac{1024\,w^2\,X^4}{3\,(k^2+X^2)^6}-\frac{1792\,X^2}{3\,(k^2+X^2)^5}+\frac{384\,w\,X^2}{(k^2+X^2)^5}+\frac{448\,w^2\,X^2}{3\,(k^2+X^2)^5}\right.\\
&-\frac{64\,w^3\,X^2}{(k^2+X^2)^5}-\frac{704\,X^4}{3\,(k^2+X^2)^5}+\frac{1088\,w\,X^4}{3\,(k^2+X^2)^5}+\frac{256}{(k^2+X^2)^4}-\frac{320\,w}{(k^2+X^2)^4}+\frac{128\,w^2}{(k^2+X^2)^4}\\
&-\frac{8\,w^4}{(k^2+X^2)^4}+\frac{1088\,X^2}{3\,(k^2+X^2)^4}-\frac{1568\,w\,X^2}{3\,(k^2+X^2)^4}+\frac{128\,w^2\,X^2}{(k^2+X^2)^4}-\frac{24\,X^4}{(k^2+X^2)^4}-\frac{96}{(k^2+X^2)^3}\\
&\left.+\frac{176\,w}{(k^2+X^2)^3}-\frac{64\,w^2}{(k^2+X^2)^3}+\frac{16\,X^2}{(k^2+X^2)^3}\right)\ln\left(\frac{4\,\alpha\,X}{k^2+X^2}\right)+\left(-\frac{8}{(k^2+X^2)^3}-\frac{16\,w}{(k^2+X^2)^3}\right.\\
&\left.+\frac{4\,w^2}{(k^2+X^2)^3}+\frac{4\,X^2}{(k^2+X^2)^3}\right)\ln\left(\frac{4\,X^2}{k^2+X^2}\right)+\left(-\frac{4\,i}{k\,X\,(k^2+X^2)^2}+\frac{4\,i\,w^2}{k\,X\,(k^2+X^2)^2}\right)\\
&\left.\cdot\left(L_2\left(-\frac{-i\,k+X}{i\,k+X}\right)-L_2\left(-\frac{i\,k+X}{-i\,k+X}\right)\right)\right]. \qquad (5.84)
\end{aligned}
$$

Because P_H is related to energy by the formula

$$E_H=\frac{e^2}{(2\pi)^4 i}\int_{C_H}d\omega\int d^3\vec{k}\,\frac{1}{\omega^2-k^2}\,P_H \qquad (5.85)$$

and $\Delta E=\alpha(m\alpha^4)/\pi\,F$, one has the relation:

$$F_H=\int_{C_H}d\omega\;\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{k^2}{\omega^2-k^2}\,\mathcal{F}(\omega,k)\,. \qquad (5.86)$$

As can be seen from (5.84), $\mathcal{F}(\omega,k)$ has a simple structure. It includes only rational, logarithmic and dilogarithmic functions. $\mathcal{F}(\omega,k)$ includes also the inverse power of α, but this term cancels out after k– and ω–integration. This simple structure of $\mathcal{F}(\omega,k)$ allows one to write a symbolic program to perform the k– and ω–integration analytically, and expand finally in ϵ.

The next step of the calculation is the integration with respect to k. I assume for a moment that X is real. By analytic continuation the value of the integral can be determined. $\mathcal{F}$ as a function of k has a branch cut on the imaginary axis, beginning at $i\,X$ in the up– and at $-i\,X$ in down–direction. Almost every term in $\mathcal{F}(\omega,k)$ has a branch cut in the upper or the lower half of the complex plane, but not in both of them. Therefore the contour of k–integration can be deformed to enclose only the pole. Below I give the table of formulae to integrate all types of terms that appear in $\mathcal{F}(\omega,k)$.

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\,\frac{1}{(X^2+k^2)^n}=$$

$$\frac{1}{2|\omega|}\frac{1}{(\omega^2+X^2)^n}+\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(X^2)^{n-1}}\;\frac{i}{2X}\;\frac{1}{\omega^2+X^2} \tag{5.87}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\;\frac{1}{(X^2+k^2)^n}\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big) &= \\ \frac{1}{2|\omega|}\frac{1}{(\omega^2+X^2)^n}\ln\Big(\frac{4\lambda X}{(X-i|\omega|)^2}\Big)& \\ +\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\Big|_{Y=X}\frac{i}{2Y}\frac{1}{\omega^2+Y^2}\ln\Big(\frac{4\lambda X}{(X+Y)^2}\Big)&\end{aligned} \tag{5.88}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\\ &\quad\cdot\frac{1}{(X^2+k^2)^n}k\bigg(\frac{1}{(X+i\,k)^l}\ln\Big(\frac{X+i\,k}{2\lambda}\Big)-\frac{1}{(X-i\,k)^l}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\bigg)=\\ &\quad-\frac{1}{(\omega^2+X^2)^n}\;\frac{1}{(X-i|\omega|)^l}\;\ln\Big(\frac{X-i|\omega|}{2\lambda}\Big)\\ &\quad+\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\Big|_{Y=X}\frac{1}{\omega^2+Y^2}\frac{1}{(X+Y)^l}\ln\Big(\frac{X+Y}{2\lambda}\Big)\end{aligned} \tag{5.89}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\\ &\qquad\cdot k^3\bigg(\frac{1}{(X+i\,k)^l}\ln\Big(\frac{X+i\,k}{2\lambda}\Big)-\frac{1}{(X-i\,k)^l}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\bigg)\\ &\qquad=-\omega^2\frac{1}{(X-i|\omega|)^l}\ln\Big(\frac{X-i|\omega|}{2\lambda}\Big)\end{aligned} \tag{5.90}$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\;\frac{1}{(X^2+k^2)^n}\ln\Big(\frac{4X^2}{X^2+k^2}\Big) &= \\ \frac{1}{2|\omega|}\frac{1}{(\omega^2+X^2)^n}\ln\Big(\frac{4X^2}{(X-i|\omega|)^2}\Big)& \\ +\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\Big|_{Y=X}\frac{i}{2Y}\;\frac{1}{\omega^2+Y^2}\;\ln\Big(\frac{4X^2}{(X+Y)^2}\Big)&\end{aligned} \tag{5.91}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\;\frac{1}{(X^2+k^2)^n}\frac{k}{i}\bigg(L_2\Big(-\frac{X-i\,k}{X+i\,k}\Big)-L_2\Big(-\frac{X+i\,k}{X-i\,k}\Big)\bigg)\\ &\quad=\frac{i}{(\omega^2+X^2)^n}L_2\Big(-\frac{X+i|\omega|}{X-i|\omega|}\Big)\\ &\quad-i\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\Big|_{Y=X}\frac{1}{\omega^2+Y^2}L_2\Big(-\frac{X-Y}{X+Y}\Big)\end{aligned} \tag{5.92}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}dk\,\frac{1}{k^2-\omega^2}\;\frac{1}{(X^2+k^2)^n}\ln^2\Big(\frac{X-i\,k}{X+i\,k}\Big)=\\ &\quad-\frac{1}{|\omega|}\frac{1}{(\omega^2+X^2)^n}\bigg(\zeta(2)+2L_2\Big(-\frac{X+i|\omega|}{X-i|\omega|}\Big)\bigg)\\ &\quad-\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\Big|_{Y=X}\frac{i}{Y}\frac{1}{\omega^2+Y^2}\bigg(\zeta(2)+2L_2\Big(-\frac{X-Y}{X+Y}\Big)\bigg)\end{aligned} \tag{5.93}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, \frac{1}{k^2-\omega^2}\, \frac{1}{(X^2+k^2)^n}\, \frac{1}{k^2} \ln^2\Big(\frac{X-i\,k}{X+i\,k}\Big) =$$
$$-\frac{1}{\omega^2|\omega|}\, \frac{1}{(\omega^2+X^2)^n}\left(\zeta(2)+2L_2\Big(-\frac{X+i|\omega|}{X-i|\omega|}\Big)\right) - 4i\ln(2)\frac{1}{X^{2n+1}}\frac{1}{\omega^2}$$
$$+\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\bigg|_{Y=X} \frac{i}{Y^3}\, \frac{1}{\omega^2+Y^2}\left(\zeta(2)+2L_2\Big(-\frac{X-Y}{X+Y}\Big)\right) \tag{5.94}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, \frac{1}{k^2-\omega^2}\, \frac{1}{(X^2+k^2)^n} k\ln\Big(\frac{X-i\,k}{X+i\,k}\Big) =$$
$$\frac{1}{(\omega^2+X^2)^n}\ln(X-i|\omega|) - \frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\bigg|_{Y=X}\frac{1}{\omega^2+Y^2}\ln(X+Y) \tag{5.95}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\, \frac{1}{k^2-\omega^2}\, \frac{1}{(X^2+k^2)^n}\, \frac{1}{k} \ln\Big(\frac{X-i\,k}{X+i\,k}\Big) =$$
$$\frac{1}{\omega^2}\frac{1}{(\omega^2+X^2)^n}\ln(X-i|\omega|) - \frac{1}{\omega^2}\frac{1}{X^{2n}}\ln(X)$$
$$+\frac{(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial(Y^2)^{n-1}}\bigg|_{Y=X}\frac{1}{\omega^2+Y^2}\frac{1}{Y^2}\ln(X+Y) \tag{5.96}$$

In the derivation of (5.93) and (5.94) I used the identity

$$L_2(-\frac{1}{y}) + L_2(-y) + \frac{1}{2}\ln^2(y) + \zeta(2) = 0\,, \tag{5.97}$$

to express $\ln^2\Big((X-i\,k)/(X+i\,k)\Big)$ as a sum of dilogarithms. Each dilogarithm has only one branch cut in the upper or lower half of the complex plane. Having the above formulae (5.87-5.96), I can do the k–integration. In the above expressions there appear $|\omega|$ and X, which needs some comments. The modulus function is defined by the square root

$$|\omega| = i\sqrt{-\omega^2} = \begin{cases} \omega & Im(\omega) > 0 \\ -\omega & Im(\omega) < 0 \end{cases} \tag{5.98}$$

and

$$X = \sqrt{X^2}\;, |\arg X| < \frac{\pi}{2}\,. \tag{5.99}$$

The square root and logarithm are assumed to have a branch cut on the negative real axis. It gives the correct analytical continuation of the result in the whole complex plane of the photon energy.

The contour C_H of the ω integration consists of two lines on the upper and the lower complex plane. Taking only the upper line, the final result is the sum of this integral and its complex conjugate. For the upper line I have $\mathrm{Im}(\omega) > 0$ and $|\omega| = \omega$. Let us denote by u the expression

$$u = \frac{X+i\,\omega}{X-i\,\omega}\,. \tag{5.100}$$

Then X and ω can be expressed as:

$$\omega = -\frac{1}{2}\frac{(1-u)^2}{u}\,, \tag{5.101}$$

$$X = \frac{i}{2}\frac{(1-u)(1+u)}{u}\,, \tag{5.102}$$

and Jacobian is

$$\frac{d\omega}{du} = \frac{1-u^2}{2u^2}\,. \tag{5.103}$$

After denoting by $\mathcal{F}(u)$,

$$\mathcal{F}(u) = \frac{d\omega}{du}\,\frac{1}{2\pi i}\int dk\;\mathcal{F}(\omega,k)\,\frac{k^2}{k^2-\omega^2}\,, \tag{5.104}$$

the result of k–integration is

$$\begin{aligned}
\mathcal{F}(u) = \alpha^{-2}&\Big[-\frac{5}{24} - \frac{1}{6\,(-1+u)^2} - \frac{1}{8\,u^2}\Big] + \Big[\frac{35}{96} + \frac{1}{24\,(-1+u)^4} \\
&+ \frac{1}{24\,(-1+u)^3} - \frac{103}{96\,(-1+u)^2} - \frac{4}{3\,(-1+u)} - \frac{1}{32\,u^2} - \frac{u}{6} - \frac{23}{32\,(1+u)^2}\Big] \\
&+\alpha\,\Big[\frac{2\,i}{3} - \frac{i}{6}\frac{1}{(-1+u)^5} - \frac{i}{4}\frac{1}{(-1+u)^4} + \frac{2\,i}{3}\frac{1}{(-1+u)^3} + \frac{89\,i}{24}\frac{1}{(-1+u)^2} \\
&+ \frac{123\,i}{32}\frac{1}{-1+u} + \frac{i}{4}\,u + \frac{9\,i}{8}\frac{1}{(1+u)^3} - \frac{9\,i}{16}\frac{1}{(1+u)^2} - \frac{43\,i}{32}\frac{1}{1+u} \\
&+ \Big(\frac{4\,i}{(-1+u)^3} + \frac{2\,i}{(-1+u)^2} - \frac{2\,i}{-1+u} + \frac{i}{u}\Big)\,\ln(1+u)\Big] \\
&+\alpha^2\,\Big[-\frac{9119}{14400} + \frac{233}{720\,(-1+u)^6} + \frac{233}{360\,(-1+u)^5} - \frac{2117}{2400\,(-1+u)^4} \\
&+ \frac{\pi^2}{2\,(-1+u)^4} + \frac{64279}{7200\,(-1+u)^3} + \frac{5\,\pi^2}{6\,(-1+u)^3} + \frac{385751}{57600\,(-1+u)^2} \\
&+ \frac{7\,\pi^2}{24\,(-1+u)^2} - \frac{1}{15\,(-1+u)} - \frac{\pi^2}{6\,(-1+u)} - \frac{33}{64\,u^2} - \frac{13}{12\,u} - \frac{59\,u}{225} \\
&+ \frac{1449}{320\,(1+u)^4} - \frac{1449}{320\,(1+u)^3} - \frac{9769}{3840\,(1+u)^2} - \frac{\pi^2}{8\,(1+u)^2} + \Big(\frac{2}{15} \\
&+ \frac{1}{3\,(-1+u)^6} + \frac{2}{3\,(-1+u)^5} - \frac{33}{5\,(-1+u)^4} - \frac{88}{5\,(-1+u)^3} - \frac{1099}{80\,(-1+u)^2} \\
&- \frac{4}{-1+u} - \frac{19\,u}{30} + \frac{9}{4\,(1+u)^4} - \frac{9}{4\,(1+u)^3} - \frac{17}{16\,(1+u)^2}\Big)\,\ln\Big(\frac{-2\,i\,\alpha\,u}{1-u}\Big) \\
&+ \Big(\frac{-1}{3\,(-1+u)^6} - \frac{2}{3\,(-1+u)^5} - \frac{3}{(-1+u)^4} - \frac{28}{3\,(-1+u)^3} + \frac{205}{48\,(-1+u)^2} \\
&- \frac{6}{-1+u} + \frac{1}{2\,u^3} + \frac{4}{3\,u^2} + \frac{7}{u} - \frac{9}{4\,(1+u)^4} + \frac{9}{4\,(1+u)^3} + \frac{17}{16\,(1+u)^2}\Big) \\
&\cdot \ln(1+u) + \Big(\frac{4}{(-1+u)^4} + \frac{8}{(-1+u)^3} + \frac{2}{(-1+u)^2} - \frac{2}{-1+u}\Big)\,L_2(-u)\Big]\,.
\end{aligned} \tag{5.105}$$

The final integral with respect to u is

$$F_H' = \int_{u_\Lambda}^{u_\epsilon} du\,\mathcal{F}(u) + c.c.\,, \tag{5.106}$$

where

$$u_\epsilon = \frac{\sqrt{2\epsilon-\epsilon^2}+i\,\epsilon}{\sqrt{2\epsilon-\epsilon^2}-i\,\epsilon} = 1 + 2i\sqrt{\frac{\epsilon}{2}} + \sqrt{\epsilon}\,O(\epsilon)\,, \tag{5.107}$$

$$\frac{1}{u_\Lambda} = \frac{\sqrt{2\Lambda - \Lambda^2} + i\,\Lambda}{\sqrt{2\Lambda - \Lambda^2} - i\,\Lambda} = -2\Lambda + 2 + O(\frac{1}{\Lambda}) \,. \tag{5.108}$$

According to previous assumptions $Im(\epsilon) > 0, Im(\Lambda) > 0$. I have introduced an upper cutoff Λ to make finite the integral of the separate terms in (5.105). After the u-integration there remains a linear divergence in Λ. This divergence originates from the change of the integration contour. The ω integration is not absolutely convergent. I mentioned in the introduction that the correct regularization is obtained, for example by the replacing of the photon propagator by

$$\frac{1}{k^2} \to \frac{1}{k^2} - \frac{1}{k^2 - M^2} \,.$$

This replacement is not applicable in this scheme of integration. Therefore I introduce the following division

$$\begin{aligned}
F_H &= \lim_{M\to\infty} \int_{C_H} d\omega \, \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \left(\frac{k^2}{\omega^2 - k^2} - \frac{k^2}{\omega^2 - k^2 - M^2} \right) \mathcal{F}(\omega, k) \\
&= \lim_{M\to\infty} \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} d\omega \, \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \left(\frac{k^2}{\omega^2 - k^2} - \frac{k^2}{\omega^2 - k^2 - M^2} \right) \mathcal{F}(\omega, k) \\
&= \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} d\omega \, \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \, \frac{k^2}{\omega^2 - k^2} \, \mathcal{F}(\omega, k) \\
&\quad - \lim_{M\to\infty} \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} d\omega \, \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \, \frac{k^2}{\omega^2 - k^2 - M^2} \, \mathcal{F}(\omega, k) \\
&= F'_H + F_{corr} \,.
\end{aligned} \tag{5.109}$$

The first term contains a symmetric integration over ω. This symmetry implies that in the complex conjugate term in (5.106), one should change also the sign of Λ. Thus, in the sum in (5.106), the divergence in Λ disappears. The second term in (5.109) denoted by F_{corr} is a correction that comes from the difference between the two ultraviolet regularizations. This term is calculated later in this section.

Coming back to the u-integration, I give below the table of formulae to integrate (5.105) with respect to u (more general integrals can be found in [39]).

$$\begin{aligned}
\int_0^{1-u_\epsilon} L_2(-u) &= -\frac{\pi^2}{12} + 2\ln(2) - 1 \\
\int_0^{1-u_\epsilon} \frac{\ln(u)}{1+u} &= -\frac{\pi^2}{12} \\
\int_0^{1-u_\epsilon} \frac{\ln(u)}{1-u} &= -\frac{\pi^2}{6} \\
\int_0^{1-u_\epsilon} \frac{\ln(1+u)}{u} &= \frac{\pi^2}{12} \\
\int_0^{1-u_\epsilon} \frac{\ln(1+u)}{1-u} &= -\frac{\pi^2}{12} + \frac{\ln^2(2)}{2} - \ln(2)\ln(u_\epsilon) \\
\int_0^{1-u_\epsilon} \frac{\ln(1-u)}{u} &= -\frac{\pi^2}{6} \\
\int_0^{1-u_\epsilon} \frac{\ln(1-u)}{1+u} &= -\frac{\pi^2}{12} + \frac{\ln^2(2)}{2} \\
\int_0^{1-u_\epsilon} \frac{L_2(-u)}{u} &= -\frac{3}{4}\zeta(3)
\end{aligned}$$

$$
\begin{aligned}
\int_0^{1-u_\epsilon} \frac{L_2(-u)}{1+u} &= \frac{\zeta(3)}{4} - \frac{\ln(2)\pi^2}{12} \\
\int_0^{1-u_\epsilon} \frac{L_2(-u)}{1-u} &= \frac{5}{8}\zeta(3) + \frac{\pi^2}{12}\ln(u_\epsilon)\,. \qquad (5.110)
\end{aligned}
$$

Together with the integration an expansion in u_ϵ is also performed. After the integration the result depends on u_ϵ. I replace u_ϵ by (5.107), expand in ϵ and leave only finite and divergent terms.

The last contribution to F_H that has not been calculated yet is F_{corr}.

$$
F_{corr} = -\lim_{M\to\infty}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} d\omega\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\,\frac{k^2}{\omega^2-k^2-M^2}\,\mathcal{F}(\omega,k)\,. \qquad (5.111)
$$

Extracting only divergent terms in $\mathcal{F}(\omega,k)$

$$
\begin{aligned}
Q_{16} &\sim \frac{1+\gamma}{\omega^2-k^2}\left(m+\frac{2m\,\omega^2\gamma}{\omega^2-k^2}\right) \\
Q_{18} &\sim -\frac{1-\gamma}{\omega^2-k^2}\left(m-\frac{2m\,\omega^2\gamma}{\omega^2-k^2}\right) \\
Q_{20} &\sim \frac{4}{3}\frac{m\,\alpha^2}{\gamma}\frac{k^2}{(\omega^2-k^2)^2} \\
Q_{21} &\sim -\frac{2\gamma}{\omega^2-k^2}\left(m+\frac{2m\,\omega^2}{\omega^2-k^2}\right) \\
Q_{22} &\sim \frac{4m\,\alpha^2}{\gamma}\frac{\omega^2}{(\omega^2-k^2)^2} \qquad (5.112)
\end{aligned}
$$

one has

$$
\delta \equiv Q_{16}+Q_{18}+Q_{20}+Q_{21}+Q_{22} = \frac{4}{3}\frac{\alpha^2}{\gamma}\,\frac{m(k^2+3\omega^2)}{(\omega^2-k^2)^2}\,. \qquad (5.113)
$$

So, that

$$
F_{corr} = -\lim_{M\to\infty}\lim_{\Lambda\to\infty}\int_{-\Lambda}^{\Lambda} d\omega\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\,\frac{k^2}{\omega^2-k^2-M^2}\,\frac{\delta}{m\,\alpha^4}\,. \qquad (5.114)
$$

After rescaling the variables $\omega\to M\,\omega$, $k\to Mk$, F_{corr} becomes

$$
\begin{aligned}
F_{corr} &= -\lim_{M\to\infty}\lim_{\Lambda\to\infty}\int_{-\Lambda/M}^{\Lambda/M} d\omega\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\,\frac{k^2}{\omega^2-k^2-1}\frac{4}{3}\frac{1}{\alpha^2\gamma}\frac{k^2+3\omega^2}{(\omega^2-k^2)^2} \\
&= -\frac{4}{3\alpha^2\gamma}\int_{-\infty}^{\infty} d\omega\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk\,\frac{k^2}{\omega^2-k^2-1}\frac{k^2+3\omega^2}{(\omega^2-k^2)^2} \\
&= \frac{1}{2\alpha^2\gamma} = \frac{1}{2\alpha^2}+\frac{1}{4}+\frac{3}{16}\alpha^2+O(\alpha)^4\,. \qquad (5.115)
\end{aligned}
$$

Finally, I obtain for the high energy part the expression:

$$
\begin{aligned}
F_H = {}& \frac{10}{9} - \frac{4\,\ln(2)}{3} - \frac{4\,\ln(\epsilon)}{3} + \alpha\left(\frac{-16\sqrt{2}}{3\sqrt{\epsilon}} + \frac{139\,\pi}{32} - 2\,\pi\,\ln(2)\right) \\
& +\alpha^2\,\ln(\alpha)\left(\frac{163}{30} - \frac{16}{3\,\epsilon} - 4\,\ln(2) - 4\,\ln(\epsilon)\right) \\
& +\alpha^2\left(-\frac{121}{60} + \frac{5\,\zeta(3)}{2} - 3\ln^2(2) - \frac{5\,\pi^2}{18} - \frac{61\,\ln(2)}{90}\right. \\
& \left. -\frac{8\,\ln(2)}{\epsilon} + \frac{16}{3\,\epsilon} - \frac{5\,\ln(\epsilon)}{3} + \frac{8\,\ln(\epsilon)}{3\,\epsilon} - \frac{22\,\ln(2)\,\ln(\epsilon)}{3} + \ln^2(\epsilon)\right)\,. \qquad (5.116)
\end{aligned}
$$

As expected, all spurious (i.e. α^{-2}) order terms disappeared. The dependence on ϵ will be canceled with the high energy part. This canceling is a check of the calculation of both F_H and F_L.

6. 1S low–energy part

The low–energy part is given by the following expression (3.7)

$$E_L = e^2 \int_{|\vec{k}|<\epsilon} d^3\vec{k}\, \frac{1}{(2\pi)^3 2k} \left(\delta^{ij} - \frac{k^i k^j}{k^2}\right)$$
$$\langle\overline{\psi}|\gamma^i\, e^{i\,\vec{k}\cdot\vec{r}} \frac{1}{\not p - \omega\,\gamma^0 - m + \frac{\alpha}{r}\,\gamma^0}\, e^{-i\,\vec{k}\cdot\vec{r}}\,\gamma^j\,|\psi\rangle, \tag{6.1}$$

As I have shown, in the expansion in $(i\,\vec{k}\cdot\vec{r})$ only the first two terms the dipole and quadrupole, contribute in order $(Z\alpha)^6$, provided that after the expansion in α the expansion in ϵ is also performed.

6.1 The nonrelativistic dipole contribution

I start the evaluation from the simplest case, i.e. from the nonrelativistic dipole contribution E_{nd}.

$$E_{nd} = -\frac{2\alpha}{3\pi m^2} \int_0^\epsilon \omega d\omega\, \langle\phi|\,\vec{p}\,\frac{1}{H-\Omega}\,\vec{p}\,|\phi\rangle, \tag{6.2}$$

where

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{\alpha}{r}, \quad \Omega = E - \omega, \quad \text{and} \quad E = -\frac{m\alpha^2}{2}\,.$$

The contribution in the nonrelativistic perturbation calculation, originating from the term $A^2/(2m)$, vanishes for $\epsilon = 0$. The matrix element

$$\langle\phi|\,\vec{p}\,(H-\Omega)^{-1}\,\vec{p}\,|\phi\rangle$$

has been calculated for the $1S$ state by Gavrila and Costescu in [40], and I present this evaluation in appendix B. Later in this section I present a different evaluation that is suited for the relativistic case. In Gavrila and Costescu notation

$$P_{nd}(\Omega) = \frac{1}{3m}\,\langle\phi|\,\vec{p}\,\frac{1}{H-\Omega}\,\vec{p}\,|\phi\rangle$$
$$= 128\frac{\tau^5}{(1+\tau)^8(2-\tau)}\;{}_2F_1(4, 2-\tau, 3-\tau; \xi)\,, \tag{6.3}$$

where

$$\xi = \left(\frac{1-\tau}{1+\tau}\right)^2, \quad \tau = \frac{\lambda}{X}, \quad \lambda = m\,\alpha, \quad X = \sqrt{-2m\Omega}, \tag{6.4}$$

and ${}_2F_1$ is the hypergeometric function [38]. This function will appear in this section quite often. For $\omega = \epsilon$ one has

$$\tau_\epsilon = \frac{\lambda}{\sqrt{-2m(E-\epsilon)}}. \tag{6.5}$$

For reasons, that will be clear later I redefine τ_ϵ to take into account some relativistic corrections

$$\tau_\epsilon = \frac{\alpha}{\sqrt{1-(\gamma-\frac{\epsilon}{m})^2}}, \tag{6.6}$$

where $\gamma = \sqrt{1-\alpha^2}$. The nonrelativistic dipole contribution to the energy shift E_{nd} becomes

$$
\begin{aligned}
E_{nd} &= -\frac{2\alpha}{\pi m}\int_0^{\epsilon} \omega d\omega \, P_{nd} \\
&= -\frac{2\alpha}{\pi m} E^2 \int_{\tau_\epsilon}^{1} d\tau \, (\frac{1}{\tau^2}-1)\,\frac{2}{\tau^3}\,P_{nd} = \frac{\alpha(m\alpha^4)}{\pi} F_{nd}\,, \qquad (6.7)
\end{aligned}
$$

where

$$
\begin{aligned}
F_{nd} &= -\int_{\tau_\epsilon}^{1} d\tau \, \frac{1-\tau^2}{\tau^5}\,P_{nd} \qquad (6.8)\\
&= -\int_{\tau_\epsilon}^{1} d\tau \, \frac{128(1-\tau)}{(1+\tau)^7(2-\tau)}\,{}_2F_1(4,2-\tau,3-\tau;\xi)\,. \qquad (6.9)
\end{aligned}
$$

The evaluation of this integral proceeds as follows. First I separate out the terms divergent at $\tau_\epsilon = 0$. The contiguous relation [38]

$$
{}_2F_1(a+1,b,c;z) = \frac{c-1}{a}\,{}_2F_1(a,b,c-1;z) - \frac{c-a-1}{a}\,{}_2F_1(a,b,c;z) \qquad (6.10)
$$

gives

$$
{}_2F_1(n+1,2-\tau,3-\tau;\xi) = \frac{2-\tau}{n}(1-\xi)^{-n} + \frac{\tau+n-2}{n}\,{}_2F_1(n,2-\tau,3-\tau;\xi) \qquad (6.11)
$$

Using this relation repeatedly one obtains,

$$
\begin{aligned}
F_{nd} = &-\int_{\tau_\epsilon}^{1} d\tau \, \frac{128(1-\tau)}{(1+\tau)^7}\left(\frac{(1-\xi)^{-3}}{3} + \frac{(1+\tau)}{6}(1-\xi)^{-2}\right.\\
&\left.+\frac{\tau(1+\tau)}{6}(1-\xi)^{-1} + \frac{\tau(1+\tau)(\tau-1)}{6}\,\frac{{}_2F_1(1,2-\tau,3-\tau;\xi)}{2-\tau}\right). \qquad (6.12)
\end{aligned}
$$

The hypergeometric function in the last line is a simple series

$$
\begin{aligned}
\frac{{}_2F_1(1,2-\tau,3-\tau;\xi)}{2-\tau} &= \frac{1}{2-\tau} + \frac{\xi}{3-\tau} + \frac{\xi^2}{4-\tau} + \cdots \\
&= -\frac{\ln(1-\xi)+\xi}{\xi^2} + \tau f(\tau)\,, \qquad (6.13)
\end{aligned}
$$

where

$$
f(\tau) = \frac{1}{2(2-\tau)} + \frac{1}{3(3-\tau)}\xi + \frac{1}{4(4-\tau)}\xi^2 + \cdots \qquad (6.14)
$$

The logarithmic singularity at $\tau = 0$ has been separated out. After the τ integration, an expansion in α up to second order and after that an expansion in ϵ should be done. Since ($\epsilon \to m\,\epsilon$)

$$
\tau_\epsilon = \frac{\alpha}{\sqrt{2\epsilon-\epsilon^2}} + O(\alpha)^3\,,
$$

it is sufficient to expand $F_{nd}(\tau_\epsilon)$ up to second order in τ_ϵ, so, that

$$
\begin{aligned}
F_{nd} &= D_r + D_s\,, \qquad (6.15)\\
D_s &= -\int_{\tau_\epsilon}^{1} d\tau \, \frac{128(1-\tau)}{(1+\tau)^7}\left(\frac{(1-\xi)^{-3}}{3} + \frac{(1+\tau)}{6}(1-\xi)^{-2}\right.\\
&\left.+\frac{\tau(1+\tau)}{6}(1-\xi)^{-1} + \frac{\tau(1+\tau)(1-\tau)}{6}\,\frac{\ln(1-\xi)+\xi}{\xi^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{83}{9} - \frac{1}{3\,{\tau_\epsilon}^2} + \frac{32\,\tau_\epsilon}{3} - 12\,{\tau_\epsilon}^2 + 8\,\ln(2) \\
&\quad + \frac{64\,{\tau_\epsilon}^2\,\ln(2)}{3} - \frac{8\,\ln(\tau_\epsilon)}{3} + \frac{32\,{\tau_\epsilon}^2\,\ln(\tau_\epsilon)}{3}\,, \qquad (6.16)
\end{aligned}
$$

$$
D_r = \int_0^1 d\tau\,\frac{64}{3}\,\frac{\tau^2(1-\tau)^2}{(1+\tau)^6}\,f(\tau) = 0.0315400367\,. \qquad (6.17)
$$

The expansion in α and then in ϵ is equivalent to the following table of replacements,

$$
\begin{aligned}
\frac{1}{{\tau_\epsilon}^2} &\to 1 \\
\frac{1}{\tau_\epsilon} &\to \frac{\alpha}{\sqrt{8\epsilon}} \\
\tau_\epsilon &\to \frac{\alpha}{\sqrt{2\epsilon}} \\
{\tau_\epsilon}^2 &\to \alpha^2\Big(\frac{1}{4} + \frac{1}{2\epsilon}\Big) \qquad (6.18) \\
\ln(\tau_\epsilon) &\to \frac{1}{2}\ln\Big(\frac{\alpha^2}{2\epsilon}\Big) + \alpha^2\Big(\frac{1}{8} - \frac{1}{4\epsilon}\Big) \\
{\tau_\epsilon}^2\ln(\tau_\epsilon) &\to \frac{\alpha^2}{8}\Big(1 + \Big(1 + \frac{2}{\epsilon}\Big)\ln\Big(\frac{\alpha^2}{2\epsilon}\Big)\Big)\,.
\end{aligned}
$$

Applying these rules I obtain

$$
\begin{aligned}
D_s &= -\frac{86}{9} + \frac{28\,\ln(2)}{3} - \frac{8\,\ln(\alpha)}{3} + \frac{4\,\ln(\epsilon)}{3} + \frac{16\sqrt{2}\,\alpha}{3\,\sqrt{\epsilon}} + \alpha^2\,\ln(\alpha)\Big(\frac{8}{3} + \frac{16}{3\,\epsilon}\Big) \\
&\quad + \alpha^2\left(-2 - \frac{16}{3\,\epsilon} + 4\,\ln(2) + \frac{8\,\ln(2)}{\epsilon} - \frac{4\,\ln(\epsilon)}{3} - \frac{8\,\ln(\epsilon)}{3\,\epsilon}\right). \qquad (6.19)
\end{aligned}
$$

For the Bethe logarithm ($A_{40} = \frac{10}{9} - \frac{4}{3}\ln k_0(1S)$) I obtain from the above and from (5.116), the expression:

$$
\ln k_0(1S) = \frac{43}{6} - 6\ln(2) - \frac{3}{4}D_r\,, \qquad (6.20)
$$

which exactly coincides with (2.11).

The evaluation of the quadrupole and relativistic dipole terms proceeds in a similar way. The main difference is in the regular part, which is more complicated, but it can be calculated with the desired precision (below 1 Hz).

6.2 The quadrupole contribution

The quadrupole contribution can be obtained from [40], where the exact nonrelativistic matrix element for the elastic scattering of a photon by H–atom was calculated.

$$
\begin{aligned}
P(\Omega) &= \frac{1}{2m}\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right)\langle\phi|e^{i\,\vec{k}\cdot\vec{r}}\,\vec{p}_i\,\frac{1}{H-\Omega}\vec{p}_j\,e^{-i\,\vec{k}\cdot\vec{r}}|\phi\rangle \\
&= 128\,\frac{\lambda^5 X^3}{[(X+\lambda)^2+k^2]^4}\,\frac{{}_2F_1(4, 2-\tau, 3-\tau; u)}{2-\tau}\,, \qquad (6.21)
\end{aligned}
$$

where λ, X, τ were defined before in (6.4), and

$$
u = \frac{(X-\lambda)^2+k^2}{(X+\lambda)^2+k^2} = \Big(\frac{X-\lambda}{X+\lambda}\Big)^2 + k^2\frac{4X\lambda}{(X+\lambda)^4} + O(k)^4\,. \qquad (6.22)
$$

Expanding $P(\Omega)$ in k^2 one obtains

$$P^{(0)}(\Omega) = P_{nd}(\Omega)\,,$$

$$P^{(2)}(\Omega) = P_q(\Omega) = k^2 \frac{\partial}{\partial k^2}\bigg|_{k^2=0}$$

$$128\,\frac{\lambda^5 X^3}{[(X+\lambda)^2+k^2]^4}\,\frac{1}{2-\tau}\,{}_2F_1\Big(4, 2-\tau, 3-\tau; \xi + k^2\frac{4X\lambda}{(X+\lambda)^4}\Big)\,. \quad (6.23)$$

Using the identity

$${}_2F_1'(4, 2-\tau, 3-\tau, \xi) = \frac{2-\tau}{\xi}(1-\xi)^{-4} - \frac{2-\tau}{\xi}\,{}_2F_1(4, 2-\tau, 3-\tau; \xi), \quad (6.24)$$

putting $k=\omega$, and changing the variables

$$\begin{aligned}\frac{\lambda}{X} &= \tau\,,\\ \frac{k^2}{\lambda^2} &= \frac{\alpha^2}{4}\left(\frac{1}{\tau^2}-2\right)^2,\end{aligned}$$

I obtain

$$P_q = \alpha^2\left(-128\frac{\tau^3}{(1+\tau)^8(2-\tau)}\,{}_2F_1(4, 2-\tau, 3-\tau, \xi) + \frac{1}{2}\right). \quad (6.25)$$

After using (6.8) F_q becomes

$$\begin{aligned}F_q &= -\int_{\tau_\epsilon}^{1} d\tau\,\frac{1-\tau^2}{\tau^5}P_q\\ &= \alpha^2\int_{\tau_\epsilon}^{1} d\tau\left(128\frac{1-\tau}{\tau^2(1+\tau)^7(2-\tau)}\,{}_2F_1(4, 2-\tau, 3-\tau; \xi) - \frac{1-\tau^2}{2\tau^5}\right). \quad (6.26)\end{aligned}$$

The evaluation proceeds as in the dipole case

$$F_q = E_r + E_s \quad (6.27)$$

$$E_s = \frac{\alpha^2}{2}\int_{\tau_\epsilon}^{1} d\tau\left(256\frac{1-\tau}{\tau^2(1+\tau)^7}\left\{\frac{(1-\xi)^{-3}}{3} + \frac{(1+\tau)}{6}(1-\xi)^{-2}\right.\right.$$
$$\left.\left. + \frac{\tau(1+\tau)}{6}(1-\xi)^{-1} + \frac{\tau(1+\tau)(1-\tau)}{6}\,\frac{\ln(1-\xi)+\xi}{\xi^2}\right\} - \frac{1-\tau^2}{\tau^5}\right) \quad (6.28)$$

$$E_r = -\alpha^2\int_0^1 d\tau\,\frac{64}{3}\frac{(1-\tau)^2}{(1+\tau)^6}\,f(\tau) \quad (6.29)$$

Using the table of replacement, that is equivalent to the expansion in α and ϵ,

$$\begin{aligned}\frac{1}{{\tau_\epsilon}^4} &\to 1\\ \frac{1}{{\tau_\epsilon}^3} &\to 0\\ \frac{1}{{\tau_\epsilon}^2} &\to 1\\ \frac{1}{\tau_\epsilon} &\to 0 \quad (6.30)\\ \ln(\tau_\epsilon) &\to \frac{1}{2}\ln\Big(\frac{\alpha^2}{2\epsilon}\Big)\\ \ln^2(\tau_\epsilon) &\to \frac{1}{4}\ln^2\Big(\frac{\alpha^2}{2\epsilon}\Big)\,,\end{aligned}$$

I obtain for E_s the following expression

$$E_s = \alpha^2 \Bigg(-\frac{82}{9} - \frac{8\,\pi^2}{3} - 4\,\ln(2) - \frac{8\ln^2(2)}{3} - \frac{32\ln^2(\alpha)}{3} - \frac{20\,\ln(\epsilon)}{3}$$

$$+ 16\,\ln(2)\,\ln(\epsilon) - \frac{8\ln^2(\epsilon)}{3} + \ln(\alpha)\left(\frac{40}{3} - 32\,\ln(2) + \frac{32\,\ln(\epsilon)}{3}\right)\Bigg) . \qquad (6.31)$$

The expression for E_r (6.29) is similar to D_r, and is calculated numerically together with the relativistic dipole part.

6.3 The relativistic dipole contribution

The dipole contribution to the energy shift is,

$$E_L^d = -\frac{2\alpha}{3\pi m} \int_0^\epsilon \omega\, d\omega\, m\, \langle \psi | \alpha^i G(E-\omega) \alpha^i | \psi \rangle \,, \qquad (6.32)$$

where

$$G(E-\omega) = \frac{1}{H - (E-\omega)} \,, \qquad (6.33)$$

and

$$H = \vec{\alpha} \cdot \vec{p} + \beta m - \frac{\alpha}{r} \,, \quad E = m\gamma \,. \qquad (6.34)$$

The term under the integral in (6.32) has a clear physical interpretation. It is proportional to the relativistic polarizability of an H-atom. Because of the selection rules, only the states with angular momentum $j = 1/2$, $j = 3/2$ contribute.

$$m\langle \psi | \alpha^i G(E-\omega) \alpha^i | \psi \rangle = P_{1/2} + P_{3/2} \,. \qquad (6.35)$$

Although the relativistic polarizability of an H-atom has been calculated before, the published results contain errors [41] , and I had to perform this calculation independently. (In the meantime I have published the correct result [33].) My result is

$$P_{1/2} = -\frac{\tau(2\gamma+1)}{3(\nu+\gamma\tau)} \Bigg\{ (2\gamma+1) - \frac{4}{\gamma-\nu+1} \left(\frac{4\tau}{(1+\tau)^2} \right)^{2\gamma+1}$$

$$\cdot \left(\frac{\tau(1+\gamma)-\nu}{1+\tau} \right)^2 {}_2F_1\Big(2\gamma+2,\ \gamma+1-\nu,\ \gamma+2-\nu;\ (\frac{1-\tau}{1+\tau})^2 \Big) \Bigg\} , \qquad (6.36)$$

$$P_{3/2} = \frac{1}{3} \tau^{2\gamma+2} \,(\frac{1+\tau}{2})^{-2(\gamma+\gamma'+1)} \left(\frac{\tau+\nu}{1+\gamma} \right) \frac{\Gamma^2(\gamma+\gamma'+1)}{\Gamma(2\gamma+1)\Gamma^2(2\gamma'+1)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\Gamma(2\gamma'+n)}{(\gamma'+n-\nu)n!} \Bigg\{ F^2(-n)\,(2\gamma'+n)(2+\tau+\gamma'+n-\nu)$$

$$+ F^2(-n+1)\, n(-2+\tau+\gamma'+n-\nu) - 2n(2\gamma'+n)\, F(-n)\, F(-n+1) \Bigg\} , \qquad (6.37)$$

where

$$\gamma = \sqrt{1-\alpha^2},\ \gamma' = \sqrt{4-\alpha^2},\ E = m\gamma,\ \kappa = \sqrt{m^2-(E-\omega)^2},$$
$$\tau = \frac{m\alpha}{\kappa},\ \nu = \frac{\alpha(E-\omega)}{\kappa} = \sqrt{\tau^2-\alpha^2}, \qquad (6.38)$$
$$\text{and}\ \ F(-n) = {}_2F_1\big(-n,\ \gamma+\gamma'+1,\ 2\gamma'+1;\ 2/(1+\tau)\big).$$

The first part of this section is devoted to the evaluation of $P_{1/2}$ and $P_{3/2}$. In this evaluation I rely on a paper of [35], where a suitable form of the Coulomb–Green function was given. For convenience I assume $\mathcal{E} = E - \omega$. The Dirac–Coulomb resolvent

$$(H - \mathcal{E})\, G(\vec{r}_1, \vec{r}_2, \mathcal{E}) = \delta^3(\vec{r}_1 - \vec{r}_2)\,, \tag{6.39}$$

can be written as follows

$$G(\vec{r}_1, \vec{r}_2, \mathcal{E}) =$$
$$\sum_{k\,\mu} \begin{pmatrix} g_k^{11}(r_1, r_2)\, \chi_k^{\mu}(\hat{r}_1)\chi_k^{\mu +}(\hat{r}_2) & -ig_k^{12}(r_1, r_2)\chi_k^{\mu}(\hat{r}_1)\chi_{-k}^{\mu +}(\hat{r}_2) \\ ig_k^{21}(r_1, r_2)\chi_{-k}^{\mu}(\hat{r}_1)\chi_k^{\mu +}(\hat{r}_2) & g_k^{22}(r_1, r_2)\, \chi_{-k}^{\mu}(\hat{r}_1)\chi_{-k}^{\mu +}(\hat{r}_2) \end{pmatrix}, \tag{6.40}$$

where

$$\chi_k^{\mu}(\theta, \phi) = \begin{pmatrix} -\frac{k}{|k|}\left(\frac{k+1/2-\mu}{2k+1}\right)^{1/2} Y_{|k+\frac{1}{2}|-\frac{1}{2}, \mu-\frac{1}{2}}(\theta, \phi) \\ \left(\frac{k+1/2+\mu}{2k+1}\right)^{1/2} Y_{|k+\frac{1}{2}|-\frac{1}{2}, \mu+\frac{1}{2}}(\theta, \phi) \end{pmatrix}. \tag{6.41}$$

and

$$\vec{\sigma} \cdot \hat{r}\, \chi_k^{\mu} = -\chi_{-k}^{\mu}\,, \tag{6.42}$$

$$(\vec{\sigma} \cdot \vec{L} + 1)\chi_k^{\mu} = -k\, \chi_k^{\mu}\,. \tag{6.43}$$

The matrix

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

fulfills the following differential equation.

$$\begin{pmatrix} \left(-\mathcal{E} + m - \dfrac{\alpha}{r_1}\right) & \left(-\dfrac{1}{r_1}\dfrac{d}{dr_1}r_1 + \dfrac{k}{r_1}\right) \\ \left(\dfrac{1}{r_1}\dfrac{d}{dr_1}r_1 + \dfrac{k}{r_1}\right) & \left(-\mathcal{E} - m - \dfrac{\alpha}{r_1}\right) \end{pmatrix} \begin{pmatrix} g_k^{11}(r_1, r_2) & g_k^{12}(r_1, r_2) \\ g_k^{21}(r_1, r_2) & g_k^{22}(r_1, r_2) \end{pmatrix}$$
$$= \frac{\delta(r_1 - r_2)}{r_1 r_2}\,. \tag{6.44}$$

The solution is (I suppress the k index):

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} =$$
$$\left(\frac{k+\gamma}{2\gamma}\right)^2 \begin{pmatrix} 1 & \frac{k-\gamma}{\alpha} \\ \frac{k-\gamma}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} h^{11} & h^{12} \\ h^{21} & h^{22} \end{pmatrix} \begin{pmatrix} 1 & \frac{k-\gamma}{\alpha} \\ \frac{k-\gamma}{\alpha} & 1 \end{pmatrix} \tag{6.45}$$

and

$$h^{11}(r_1, r_2) = 2\left(\frac{Ek + m\gamma}{k+\gamma}\right)(2\kappa)^{2\gamma+1}(r_1 r_2)^{\gamma} e^{-\kappa(r_1+r_2)}$$
$$\cdot \sum_{n=0}^{\infty} \frac{1}{n+1+\gamma-\nu}\, \frac{n!}{\Gamma(2\gamma+2+n)}\, L_n^{2\gamma+1}(2\kappa r_1)\, L_n^{2\gamma+1}(2\kappa r_2)\,, \tag{6.46}$$

$$h^{22}(r_1, r_2) = 2\left(\frac{Ek - m\gamma}{k+\gamma}\right)(2\kappa)^{2\gamma-1}(r_1 r_2)^{\gamma-1}\, e^{-\kappa(r_1+r_2)}$$
$$\cdot \sum_{n=0}^{\infty} \frac{1}{n+\gamma-\nu}\, \frac{n!}{\Gamma(2\gamma+n)}\, L_n^{2\gamma-1}(2\kappa r_1)\, L_n^{2\gamma-1}(2\kappa r_2)\,, \tag{6.47}$$

$$h^{21}(r_1,r_2)=h^{12}(r_2,r_1)=\frac{1}{k+\gamma}\biggl(\frac{\delta(r_1-r_2)}{\sqrt{r_1r_2}}+(2\kappa)^{2\gamma+1}r_1^{\gamma-1}r_2^{\gamma}$$
$$\cdot e^{-\kappa(r_1+r_2)}\sum_{n=0}^{\infty}\frac{1}{n+1+\gamma-\nu}\,\frac{(n+1)!}{\Gamma(2\gamma+1+n)}\,L_{n+1}^{2\gamma-1}(2\kappa r_1)\,L_n^{2\gamma+1}(2\kappa r_2)\biggr),\qquad(6.48)$$

where $\gamma=\sqrt{k^2-\alpha^2}$. The S–states have index $k=-1$. After performing the angular integrals, I obtain

$$\begin{aligned}P_{3/2} &= m\langle\psi_{-1}|\alpha^i G_{-2}(\mathcal{E})\alpha^i|\psi_{-1}\rangle\\ &= \frac{8}{3}m\int_0^\infty r_1^2\,dr_1\int_0^\infty r_2^2\,dr_2\,f_2(r_1)\,g_{-2}^{11}(r_1,r_2)\,f_2(r_2)\,, \qquad(6.49)\\ P_{1/2} &= m\langle\psi_{-1}|\alpha^i G_1(\mathcal{E})\alpha^i|\psi_{-1}\rangle =\end{aligned}$$
$$m\int_0^\infty r_1^2\,dr_1\int_0^\infty r_2^2\,dr_2\biggl(\frac{1}{3}\,f_2(r_1)\,g_1^{11}(r_1,r_2)\,f_2(r_2)+f_2(r_1)\,g_1^{12}(r_1,r_2)\,f_1(r_2)$$
$$+f_1(r_1)\,g_1^{21}(r_1,r_2)\,f_2(r_2)+3\,f_1(r_1)\,g_1^{22}(r_1,r_2)\,f_2(r_2)\biggr)\,,\qquad(6.50)$$

where f_1, f_2 are defined by

$$\psi=\begin{pmatrix} f_1\,\chi_k^\mu\\ if_2\,\chi_{-k}^\mu\end{pmatrix}\,.\qquad(6.51)$$

Upon introducing g_1, g_2, defined by

$$\begin{pmatrix} g_1\\ g_2\end{pmatrix}=\begin{pmatrix} 1 & (1+\gamma)/\alpha\\ (1+\gamma)/\alpha & 1\end{pmatrix}\begin{pmatrix} f_1\\ f_2\end{pmatrix}\,,\qquad(6.52)$$

$P_{1/2}, P_{3/2}$ can be transformed to the form suited for the next integration

$$P_{1/2}=m\int_0^\infty r_1^2\,dr_1\int_0^\infty r_2^2\,dr_2\,\frac{\alpha^2}{12\gamma^4}\,(g_1(r_1),g_2(r_1))\begin{pmatrix} 1-2\gamma & \alpha\\ -\alpha & -(1+2\gamma)\end{pmatrix}$$
$$\cdot\begin{pmatrix} h_1^{11}(r_1,r_2) & h_1^{12}(r_1,r_2)\\ h_1^{21}(r_1,r_2) & h_1^{22}(r_1,r_2)\end{pmatrix}\begin{pmatrix} 1-2\gamma & -\alpha\\ \alpha & -(1+2\gamma)\end{pmatrix}\begin{pmatrix} g_1(r_2)\\ g_2(r_2)\end{pmatrix},\qquad(6.53)$$

$$P_{3/2} = m\int_0^\infty r_1^2\,dr_1\int_0^\infty r_2^2\,dr_2\,\frac{\alpha^4}{6\gamma^2\gamma'^2(1+\gamma)(2+\gamma')}\,(g_1(r_1),g_2(r_1))$$
$$\cdot\begin{pmatrix} -\sqrt{(1+\gamma)(2-\gamma')} & \sqrt{(1+\gamma)(2+\gamma')}\\ \sqrt{(1-\gamma)(2-\gamma')} & -\sqrt{(1-\gamma)(2+\gamma')}\end{pmatrix}\begin{pmatrix} h_{-2}^{11}(r_1,r_2) & h_{-2}^{12}(r_1,r_2)\\ h_{-2}^{21}(r_1,r_2) & h_{-2}^{22}(r_1,r_2)\end{pmatrix}$$
$$\cdot\begin{pmatrix} -\sqrt{(1+\gamma)(2-\gamma')} & \sqrt{(1-\gamma)(2-\gamma')}\\ \sqrt{(1+\gamma)(2+\gamma')} & -\sqrt{(1-\gamma)(2+\gamma')}\end{pmatrix}\begin{pmatrix} g_1(r_2)\\ g_2(r_2)\end{pmatrix}.\qquad(6.54)$$

For 1S–state

$$\begin{aligned} g_1 &= 0\,, \qquad(6.55)\\ g_2 &= \frac{4\gamma\,\lambda^{3/2}}{\sqrt{\Gamma(1+2\gamma)(1-\gamma)}}(2\lambda r)^{\gamma-1}e^{-\lambda r}\,. \qquad(6.56)\end{aligned}$$

The integrals with respect to r_1, r_2 can be done by means of the integration formula [42]

$$\int_0^\infty dt\, e^{-st}\, t^{\gamma-1}\, L_n^\mu(t) = \frac{\Gamma(\gamma)\Gamma(n+\mu+1)}{n!\,\Gamma(\mu+1)} s^{-\gamma}\; {}_2F_1\Big(-n, \gamma, 1+\mu; \frac{1}{s}\Big)\,. \quad (6.57)$$

In the case of $P_{1/2}$, the hypergeometric function, that arises from the r–integration has the same second and third parameter.

$$ {}_2F_1(-n, b, b, z) = (1-z)^{-n}. \quad (6.58)$$

Thus, the k-series in (6.46), (6.47), and (6.48) can be summed up, leading to another hypergeometric function and to the expression (6.36). In the case of $P_{3/2}$ several hypergeometric functions with different parameters appear after the integration. Using the contiguous relations [38] they can be reduced to two hypergeometric functions, giving the expression (6.37). This calculation is quite lengthly, the problem being the rearrangement of all terms and their simplification. As a test of correctness of $P_{1/2}$ and $P_{3/2}$ I considered the nonrelativistic limit that coincides with the Gavrila–Costescu result, and the identity $P_{1/2} + P_{3/2} = 0$ that holds at E equal to ground state energy.

I proceed now to the evaluation of the relativistic dipole $j = 1/2$ contribution.

$$\Delta E_{1/2} = -\frac{2\alpha}{3\pi m}\int_0^\epsilon \omega\, d\omega\, P_{1/2}\,, \quad (6.59)$$

It is convenient to change the integration variable ω to τ (defined in (6.38)).

$$\tau = \frac{\lambda}{\sqrt{m^2 - (E-\omega)^2}}\,. \quad (6.60)$$

So, that

$$\omega = m\alpha^2 \frac{(1-\tau^2)}{\tau(\nu+\gamma\tau)}\,, \quad (6.61)$$

$$d\omega = -\frac{m\alpha^2}{\nu\tau^2}\, d\tau\,. \quad (6.62)$$

Thus,

$$\begin{aligned} E_{1/2} &= \alpha\frac{m\alpha^4}{\pi} F_{1/2}\,, & (6.63)\\ F_{1/2} &= -\frac{2}{3}\int_{\tau_\epsilon}^1 d\tau\, \frac{1-\tau^2}{\nu\tau^3(\nu+\gamma\tau)}\, P_{1/2} & (6.64)\\ &= \frac{2}{9}\int_{\tau_\epsilon}^1 d\tau\, \frac{(2\gamma+1)(1-\tau^2)}{\nu\tau^2(\nu+\gamma\tau)^2}\Bigg[(2\gamma+1) - \frac{4}{\gamma-\nu+1}\left(\frac{\tau(1+\gamma)-\nu}{1+\tau}\right)^2 & \\ &\quad\cdot\left(\frac{4\tau}{(1+\tau)^2}\right)^{2\gamma+1} {}_2F_1(2\gamma+2, \gamma+1-\nu, \gamma+2-\nu; \xi)\Bigg]\,, & (6.65) \end{aligned}$$

where τ_ϵ is defined in (6.6). I have to remove the nonrelativistic contribution, which has been calculated before . It is obtained by putting $\nu = \tau$, and $\gamma = 1$. It is important that one can expand in α^2 and take only the α^2 coefficient of the expression under the integral in (6.65), before the integration is performed. Higher terms in the α^2 expansion do not contribute in the desired order, provided that the expansion in ϵ is performed. It is convenient before an expansion in α^2 to separate out the term singular in $\tau_\epsilon = 0$. With the use of the contiguous relation [38]

$$ {}_2F_1(a, c-1, c, \xi) = \frac{c-1}{a-1}(1-\xi)^{-a+1} + \frac{a-c}{a-1}\, {}_2F_1(a-1, c-1, c; \xi) \quad (6.66)$$

I obtain

$$\frac{{}_2F_1(2\gamma+2,\gamma+1-\nu,\gamma+2-\nu;\xi)}{\gamma+1-\nu} =$$

$$\frac{(1-\xi)^{-(2\gamma+1)}}{2\gamma+1} + \frac{(\gamma+\nu)}{(2\gamma+1)2\gamma}(1-\xi)^{-2\gamma} + \frac{(\gamma+\nu)(\gamma+\nu-1)}{(2\gamma+1)2\gamma(2\gamma-1)}(1-\xi)^{-2\gamma+1}$$

$$+\frac{(\gamma+\nu)(\gamma+\nu-1)(\gamma+\nu-2)}{(2\gamma+1)2\gamma(2\gamma-1)} \cdot \frac{{}_2F_1(2\gamma-1,\gamma+1-\nu,\gamma+2-\nu;\xi)}{\gamma+1-\nu}. \quad (6.67)$$

Since

$$\nu = \tau\Big(1-\frac{\alpha^2}{2\tau^2}\Big) + O(\alpha^4) \text{ and } \gamma = 1-\frac{\alpha^2}{2}+O(\alpha^4)$$

the hypergeometric function in the last expression should also be expanded in parameters.

$$\frac{{}_2F_1(2\gamma-1,\gamma+1-\nu,\gamma+2-\nu;\xi)}{\gamma+1-\nu} =$$

$$\frac{1}{2-\tau-\frac{\alpha^2}{2}(1-\frac{1}{\tau})}\,{}_2F_1\Big(1-\alpha^2, 2-\tau-\frac{\alpha^2}{2}(1-\frac{1}{\tau}), 3-\tau-\frac{\alpha^2}{2}(1-\frac{1}{\tau});\xi\Big)$$

$$= -\frac{\ln(1-\xi)+\xi}{\xi^2} + \tau f(\tau) - \alpha^2 h(\tau) + \frac{\alpha^2}{2}(1-\frac{1}{\tau})g(\tau)\,, \quad (6.68)$$

where $f(\tau)$ was defined previously in (6.14),

$$h(\tau) = \sum_{i=0}^{\infty} \frac{(\Psi(i+1)+C)\,\xi^i}{2+i-\tau}\,, \quad (6.69)$$

$$g(\tau) = \sum_{i=0}^{\infty} \frac{\xi^i}{(2+i-\tau)^2}\,, \quad (6.70)$$

$\Psi(z)$ is the logarithmic derivative of the Gamma function $\Psi(z) = \Gamma'(z)/\Gamma(z)$

$$\Psi(i+1) = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}-C\,, \quad (6.71)$$

and C is the Euler constant. Separating out the nonrelativistic contribution, I have

$$F_{1/2} = F_{1/2nr} + F_{1/2r}\,, \quad (6.72)$$

$$F_{1/2r} = H_r + H_s\,, \quad (6.73)$$

where the regular part H_r, is finite for $\tau_\epsilon = 0$ and includes the f, g, h functions

$$H_r = \frac{32}{9}\alpha^2 \int_0^1 d\tau \frac{1-\tau}{(1+\tau)^6}$$

$$\cdot\Big[-2\tau(1-\tau)\,h(\tau) - (1-\tau)^2\,g(\tau) + (3-4\tau+5\tau^2-2\tau^3)\,f(\tau)\Big] \quad (6.74)$$

The remaining part H_s, that does not depend on f, g, h functions, can be integrated analytically.

$$H_s = \int_{\tau_\epsilon}^1 d\tau\,\alpha^2 \Big[\frac{4}{9\,(-1+\tau)} - \frac{4}{9\,\tau^5} + \frac{19}{9\,\tau^3} - \frac{64}{9\,\tau^2} + \frac{100}{9\,\tau} + \frac{112}{9\,(1+\tau)^4}$$

$$-\frac{56}{9\,(1+\tau)^3} - \frac{44}{9\,(1+\tau)^2} - \frac{104}{9\,(1+\tau)} + \Big(\frac{16}{9\,(-1+\tau)^3} - \frac{32}{9\,(-1+\tau)^2}$$

$$+\frac{4}{-1+\tau}-\frac{32}{3\,\tau}-\frac{64}{9\,(1+\tau)^4}+\frac{64}{9\,(1+\tau)^3}+\frac{56}{9\,(1+\tau)^2}+\frac{20}{3\,(1+\tau)}\Big)$$

$$\cdot\ln\Big(\frac{4\,\tau}{(1+\tau)^2}\Big)+\Big(\frac{16}{9\,(-1+\tau)^2}-\frac{16}{9\,(1+\tau)^2}\Big)\ln^2\Big(\frac{4\,\tau}{(1+\tau)^2}\Big)\Big]. \quad (6.75)$$

Expanding around $\tau_\epsilon = 0$ and applying the replacements (6.30) I obtain

$$H_s = \alpha^2\Big[-\frac{659}{81}+\frac{49\,\pi^2}{27}+\frac{362\,\ln(2)}{27}-\frac{52\ln^2(2)}{9}+\frac{16\ln^2(\alpha)}{3}+\frac{50\,\ln(\epsilon)}{9}$$

$$-\,8\,\ln(2)\,\ln(\epsilon)+\frac{4\ln^2(\epsilon)}{3}+\ln(\alpha)\Big(-\frac{100}{9}+16\,\ln(2)-\frac{16\,\ln(\epsilon)}{3}\Big)\Big]. \quad (6.76)$$

The evaluation of $F_{3/2}$ is much more complicated, because an infinite sum of hypergeometric functions appears. I will describe it in more detail. I have the following expression for $F_{3/2}$

$$F_{3/2} = -\frac{2}{3}\int_{\tau_\epsilon}^{1} d\tau\,\frac{1-\tau^2}{\nu\,\tau^3(\nu+\gamma\,\tau)}\,P_{3/2}$$

$$= -\frac{32}{9}\int_{\tau_\epsilon}^{1} d\tau\,\frac{1-\tau^2}{\nu\,\tau^3(\nu+\gamma\,\tau)}\,\tau^{2\gamma+2}\left(\frac{1+\tau}{2}\right)^{-2(\gamma+\gamma'+1)}\left(\frac{\tau+\nu}{1+\gamma}\right)$$

$$\cdot\frac{\Gamma^2(\gamma+\gamma'+1)}{\Gamma(2\gamma+1)\Gamma^2(2\gamma'+1)}\sum_{n=0}^{\infty}\frac{\Gamma(2\gamma'+n)}{(\gamma'+n-\nu)n!}\,S_n\,, \quad (6.77)$$

$$S_n = \frac{1}{16}\Big\{F^2(-n)\,(2\gamma'+n)(2+\tau+\gamma'+n-\nu)$$

$$+\,F^2(-n+1)\,n(-2+\tau+\gamma'+n-\nu)-2n(2\gamma'+n)\,F(-n)\,F(-n+1)\Big\} \quad (6.78)$$

where

$$F(-n) = {}_2F_1\Big(-n,\gamma+\gamma'+1,2\gamma'+1;\frac{2}{1+\tau}\Big).$$

The nonrelativistic limits of $F_{3/2}$ together with $F_{1/2}$ gives F_{nd} (up to the term that vanishes at $\epsilon = 0$). Using the contiguous relation

$$2\gamma'\,{}_2F_1\Big(-n,\gamma+\gamma'+1,2\gamma';\frac{2}{1+\tau}\Big) =$$

$$(2\gamma'+n)\,{}_2F_1\Big(-n,\gamma+\gamma'+1,2\gamma'+1;\frac{2}{1+\tau}\Big)$$

$$-\,n\,{}_2F_1\Big(-n+1,\gamma+\gamma'+1,2\gamma'+1;\frac{2}{1+\tau}\Big)\,. \quad (6.79)$$

S_n can be rewritten in a form better suited to further evaluation.

$$S_n = \frac{1}{16}\Big\{4\gamma'^2\,{}_2F_1^2\Big(-n,\gamma+\gamma'+1,2\gamma';\frac{2}{1+\tau}\Big)$$

$$+\,F^2(-n)\,(2\gamma'+n)(2-\gamma'+\tau-\nu)+F^2(-n+1)\,n(\gamma'-2+\tau-\nu)\Big\} \quad (6.80)$$

With the use of

$$\gamma = 1-\frac{\alpha^2}{2}+O(\alpha^4)\,, \quad (6.81)$$

$$\gamma' = 2(1-\frac{\alpha^2}{8})+O(\alpha^4)\,, \quad (6.82)$$

$$\nu = \tau(1-\frac{\alpha^2}{2\tau^2})+O(\alpha^4)\,, \quad (6.83)$$

the relativistic correction $F_{3/2r}$ becomes

$$\begin{aligned}
F_{3/2r} \;&=\; -\frac{256}{3}\alpha^2 \int_{\tau_\epsilon}^{1} d\tau\, \frac{1-\tau}{(1+\tau)^7} \sum_{n=0}^{\infty} \frac{\Gamma(4+n)}{\Gamma(4)n!}\, \frac{1}{2+n-\tau} \\
&\quad \Bigg\{ \Bigg[\frac{1}{6} + \frac{3}{2}\ln\Big(\frac{1+\tau}{2}\Big) - \ln(\tau) - \frac{1}{2}\Big(\Psi(4+n) - \Psi(4)\Big) \\
&\quad + \frac{1}{2+n-\tau}\Big(\frac{1}{4} - \frac{1}{2\tau}\Big) + \frac{1}{2\tau^2}\Bigg] \Big(\frac{1-\tau}{1+\tau}\Big)^{2n} \\
&\quad - \frac{1}{2}\Big(\frac{\tau-1}{\tau+1}\Big)^{n}\, {}_2F_1^{(,2)}\Big(-n, 4, 4; \frac{2}{1+\tau}\Big) \\
&+ \Bigg[F^2(-n)\,\frac{4+n}{16}\Big(\frac{1}{4} + \frac{1}{2\tau}\Big) + F^2(-n+1)\,\frac{n}{16}\Big(-\frac{1}{4} + \frac{1}{2\tau}\Big)\Bigg]\Bigg\}, \qquad (6.84)
\end{aligned}$$

where $F(-n) = {}_2F_1(-n, 4, 5, 2/(1+\tau))$ and $(,2)$ denotes the derivative with respect to the second parameter. Although this infinite sum can be transformed to the hypergeometric and Apple functions and their derivatives, such a form is not quite suited for the separation of the divergence in $\tau_\epsilon = 0$ and later numerical integration. Instead I use the summation formula [38]

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{n!} s^n\, {}_2F_1(-n, b, c; z) = \Gamma(\lambda)(1-s)^{-\lambda}\, {}_2F_1\Big(\lambda, b, c; -\frac{s\,z}{1-s}\Big) \qquad (6.85)$$

to sum up only the terms that give divergence at $\tau_\epsilon = 0$. To achieve this I apply the identity

$$\frac{\Gamma(4+n)}{n!}\frac{1}{2+n-\tau} = \frac{\Gamma(n+3)}{n!} + (1+\tau)\frac{\Gamma(n+2)}{n!} + \tau(1+\tau)\frac{\Gamma(n+1)}{n!} - \frac{\tau(1-\tau^2)}{2+n-\tau} \qquad (6.86)$$

and sum up the terms without $(2+n-\tau)$ in the denominator. In the terms where n appears only in the power function, the geometric-like series can be easily summed up. For the term in (6.84) with τ^2 in the denominator an additional subtraction should be made,

$$\begin{aligned}
\sum_{n}^{\infty} \xi^n \frac{1}{2+n-\tau} \;&=\; \sum_{n}^{\infty} \xi^n \Big(\frac{1}{2+n} + \frac{\tau}{(2+n)(2+n-\tau)}\Big) \\
&= -\Big(\frac{\ln(1-\xi) + \xi}{\xi^2}\Big) + \sum_{n}^{\infty} \xi^n \frac{\tau}{(2+n)(2+n-\tau)}. \qquad (6.87)
\end{aligned}$$

The summation is more difficult where n enters the expression in a more complicated way. Differentiating the identity (6.85) with respect to b, with $s = \frac{\tau-1}{\tau+1}$, $b = c = 4$, and $z = \frac{2}{1+\tau}$ one obtains:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\Gamma(n+i)}{n!} \Big(\frac{\tau-1}{\tau+1}\Big)^n\, {}_2F_1^{(,2)}\Big(-n, 4, 4; \frac{2}{1+\tau}\Big) = \\
&\qquad \Gamma(i)\Big(\frac{2}{1+\tau}\Big)^{-i}\, {}_2F_1^{(,2)}\Big(i, 4, 4; \frac{1-\tau}{1+\tau}\Big). \qquad (6.88)
\end{aligned}$$

The identity (which can be checked by an explicit expansion in z)

$$\begin{aligned}
&{}_2F_1^{(,2)}(1, 4, 4; z) = \\
&\quad -\frac{11}{6}\frac{1}{1-z} - \frac{1}{z^2} - \frac{3}{2z} - \ln(1-z)\Big(\frac{1}{1-z} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}\Big), \qquad (6.89)
\end{aligned}$$

together with the recurrence relation

$$ {}_2F_1(a+1,b,c;z) = {}_2F_1(a,b,c;z) + \frac{z}{a}\, {}_2F_1'(a,b,c;z) \tag{6.90} $$

allows to rewrite the result of this summation in terms of rational and logarithmic functions. Thus the τ-integration can be done analytically. The next term, where some trouble in the summation can appear is the square of the hypergeometric functions. The identity

$$ \begin{aligned} {}_2F_1(-n,4,5,z) = \frac{4}{z^4}\Big(& \frac{(1-z)^{n+4}-1}{n+4} - \frac{3((1-z)^{n+3}-1)}{n+3} \\ & + \frac{3((1-z)^{n+2}-1)}{n+2} - \frac{(1-z)^{n+1}-1}{n+1}\Big) \end{aligned} \tag{6.91} $$

allows the summation and later the τ–integration to be performed. For the terms with a Ψ–function which are singular in $\tau_\epsilon = 0$, the summation formula can be used,

$$ \sum_{n=0}^{\infty}\Big(\Psi(n+4)-\Psi(4)\Big)\xi^n\,\frac{\Gamma(n+i)}{n!} = \Gamma(i)\, {}_2F_1{}^{(,2)}(i,4,4;\xi)\,. \tag{6.92} $$

This can be checked by an expansion of the hypergeometric function into a power series, and by a term by term differentiation, so that the complete singular term A_s

$$ F_{3/2r} = A_s + A_r\,. \tag{6.93} $$

can be calculated analytically. Although A_s is quite complicated, it consists of many terms, which can all be handled by symbolic procedures. Using the table of replacements (6.30), I obtain for A_s

$$ \begin{aligned} A_s = \alpha^2\Big(& -\frac{2095}{108} + \frac{11\,\pi^2}{9} + \frac{289\,\ln(2)}{27} - \frac{77\ln^2(2)}{9} + \frac{4\ln^2(\alpha)}{3} + \\ \ln(\alpha)\Big(-\frac{74}{9} + \frac{4\,\ln(2)}{3} - \frac{4\,\ln(\epsilon)}{3}\Big) & + \frac{37\,\ln(\epsilon)}{9} - \frac{2\,\ln(2)\,\ln(\epsilon)}{3} + \frac{\ln^2(\epsilon)}{3}\Big). \end{aligned} \tag{6.94} $$

The rest, A_r, the regular part of $F_{3/2r}$ has the following form

$$ \begin{aligned} A_r \;=\; & \int_0^1 d\tau\,\alpha^2 \frac{16}{9}\,\frac{(-1+\tau)^2}{(1+\tau)^6}\Big(2\,(-2+\tau)\,g(\tau) - 4\tau\,p(\tau) + (2+\tau)\,q(\tau) \\ & - 4\tau\,r(\tau) + \frac{4}{3}\,\frac{\tau^2}{1-\tau^2}\,f(\tau)\,\Big(-7+3\,\tau+\tau^2+9\,(1-\tau^2)\,\ln(2) \\ & + 6\,(1-\tau^2)\,\ln(\tau) - 9\,(1-\tau^2)\,\ln(1+\tau)\Big)\Big), \end{aligned} \tag{6.95} $$

where f, g were defined in (6.14),(6.70), and

$$ \begin{aligned} p(\tau) &= \sum_{n=0}^{\infty}\frac{1}{2+n-\tau}\Big(\frac{\tau-1}{\tau+1}\Big)^n {}_2F_1^{(,2)}\Big(-n,4,4,\frac{2}{1+\tau}\Big)\,, \\ q(\tau) &= \sum_{n=0}^{\infty}\frac{1}{2+n-\tau}\Big(\frac{\tau-1}{\tau+1}\Big)^n {}_2F_1\Big(-n,4,5,\frac{2}{1+\tau}\Big)\,, \\ r(\tau) &= \sum_{n=0}^{\infty}\frac{\xi^n}{2+n-\tau}\Big(\Psi(4+n)-\Psi(4)\Big)\,. \end{aligned} \tag{6.96} $$

Taking the sum of all regular parts, I obtain

$$\alpha^2 F_r = E_r + H_r + A_r =$$
$$\int_0^1 d\tau \, \frac{32\,(1-\tau)^2}{9(1+\tau)^6} \Big((-3 + 2\,\tau)\, g(\tau) - 2\,\tau\, h(\tau) - 2\,\tau\, p(\tau) + (1 + \frac{\tau}{2})\, q(\tau)$$
$$-2\,\tau\, r(\tau) + f(\tau) \Big(-9 - \frac{3}{-1+\tau} - \tau + \frac{8\,\tau^2}{3} + \frac{3}{1+\tau} - 10\,\tau^2 \ln(2)$$
$$-6\,\tau^2 \ln(\tau) + 10\,\tau^2 \ln(1+\tau) \Big) \Big) \,. \tag{6.97}$$

This regular part is calculated numerically by a Gaussian integration procedure. All functions f, g, h, p, q, r converge at least like $(1-\tau)^n/n^2$, because all the slower decreasing terms have been separated out and calculated analytically. The value of the function $F_r(t)$ under the integral in (6.97) was calculated by performing the summation from the end i.e. from the highest n to $n=1$ in all functions f, g, h, p, q, r. The largest value of n was determined by the condition that the n–th term in the sum is smaller than a specified value, for example 10^{-10}. It is helpful in the evaluation of the p function to derive a recurrence relation. Differentiating over b the contiguous relation

$$(c-a)\, {}_2F_1(a-1, b, c; z) + (2a - c - az + bz)\, {}_2F_1(a, b, c; z)$$
$$+\, a(z-1)\, {}_2F_1(a+1, b, c; z) = 0 \tag{6.98}$$

at $a = -n$, $b = c = 4$, I obtain

$$f_p(n) \equiv {}_2F_1^{(,2)}(-n, 4, 4, z)) \tag{6.99}$$
$$f_p(n+1) = \frac{2n + 4 - nz - 4z}{4+n}\, f_p(n) + \frac{n(z-1)}{4+n}\, f_p(n-1) - \frac{z(1-z)^n}{4+n} \,. \tag{6.100}$$

Together with initial values $f_p(0) = 0, f_p(1) = -z/4$, this is the desired relation, and the numerical integration can be easily performed. The result of this integration is

$$F_r = -0.819858 \,. \tag{6.101}$$

At this point all terms up to α^2 order have been calculated. Summing up the low–energy part I obtain

$$F_L = (D_s + H_s + E_s + A_s) + D_r + \alpha^2 F_r =$$
$$= D_r - \frac{86}{9} + \frac{16\sqrt{2}\,\alpha}{3\sqrt{\epsilon}} + \frac{28\ln(2)}{3} - \frac{8\ln(\alpha)}{3} + \frac{4\ln(\epsilon)}{3} +$$
$$\alpha^2 \Big(F_r - \frac{12521}{324} - \frac{16}{3\,\epsilon} + \frac{10\,\pi^2}{27} + \frac{217\ln(2)}{9} + \frac{8\ln(2)}{\epsilon} - 17\ln^2(2)$$
$$- \frac{10\ln(\alpha)}{3} + \frac{16\ln(\alpha)}{3\,\epsilon} - \frac{44\ln(2)\ln(\alpha)}{3} - 4\ln^2(\alpha) + \frac{5\ln(\epsilon)}{3}$$
$$- \frac{8\ln(\epsilon)}{3\,\epsilon} + \frac{22\ln(2)\ln(\epsilon)}{3} + 4\ln(\alpha)\ln(\epsilon) - \ln^2(\epsilon) \Big) \,. \tag{6.102}$$

with D_r and F_r given in (6.17) and (6.101).

The final result

$$F = F_L + F_H =$$
$$D_r - \frac{76}{9} + 8\ln(2) - \frac{8\ln(\alpha)}{3} + \alpha \left(\frac{139\,\pi}{32} - 2\,\pi\ln(2) \right)$$

$$+\alpha^2\left(F_r - \frac{16468}{405} + \frac{5\,\pi^2}{54} + \frac{703\,\ln(2)}{30} - 20\ln^2(2) + \frac{5\,\zeta(3)}{2}\right.$$
$$\left. + \left(\frac{21}{10} - \frac{56\,\ln(2)}{3}\right)\ln(\alpha) - 4\ln^2(\alpha)\right) \tag{6.103}$$

does not depend on ϵ, as expected. All above coefficients coincide with those previously known and

$$A_{60}(1S) = -30.92890(1)\,. \tag{6.104}$$

coincides with (2.12), but is much more precise.

7. 2S high–energy part

In the evaluation of the binding corrections for 2S state I have two choices: to repeat all the calculations only with a changed wave function and energy, or use the result for 1S and calculate the difference $F_{2S} - F_{1S}$ –i.e. those corrections which do not scale like $1/n^3$. In the high–energy part I chose the second way, while in the low energy part the first way is more suitable. Why can the evaluation of the difference $(F_H)_{2S} - (F_H)_{1S}$ be simpler than the evaluation of separate terms? The main difficulty in the evaluation of the high–energy part is the appearance of nonanalytic terms. One cannot expand in $\vec{p}$ in the denominator of the electron propagator to obtain for example $(\alpha/r)^3$ because the expression $\langle\psi|(\frac{\alpha}{r})^3|\psi\rangle$ is infinite. In contrast, the difference

$$8\langle\psi_{2S}|\left(\frac{\alpha}{r}\right)^3|\psi_{2S}\rangle - \langle\psi_{1S}|\left(\frac{\alpha}{r}\right)^3|\psi_{1S}\rangle \tag{7.1}$$

is finite, provided that for separate terms a regularization is introduced (that cancels out in the difference). The finiteness of such a difference can be explained by considering the behavior of the wave function around the center. The difference of the nonrelativistic wave functions is

$$8\,\phi_{2S}^2(r) - \phi_{1S}^2(r) = O(r^2) \tag{7.2}$$

and also for the relativistic corrections to the wave function the logarithmic singularity cancels out. It so happens, that one can expand in the momentum $\vec{p}$ in the electron propagator and the difference for all the terms up to $(Z\alpha)^6$ order is finite, which enormously simplifies the calculation of F_H. In fact many terms depend only on the value of the wave function at the center, which behaves like $n^{-3/2}$, thus they cancel out in the difference.

I will explain the evaluation on selected examples. I use the following notation for E and λ

$$E = \begin{cases} m\gamma & \text{for } 1S \\ m\sqrt{\frac{1+\gamma}{2}} & \text{for } 2S \end{cases} \tag{7.3}$$

$$\lambda = \begin{cases} m\alpha \text{ for } 1S \\ \frac{1}{2}m\alpha \text{ for } 2S \end{cases} \tag{7.4}$$

and

$$\langle\langle\phi|Q|\phi\rangle\rangle \equiv 8\langle\phi|Q|\phi\rangle_{2S} - \langle\phi|Q|\phi\rangle_{1S},$$

where Q is any operator. Let us start the evaluation from the three vertex term. I put $\vec{p} = 0$ in P_{31} and obtain

$$P_H^{31} \approx \alpha^6\langle\langle\phi|\frac{1}{(\alpha r)^3}|\phi\rangle\rangle = \alpha^6\left(8\langle\phi|\frac{1}{(\alpha r)^3}|\phi\rangle_{2S} - \langle\phi|\frac{1}{(\alpha r)^3}|\phi\rangle_{1S}\right). \tag{7.5}$$

To calculate this expression I regularize the r integral by multiplying it by r^η, integrate separately both terms, expand in η, and calculate the difference

$$\langle\langle\phi|\frac{1}{(\alpha r)^3}|\phi\rangle\rangle = 8\left(\frac{1}{2\eta}-\frac{3}{8}-\frac{C}{2}\right)-\left(\frac{4}{\eta}-4C-4\ln(2)\right)=4\ln(2)-3\,, \quad (7.6)$$

The evaluation of P_H^{32} is simple. It depends on the value of the wave function at $r=0$, thus it vanishes.

Expanding in $\vec{p}$ the double vertex part I obtain the terms:

$$\langle\langle\phi|\vec{p}^{\,2}\Big(\frac{\alpha}{r}\Big)^2|\phi\rangle\rangle,\ \langle\langle\phi|\frac{\alpha}{r}\,\vec{p}^{\,2}\,\frac{\alpha}{r}|\phi\rangle\rangle,\ \langle\langle\phi|\vec{p}\,\frac{\alpha}{r}\,\vec{p}\,\frac{\alpha}{r}|\phi\rangle\rangle,\ \langle\langle\phi|\vec{p}\,\Big(\frac{\alpha}{r}\Big)^2\,\vec{p}\,|\phi\rangle\rangle\,.$$

All of them can be calculated as in the previous case. These integrals and other that are needed in this calculation are given below (in atomic units).

$$\begin{aligned}
\langle\langle\phi|\frac{1}{r^3}|\phi\rangle\rangle &= 4\ln(2)-3\\
\langle\langle\phi|\frac{1}{r^4}|\phi\rangle\rangle &= 5-8\ln(2)\\
\langle\langle\phi|\vec{p}^{\,2}\frac{1}{r^2}|\phi\rangle\rangle &= 8\ln(2)-\frac{9}{2}\\
\langle\langle\phi|\frac{1}{r}\vec{p}^{\,2}\frac{1}{r}|\phi\rangle\rangle &= \frac{1}{2}\\
\langle\langle\phi|\frac{1}{r}\vec{p}^{\,4}|\phi\rangle\rangle &= \frac{25}{8}\\
\langle\langle\phi|\vec{p}^{\,2}\frac{1}{r}\vec{p}^{\,2}|\phi\rangle\rangle &= 16\ln(2)-\frac{55}{8}\\
\langle\langle\phi|\vec{p}^{\,6}|\phi\rangle\rangle &= \frac{77}{8} \qquad (7.7)\\
\langle\langle\phi|\vec{p}\,\frac{1}{r}\vec{p}\,\frac{1}{r}|\phi\rangle\rangle &= 4\ln(2)-2\\
\langle\langle\phi|\vec{p}^{\,2}\;\vec{p}\,\frac{1}{r}\vec{p}\,|\phi\rangle\rangle &= 8\ln(2)-\frac{27}{8}\\
\langle\langle\phi|\frac{1}{r}\vec{p}^{\,2}|\psi_{ru}\rangle\rangle &= \alpha^2\Big(\frac{49}{64}+2\ln(2)\Big)\\
\langle\langle\phi|\vec{p}^{\,4}|\psi_{ru}\rangle\rangle &= \alpha^2\Big(\frac{107}{64}+4\ln(2)\Big)
\end{aligned}$$

There are the terms in the single vertex part, which are calculated using the other regularization.

$$\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\frac{1}{Y^2+(\vec{p}-\vec{k})^2}\,\frac{1}{Y^2+(\vec{p}\,'-\vec{k})^2}\,\phi(\vec{p})\,\frac{4\pi\alpha}{q^2}\,\phi(\vec{p}\,')\,q^2\,. \quad (7.8)$$

After an expansion in $\vec{p}$ there appears a delta function at $r=0$ in the above expression. I regularize this expression by,

$$\delta^3(\vec{r}\,)\to\delta^3(\vec{r}-\vec{\eta}\,) \quad (7.9)$$

to obtain

$$4\pi\alpha\left[\frac{\phi^2(\eta)}{(Y^2+k^2)^2}+2\phi(\eta)(\vec{p}^{\,2}\phi)(\eta)\frac{1}{(Y^2+k^2)^3}\left(\frac{4}{3}\frac{k^2}{Y^2+k^2}-1\right)\right]. \quad (7.10)$$

After using the Schrödinger equation

$$\vec{p}^{\,2}\phi=\Big(-\lambda^2+\frac{2\alpha}{r}\Big)\phi \quad (7.11)$$

this becomes,

$$\frac{4\pi\alpha\,\phi^2(\eta)}{(Y^2+k^2)^2}\left(1+\frac{2}{Y^2+k^2}\left\{-\lambda^2+\frac{2\alpha}{\eta}\right\}\left(\frac{4}{3}\frac{k^2}{Y^2+k^2}-1\right)\right]. \tag{7.12}$$

The second term in braces does not contribute because

$$\lim_{\eta\to 0}\frac{1}{\eta}\left(8\,\phi_{2S}^2(\eta)-\phi_{1S}^2(\eta)\right)=0\,. \tag{7.13}$$

Thus, after an expansion in η I obtain

$$\frac{4\alpha\lambda^3}{(Y^2+k^2)^2}\left[1-\frac{2\lambda^2}{Y^2+k^2}\left(\frac{4}{3}\frac{k^2}{Y^2+k^2}-1\right)\right]. \tag{7.14}$$

To check this result, I have integrated (7.8) exactly, using K_i integrals. The next term that needs some care appears in P_{12}.

$$8\pi\alpha\langle\langle\phi|\delta^3(\vec{r})|\psi_{ru}\rangle\rangle=$$

$$\lim_{\eta\to 0}8\pi\alpha\left(8\phi_{2S}(\eta)(\psi_{ru})_{2S}(\eta)-\phi_{1S}(\eta)(\psi_{ru})_{1S}(\eta)\right)=\frac{\alpha^6}{4}(3+16\ln(2)). \tag{7.15}$$

Using (7.7) and table 1 in appendix B I obtain the following expression for the high energy part

$$8\mathcal{F}_{2S}(\omega,k)-\mathcal{F}_{1S}(\omega,k)=m\alpha^{-4}\Big(8(P_H)_{2S}-(P_H)_{1S}\Big)= \tag{7.16}$$

$$\alpha^{-2}\left\{\frac{8\,X^2}{3\,(k^2+X^2)^3}+\frac{8\,\omega\,X^2}{3\,(k^2+X^2)^3}-\frac{8}{3\,(k^2+X^2)^2}-\frac{2\,\omega}{3\,(k^2+X^2)^2}+\frac{2\,\omega^2}{(k^2+X^2)^2}\right.$$

$$\left.-\frac{4\,X^2}{3\,(k^2+X^2)^2}+\frac{4}{3\,(k^2+X^2)}-\frac{\omega}{k^2+X^2}\right\}-\frac{48\,X^4}{5\,(k^2+X^2)^5}-\frac{48\,\omega\,X^4}{5\,(k^2+X^2)^5}$$

$$+\frac{96\,X^2}{5\,(k^2+X^2)^4}+\frac{36\,\omega\,X^2}{5\,(k^2+X^2)^4}-\frac{12\,\omega^2\,X^2}{(k^2+X^2)^4}+\frac{24\,X^4}{5\,(k^2+X^2)^4}-\frac{48}{5\,(k^2+X^2)^3}$$

$$+\frac{12\,\omega}{5\,(k^2+X^2)^3}+\frac{9\,\omega^2}{(k^2+X^2)^3}-\frac{3\,\omega^3}{(k^2+X^2)^3}-\frac{38\,X^2}{5\,(k^2+X^2)^3}+\frac{8\,\omega\,X^2}{(k^2+X^2)^3}$$

$$+\frac{14}{5\,(k^2+X^2)^2}-\frac{29\,\omega}{4\,(k^2+X^2)^2}+\frac{9\,\omega^2}{4\,(k^2+X^2)^2}-\frac{X^2}{(k^2+X^2)^2}+\frac{1}{k^2+X^2}$$

$$-\frac{3\,\omega}{8\,(k^2+X^2)}+\alpha^2\left\{\frac{176\,X^6}{(k^2+X^2)^7}+\frac{176\,\omega\,X^6}{(k^2+X^2)^7}-\frac{3952\,X^4}{15\,(k^2+X^2)^6}-\frac{220\,\omega\,X^4}{(k^2+X^2)^6}\right.$$

$$+\frac{652\,\omega^2\,X^4}{15\,(k^2+X^2)^6}-\frac{88\,X^6}{(k^2+X^2)^6}+\frac{784\,X^2}{15\,(k^2+X^2)^5}+\frac{168\,\omega\,X^2}{(k^2+X^2)^5}-\frac{3154\,\omega^2\,X^2}{15\,(k^2+X^2)^5}$$

$$+\frac{58\,\omega^3\,X^2}{(k^2+X^2)^5}+\frac{1406\,X^4}{15\,(k^2+X^2)^5}+\frac{1024\,\omega\,X^4}{15\,(k^2+X^2)^5}+\frac{176}{5\,(k^2+X^2)^4}-\frac{72\,\omega}{(k^2+X^2)^4}$$

$$+\frac{394\,\omega^2}{5\,(k^2+X^2)^4}-\frac{71\,\omega^3}{4\,(k^2+X^2)^4}-\frac{23\,\omega^4}{4\,(k^2+X^2)^4}+\frac{96\,X^2}{5\,(k^2+X^2)^4}-\frac{3467\,\omega\,X^2}{60\,(k^2+X^2)^4}$$

$$-\frac{41\,\omega^2\,X^2}{4\,(k^2+X^2)^4}-\frac{3\,X^4}{5\,(k^2+X^2)^4}-\frac{164}{15\,(k^2+X^2)^3}+\frac{1471\,\omega}{60\,(k^2+X^2)^3}-\frac{4\,\omega^2}{(k^2+X^2)^3}$$

$$-\frac{35\,\omega^3}{8\,(k^2+X^2)^3}-\frac{527\,X^2}{40\,(k^2+X^2)^3}+\frac{97\,\omega\,X^2}{4\,(k^2+X^2)^3}+\frac{251}{40\,(k^2+X^2)^2}-\frac{523\,\omega}{32\,(k^2+X^2)^2}$$

$$+\frac{65\,\omega^2}{32\,(k^2+X^2)^2}-\frac{13\,X^2}{16\,(k^2+X^2)^2}+\frac{13}{16\,(k^2+X^2)}-\frac{13\,\omega}{64\,(k^2+X^2)}+\left(\frac{-1024\,X^4}{3\,(k^2+X^2)^6}\right.$$

$$+\frac{1024\,\omega^2\,X^4}{3\,(k^2+X^2)^6}+\frac{1792\,X^2}{3\,(k^2+X^2)^5}-\frac{384\,\omega\,X^2}{(k^2+X^2)^5}-\frac{448\,\omega^2\,X^2}{3\,(k^2+X^2)^5}+\frac{64\,\omega^3\,X^2}{(k^2+X^2)^5}$$

$$+\frac{2752\,X^4}{15\,(k^2+X^2)^5}-\frac{6208\,\omega\,X^4}{15\,(k^2+X^2)^5}-\frac{256}{(k^2+X^2)^4}+\frac{320\,\omega}{(k^2+X^2)^4}-\frac{128\,\omega^2}{(k^2+X^2)^4}$$

$$+\frac{8\,\omega^4}{(k^2+X^2)^4}-\frac{3904\,X^2}{15\,(k^2+X^2)^4}+\frac{9056\,\omega\,X^2}{15\,(k^2+X^2)^4}-\frac{448\,\omega^2\,X^2}{3\,(k^2+X^2)^4}+\frac{184\,X^4}{5\,(k^2+X^2)^4}$$

$$+\frac{224}{5\,(k^2+X^2)^3}-\frac{2608\,\omega}{15\,(k^2+X^2)^3}+\frac{280\,\omega^2}{3\,(k^2+X^2)^3}-\frac{824\,X^2}{15\,(k^2+X^2)^3}+\frac{152}{15\,(k^2+X^2)^2}\Bigg)$$

$$\cdot\ln(2)\Bigg\}. \tag{7.17}$$

As was expected, there are no non–analytic terms in α^2 in this expression. The further evaluation proceeds in the same way as in the 1S case, so I shall not describe it. There is a difference in the correction term in comparison with the 1S state. Its value is

$$(F_{corr})_{2S}-(F_{corr})_{1S}=\frac{1}{2\alpha^2}+\frac{3}{8}+\frac{39}{128}\alpha^2\,. \tag{7.18}$$

The final result for the difference is

$$\begin{aligned} F_\Delta = (F_H)_{2S}-(F_H)_{1S} = \\ \alpha^2\Big(\frac{30923}{7200}-\frac{10}{3\,\epsilon}-\frac{523\,\ln(2)}{60}+\frac{16\,\ln(2)}{3\,\epsilon} \\ +4\ln^2(2)-\frac{197\,\ln(\epsilon)}{60}+4\,\ln(2)\,\ln(\epsilon)\Big)\,. \end{aligned} \tag{7.19}$$

8. 2S low–energy part

The evaluation of the low–energy part F_L for the 2S state follows the method applied to the 1S state. I have the nonrelativistic dipole F_{nd}, the nonrelativistic quadrupole F_q and the relativistic dipole F_{rd} contributions. In contrast to the previous case the quantities $P_{nd}, P_q, P_{1/2}, P_{3/2}$ have not been calculated to my knowledge at all for 2S state, thus I had to perform these evaluations. Because the 2S-wave function is more complicated, the calculation involves more terms. Having calculated these quantities, the remainder of the evaluation is almost identical to the 1S case.

I start the evaluation from P_{nd} and P_q. I use the notation:

$$l=\frac{m\,\alpha}{2},\quad t=\frac{l}{X},\quad X=\sqrt{-2m(E-\omega)},\quad E=-\frac{m\,\alpha^2}{8}=-\frac{l^2}{2m}$$

$$\text{and}\;\; G=\frac{1}{H-(E-\omega)}\,.$$

I have to evaluate P, (for a moment $\vec{k}\neq\vec{k}'$).

$$\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda}\,P = \epsilon^i_{k\lambda}\,\epsilon^j_{k'\lambda}\,\frac{1}{m}\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,p_i\,p'_j\,\phi(\vec{p}+\vec{k})\,\phi(\vec{p}\,'+\vec{k}')\,G(\vec{p},\vec{p}\,',E-\omega). \tag{8.1}$$

The 2S wave function in momentum representation is

$$\phi(p)=l^{3/2}\,\frac{\partial}{\partial l}\,\frac{16\,\pi\,l^2}{(p^2+l^2)^2}\,\frac{1}{\sqrt{4\pi}}\,. \tag{8.2}$$

So, that

$$\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda}\,P=\frac{64\pi l^3}{m}\,\frac{\partial}{\partial\lambda}\,\frac{\partial}{\partial\mu}\bigg|_{\lambda=\mu=l}\lambda^2\,\mu^2\,\frac{1}{4}\,\epsilon^i_{k\lambda}\,\epsilon^j_{k'\lambda}\,\frac{\partial}{\partial k^i}\,\frac{\partial}{\partial k'^j}$$

$$\bigg\{\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\,\frac{1}{(\vec{p}\,'+\vec{k}')^2+\mu^2}\,G(\vec{p},\vec{p}\,',E-\omega)\bigg\}. \tag{8.3}$$

The term in braces, denoted by I, is calculated in (B11); using this result I obtain,

$$\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda}\,P =$$
$$8X\,l^3\,\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\bigg|_{\lambda=\mu=l}\lambda^2\mu^2\,\epsilon^i_{k\lambda}\,\epsilon^j_{k'\lambda}\,\frac{\partial}{\partial k^i}\frac{\partial}{\partial k'^j}\int_0^1 d\rho\,\rho^{-\tau}\frac{1}{a^2-b}\frac{a_2}{2\rho^2},\quad (8.4)$$

where a_2, a, b are defined in appendix B. Because $\vec{\epsilon}_k$ is orthogonal to $\vec{k}$, I can replace

$$\frac{\partial}{\partial k^i}\frac{\partial}{\partial k'^j}\to\frac{\partial^2(\vec{k}\vec{k}')}{\partial k^i\partial k'^j}\frac{\partial}{\partial(\vec{k}\vec{k}')}=\delta^{ij}\,\frac{\partial}{\partial(\vec{k}\vec{k}')}\bigg|_{k^2,k'^2}.\quad (8.5)$$

a_2 does not depend on $\vec{k}\vec{k}'$ hence the derivative is

$$\frac{\partial a^2-b}{\partial(\vec{k}\vec{k}')}\bigg|_{\vec{k}=\vec{k}'}=-\frac{2X^2}{\rho}a_2\,.\quad (8.6)$$

Thus the expression for P becomes,

$$P=8X^3l^3\,\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\bigg|_{\lambda=\mu=l}\lambda^2\mu^2\int_0^1 d\rho\,\rho^{-\tau-3}\frac{a_2^2}{(a^2-b)^2}\,.\quad (8.7)$$

I have put here $\vec{k}=\vec{k}'$. Expanding the term under the ρ integral in $(\lambda-\mu)^2$, and using the integral representation of the hypergeometric function,

$${}_2F_1(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t\,z)^{-a}dt\quad (8.8)$$

with $a=n$, $b=n-2-\tau$, $c=n-1-\tau$, and $n\in\mathcal{N}$, I obtain

$$\begin{aligned}P\;=\;&8X^3l^3\left[\frac{\partial^2}{\partial\mu\,\partial\lambda}\bigg|_{\lambda=\mu=l}\frac{2^4\lambda^2\mu^2}{[(X+\lambda)^2+k^2][(X+\mu)^2+k^2]}\right.\\&\cdot\frac{1}{2-\tau}\,{}_2F_1\left(4,2-\tau,3-\tau;\frac{1}{2}\frac{(X-\mu)^2+k^2}{(X+\mu)^2+k^2}+\frac{1}{2}\frac{(X-\lambda)^2+k^2}{(X+\lambda)^2+k^2}\right)\\&+\frac{2^8X^2l^4}{[(X+l)^2+k^2]^6}\frac{1}{3-\tau}\,{}_2F_1\left(5,3-\tau,4-\tau;\frac{(X-l)^2+k^2}{(X+l)^2+k^2}\right)\\&\left.+\frac{2^{10}X^2k^2l^6}{[(X+l)^2+k^2]^8}\frac{1}{4-\tau}\,{}_2F_1\left(6,4-\tau,5-\tau;\frac{(X-l)^2+k^2}{(X+l)^2+k^2}\right)\right].\quad (8.9)\end{aligned}$$

Performing the differentiation, using the relations

$$\begin{aligned}{}_2F_1'(a,b,c;x)&=\frac{b}{x}\Big({}_2F_1(a,b+1,c;x)-{}_2F_1(a,b,c;x)\Big),\\{}_2F_1(a+1,b+1,c+1;x)&=\frac{c}{a\,x}\Big({}_2F_1(a,b+1,c;x)-{}_2F_1(a,b,c;x)\Big),\\{}_2F_1(a,b,b;x)&=(1-x)^{-a},\end{aligned}\quad (8.10)$$

and expanding in k^2, I obtain

$$\begin{aligned}P_{nd}&=\frac{2^8\,t^5}{(1-t)(1+t)^8}\,{}_2F_1(4,2-2\,t,3-2\,t,\xi)-\frac{2\,t^2}{(1-t)(1+t)},\\P_q&=\frac{k^2}{X^2}\left\{\frac{2^{10}\,t^5\,(-5+t^4)}{5(1-t)^4(1+t)^{11}}\,{}_2F_1(4,2-2\,t,3-2\,t,\xi)\right.\\&\left.\quad-\frac{2\,t^2\,(-25+6\,t^2+3\,t^4)}{5(1-t)^4(1+t)^4}\right\}.\end{aligned}\quad (8.11)$$

Having P_{nd} and P_q with $k=\omega$ the contribution to energy is

$$\Delta E = -\frac{2\alpha}{\pi m}\int_0^{\epsilon} \omega d\omega\ P. \tag{8.12}$$

With

$$t = \frac{1}{\sqrt{1+\frac{2m\omega}{l^2}}}, \quad \frac{k^2}{X^2} = \frac{l^2}{4m^2}\frac{(1-t^2)^2}{t^2}, \quad \text{and} \quad \xi = \left(\frac{1-t}{1+t}\right)^2,$$

F becomes

$$F = -\frac{1}{2}\int_{t_\epsilon}^{1} dt\ \frac{1-t^2}{t^5}\ P \tag{8.13}$$

As in the 1S case I use the relativistic expression for t_ϵ ($\epsilon \to m\epsilon$).

$$t_\epsilon = \frac{\sqrt{\frac{1-\gamma}{2}}}{\sqrt{1-(\sqrt{\frac{1-\gamma}{2}}-\epsilon)^2}}. \tag{8.14}$$

Further evaluation is the same as in the 1S case. The result for the dipole contribution is:

$$F_{nd} = D_s + D_r\,, \tag{8.15}$$

$$D_s = -\frac{3497}{45} + \frac{16\sqrt{2}\,\alpha}{3\sqrt{\epsilon}} + 108\ \ln(2) - \frac{8\ \ln(\alpha)}{3} + \frac{4\ \ln(\epsilon)}{3}$$
$$+\alpha^2\,(\frac{8}{3}+\frac{16}{3\,\epsilon})\ \ln(\alpha) + \alpha^2\left(-\frac{1}{6} - \frac{2}{\epsilon} + \frac{4\ \ln(2)}{3} + \frac{8\ \ln(2)}{3\,\epsilon} - \frac{4\ \ln(\epsilon)}{3} - \frac{8\ \ln(\epsilon)}{3\,\epsilon}\right).$$

$$D_r = \int_0^1 dt\,\frac{512\,f(t)\ (1-2\,t)\ (1-t)\ t^2\ (1+2\,t)}{3\,(1+t)^7} = 0.02638533\,, \tag{8.16}$$

where

$$f(t) = \sum_{k=1}^{\infty}\frac{\xi^k}{(2+k)(2+k-2t)}\,. \tag{8.17}$$

The quadrupole contribution is

$$F_q = E_s + E_r\,, \tag{8.18}$$

$$E_s = \frac{-32\,\alpha^2\ln^2(\alpha)}{3} + \alpha^2\ \ln(\alpha)\left(-\frac{7}{5} - \frac{32\ \ln(2)}{3} + \frac{32\ \ln(\epsilon)}{3}\right)$$
$$+\alpha^2\left(-\frac{204227}{12600} - \frac{8\,\pi^2}{3} - \frac{47\ \ln(2)}{6} + \frac{56\,\ln^2(2)}{3} + \frac{7\ \ln(\epsilon)}{10}\right.$$
$$\left. + \frac{16\ \ln(2)\ \ln(\epsilon)}{3} - \frac{8\,\ln^2(\epsilon)}{3}\right),$$

$$E_r = \int_0^1 dt\,\frac{-128\,\alpha^2\,f(t)\ (-1+2\,t)\ (1+2\,t)\ (-5+t^4)}{15\,(1+t)^8}\,. \tag{8.19}$$

In the evaluation of $P_{1/2}$ and $P_{3/2}$ I use the formulae (6.53) and (6.54) with the relativistic wave function and energy E for the 2S state. As a result of r integration, I obtain

$$P_{1/2} = \frac{E(E+\gamma)(E-\omega+\gamma)}{24\gamma^4\Gamma(2\gamma+1)}(1-\xi)^{2\gamma+2}\,t$$
$$\sum_{k=0}^{\infty}\frac{\Gamma(2\gamma+2+k)}{k!(k+1+\gamma-\nu)}\,\xi^k\left[A_1\frac{4t}{1+t}(1+\gamma+\frac{k}{1-t})+B_1\right]^2 + R_1 \tag{8.20}$$

$$\begin{aligned}
A_1 &= -\frac{\gamma^2(3+4E)}{2E(E+\gamma)} + \frac{\gamma(1+2E)}{\gamma+E-\omega}\Big(\frac{1+2\gamma}{2E} - \frac{E-\gamma}{1+2\gamma}\Big) \\
&= \frac{1}{2} + \alpha^2\Big(-\frac{19}{64} + \frac{9}{64\,t^2}\Big) + O(\alpha^4)\,, \qquad (8.21)\\
B_1 &= 4\gamma^2 + \frac{2\gamma}{\gamma+E-\omega}\Big\{-(1+2\gamma)(\gamma+E) + \frac{E-\gamma}{1+2\gamma}\Big\} \\
&= -2 + \alpha^2\left(\frac{3}{2} - \frac{3}{8\,t^2}\right), \qquad (8.22)\\
R_1 &= -\frac{\alpha^2}{6\gamma^2(1+\gamma)(E-\omega+\gamma)}\int_0^1 r^2dr\,\Big(\alpha g_1(r) - (1+2\gamma)g_2(r)\Big)^2 \qquad (8.23)
\end{aligned}$$

where

$$\begin{aligned}
&\nu = \frac{(E-\omega)\alpha}{\sqrt{m^2-(E-\omega)^2}},\quad t = \frac{l}{\sqrt{m^2-(E-\omega)^2}},\quad E = \sqrt{\frac{1+\gamma}{2}}, \\
&l = \sqrt{\frac{1-\gamma}{2}},\quad \gamma' = \sqrt{4-\alpha^2}\,. \qquad (8.24)
\end{aligned}$$

R_1 does not give a contribution to $F_{1/2}$ after an expansion in ϵ is made, therefore I do not calculate it.

The result for $P_{3/2}$ is

$$\begin{aligned}
P_{3/2} \;=\; & \frac{1}{6\gamma^2E(E+\gamma)(2(E-\omega)+\gamma')}\,\frac{\Gamma^2(\gamma+\gamma'+1)}{\Gamma(2\gamma+1)\Gamma^2(2\gamma')} \\
& \cdot t^{2\gamma+3}\Big(\frac{1+t}{2}\Big)^{-2(\gamma+\gamma'+1)}\sum_{k=0}^{\infty}\frac{\Gamma(k+2\gamma')}{k!(k+\gamma'-\nu)} \\
& \Big\{A_2\,\frac{1+t}{t}\,{}_2F_1\Big(-k,\gamma+\gamma',2\gamma';\frac{2}{1+t}\Big) \\
& + B_2\,{}_2F_1\Big(-k,\gamma+\gamma'+1,2\gamma';\frac{2}{1+t}\Big) \\
& + C_2\,\frac{t}{1+t}\,{}_2F_1\Big(-k,\gamma+\gamma'+2,2\gamma';\frac{2}{1+t}\Big)\Big\}^2. \qquad (8.25)
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sqrt{2-\gamma'}\,(E+\gamma)\,\gamma\,l = \frac{\alpha^2}{2} + O(\alpha^4)\,, \qquad (8.26)\\
B_2 &= \frac{\gamma}{\gamma'}\sqrt{2+\gamma'}\,(E+\gamma)(2(E-\omega)+\gamma') - \sqrt{2-\gamma'} \\
&\quad\cdot\Big[(E+\gamma)\,\gamma\,\Big(\frac{\alpha(E-\omega)}{\gamma'} + l\Big) + l\,\gamma\,(\gamma+\gamma'+1)\Big] \\
&= 8 + \alpha^2\Big(-\frac{33}{4} - \frac{1}{2t^2}\Big), \qquad (8.27)\\
C_2 &= \frac{\gamma}{\gamma'}\,(\gamma+\gamma'+1)\Big[-\sqrt{2+\gamma'}\,(2(E-\omega)+\gamma') + \sqrt{2-\gamma'}(\gamma'\,l + \alpha\,(E-\omega))\Big] \\
&= -16 + \alpha^2\Big(\frac{25}{2} + \frac{1}{t^2}\Big). \qquad (8.28)
\end{aligned}$$

The expression for the energy shift (6.59)

$$\Delta E = -\frac{2\alpha}{3\pi m}\int_0^{\epsilon}\omega d\omega\, P \qquad (8.29)$$

gives for F

$$F = -\frac{8}{3(1+\gamma)^2}\int_{t_\epsilon}^{1} dt\, \frac{1-t^2}{t^3\,\nu\left(t+\dfrac{\nu}{(1+\gamma)}\right)}\,P\,, \tag{8.30}$$

with t_ϵ defined previously in (8.14). As a check of r–integration I have

$$F_{nd} = (F_{1/2})_{nrel} + (F_{3/2})_{nrel}\,. \tag{8.31}$$

The further evaluation proceeds in the same way as in the 1S case. The final result for $F_{1/2}$, after the separation of nonrelativistic term, is:

$$F_{1/2r} \;=\; H_s + H_r\,, \tag{8.32}$$

$$\begin{aligned} H_s \;=\;& \alpha^2\left(-\frac{668167}{5400} + \frac{175\,\pi^2}{27} + \frac{14771\,\ln(2)}{90} - \frac{724\ln^2(2)}{9} - \frac{13\,\ln(\alpha)}{3}\right.\\ &+ \frac{16\,\ln(2)\,\ln(\alpha)}{3} + \frac{16\ln^2(\alpha)}{3} + \frac{13\,\ln(\epsilon)}{6} - \frac{8\,\ln(2)\,\ln(\epsilon)}{3}\\ &\left.- \frac{16\,\ln(\alpha)\,\ln(\epsilon)}{3} + \frac{4\ln^2(\epsilon)}{3}\right), \end{aligned} \tag{8.33}$$

$$\begin{aligned} H_r \;=\;& \int_0^1 dt\,\alpha^2\,\frac{64}{9}\,\frac{1-t}{(1+t)^7}\\ &\Big\{g(t)\,(1-t)^2\,(-1+2\,t)\,(1+2\,t) + 4\,h(t)\,t\,(-1+2\,t)\,(1+2\,t)\\ &+ f(t)\left[3 - 2\,t + 14\,t^2 + 24\,t^3 - 64\,t^4 - 8\,t^2\,(1-4\,t^2)\ln\left(\frac{4\,t}{(1+t)^2}\right)\right]\Big\}, \end{aligned} \tag{8.34}$$

where f is defined in (8.17), and

$$g(t) \;=\; \sum_{k=1}^{\infty}\frac{\xi^k}{(k+2-2t)^2}\,, \tag{8.35}$$

$$h(t) \;=\; \sum_{k=1}^{\infty}\frac{\Psi(k+1)+C}{k+2-2t}\,\xi^k\,. \tag{8.36}$$

The result for $F_{3/2}$ is

$$F_{3/2r} \;=\; A_s + A_r\,, \tag{8.37}$$

$$\begin{aligned} A_s \;=\;& \alpha^2\left(-\frac{5393593}{32400} + \frac{80\,\pi^2}{9} + \frac{91061\,\ln(2)}{540} - \frac{797\ln^2(2)}{9} - \frac{41\,\ln(\alpha)}{6}\right.\\ &- \frac{4\,\ln(2)\,\ln(\alpha)}{3} + \frac{4\ln^2(\alpha)}{3} + \frac{41\,\ln(\epsilon)}{12} + \frac{2\,\ln(2)\,\ln(\epsilon)}{3}\\ &\left.- \frac{4\,\ln(\alpha)\,\ln(\epsilon)}{3} + \frac{\ln^2(\epsilon)}{3}\right), \end{aligned} \tag{8.38}$$

$$\begin{aligned} A_r \;=\;& \alpha^2\int_0^1 dt\frac{64}{9(1+t)^7}\Big\{4\,h(t)\,(1-2\,t)\,(-1+t)\,t\,(1+2\,t)\\ &+ 4\,p(t)\,(1-2\,t)\,(-1+t)\,t\,(1+2\,t)\\ &+ 2\,g(t)\,(1-2\,t)\,(1-t)\,(1+2\,t)\,(-1+t-t^2)\\ &+ q(t)\,(-1-2\,t)\,(-1+2\,t)\,(1+t^2) \end{aligned}$$

$$+\frac{2}{3}\,f(t)\,(-1+t)\,t\,\Big(3-47\,t-36\,t^2+68\,t^3+36\,t\,\ln(2)-144\,t^3\,\ln(2)$$
$$+24\,t\,\ln(t)-96\,t^3\,\ln(t)-36\,t\,\ln(1+t)+144\,t^3\,\ln(1+t)\Big)\Big\}, \quad (8.39)$$

where

$$p(t)=\sum_{k=1}^{\infty}\frac{1}{2+k-2\,t}\left(\frac{t-1}{t+1}\right)\,{}_2F_1^{(,2)}\Big(-k,4,4,\frac{2}{1+t}\Big) \tag{8.40}$$

$$q(t)=\sum_{k=1}^{\infty}\frac{1}{2+k-2\,t}\left(\frac{t-1}{t+1}\right)\,{}_2F_1\Big(-k,3,4,\frac{2}{1+t}\Big) \tag{8.41}$$

Summing up all contributions to the low energy part one obtains

$$\begin{aligned}
F_L &= (D_s+H_s+E_s+A_s)+D_r+\alpha^2 F_r \\
&= -\frac{3497}{45}+D_r+\frac{16\sqrt{2}\,\alpha}{3\,\sqrt{\epsilon}}+108\,\ln(2)-\frac{8\,\ln(\alpha)}{3}\frac{4\,\ln(\epsilon)}{3}+ \\
&\quad +\alpha^2\Big(-\frac{69532051}{226800}-\frac{2}{\epsilon}+F_r+\frac{343\,\pi^2}{27}+\frac{176177\,\ln(2)}{540}+\frac{8\,\ln(2)}{3\,\epsilon} \\
&\quad -\frac{451\ln^2(2)}{3}-\frac{99\,\ln(\alpha)}{10}+\frac{16\,\ln(\alpha)}{3\,\epsilon}-\frac{20\,\ln(2)\,\ln(\alpha)}{3}-4\ln^2(\alpha) \\
&\quad +\frac{99\,\ln(\epsilon)}{20}-\frac{8\,\ln(\epsilon)}{3\,\epsilon}+\frac{10\,\ln(2)\,\ln(\epsilon)}{3}+4\,\ln(\alpha)\,\ln(\epsilon)-\ln^2(\epsilon)\Big),
\end{aligned} \tag{8.42}$$

where D_r is given in (8.16), and

$$\begin{aligned}
F_r &= \frac{1}{\alpha^2}\int_0^1 dt\ \ (H_r+E_r+A_r) \\
&= \int_0^1 dt\frac{64}{9(1+t)^7}\Big[(1-2\,t)\,(1+2\,t)\,\Big(8\,h(t)\,(-1+t)\,t+4\,p(t)\,(-1+t)\,t \\
&\quad +q(t)\,(1+t^2)+g(t)\,(-1+t)\,(3-4\,t+3\,t^2)\Big) \\
&\quad +f(t)\,\Big(\frac{-(45+60\,t-995\,t^2-780\,t^3+2302\,t^4+720\,t^5-1568\,t^6)}{15\,(1+t)} \\
&\quad +8\,(1-2\,t)\,(-1+t)\,t^2\,(1+2\,t)\,\ln\Big(\frac{32\,t^3}{(1+t)^5}\Big)\Big)\Big]=-1.066519\,.
\end{aligned} \tag{8.43}$$

The final result is the sum

$$\begin{aligned}
F(2S) &= F_H(1S)+F_\Delta+F_L \\
&= D_r-\frac{383}{5}+\frac{320\,\ln(2)}{3}-\frac{8\,\ln(\alpha)}{3}+\alpha\,\Big(\frac{139\,\pi}{32}-2\,\pi\,\ln(2)\Big) \\
&\quad +\alpha^2\,\Big(F_r-\frac{138030713}{453600}+\frac{671\,\pi^2}{54}+\frac{42776\,\ln(2)}{135}-\frac{448\ln^2(2)}{3} \\
&\quad +\frac{5\,\zeta(3)}{2}-\ln(\alpha)\Big(\frac{67}{15}+\frac{32\,\ln(2)}{3}\Big)-4\ln^2(\alpha)\Big)\,.
\end{aligned} \tag{8.44}$$

The dependence on ϵ cancels out. All coefficients are in agreement with the previously known and the numerical value

$$A_{60}(2S)=-31.84047(1)\,. \tag{8.45}$$

is in agreement with (2.12).

Acknowledgment

I express my thanks to my thesis advisor Prof. I. Białynicki–Birula for encouragement during the stages of this work, and for detailed reviewing of the dissertation on which this paper is based. I also appreciate the inspiration given to me by Prof. T.W. Hänsch, whose high precision experiments were an important support of this work.

A. General notation

I use the notation from [47] and the units $\hbar = c = 1$. The metric tensor is$g^{00} = 1\,,\ g^{ii} = -1$. The Dirac equation is

$$(\not{p} - e\not{A} - m)\psi = 0 \tag{A1}$$

where $\not{q} = q \cdot \gamma = q^\mu \gamma_\mu$, and ψ is a four component wave function.
γ matrices fulfill the relation

$$[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}. \tag{A2}$$

They can be expressed in terms of matrices β and $\vec{\alpha}$,

$$\gamma^0 = \beta\,,\quad \vec{\gamma} = \beta\,\vec{\alpha}. \tag{A3}$$

The Dirac representation is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}. \tag{A4}$$

For the Coulomb field $A_0 = -\alpha/r$, the Dirac equation

$$(\vec{\alpha} \cdot \vec{p} + \beta m - \frac{\alpha}{r} - E)\psi = 0 \tag{A5}$$

can be rewritten using $K = \beta(\vec{\sigma} \cdot \vec{L} + 1)$,

$$\left\{\vec{\alpha} \cdot \hat{r}\Big[-\frac{i}{r}\frac{\partial}{\partial r} r + \frac{i}{r}\beta K\Big] + \beta m - \frac{\alpha}{r} - E\right\}\psi = 0. \tag{A6}$$

The total angular momentum operator $\vec{J} = \vec{L} + \frac{1}{2}\vec{\sigma}$, and K commute with the Dirac–Coulomb operator. Thus the solution of (A6) can be chosen to be the eigenstate of J^2, J_z and K with the eigenvalues $j(j+1), m, -k$, and

$$k = \begin{cases} l & l = j + \frac{1}{2} \\ -(l+1) & l = j - \frac{1}{2} \end{cases} \tag{A7}$$

The solution of (A6) can be written in the form,

$$\psi(\vec{r}) = \begin{pmatrix} f_1(r)\,\chi_k^m(\theta, \phi) \\ i f_2(r)\,\chi_{-k}^m(\theta, \phi) \end{pmatrix}. \tag{A8}$$

Spherical functions χ are defined in (6.41), and they have the property

$$\vec{\sigma} \cdot \hat{r}\ \chi_k^m(\theta, \phi) = \chi_{-k}^m(\theta, \phi)\,. \tag{A9}$$

Introducing new functions

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-k+\gamma}{\alpha} \\ \frac{-k+\gamma}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{A10}$$

where $\gamma = \sqrt{k^2 - \alpha^2}$, the solution of (A6) is

$$g_1(r) = N_1\,(2\kappa r)^\gamma\, e^{-\kappa r}\, L_{N-1}^{2\gamma+1}(2\kappa r)\,, \tag{A11}$$

$$g_2(r) = N_2\,(2\kappa r)^{\gamma-1}\, e^{-\kappa r}\, L_N^{2\gamma-1}(2\kappa r)\,, \tag{A12}$$

where $\kappa = \sqrt{m^2 - E^2}$, $N = \alpha E/\kappa - \gamma$ is the integer, and L is the Laguerre polynomial. The eigenvalue is

$$E = \frac{m}{\sqrt{1 + \dfrac{\alpha^2}{(N+\gamma)^2}}} \,. \tag{A13}$$

The normalization constants: for $N \neq 0$

$$N_1 = \frac{k}{|k|} \frac{2\kappa^2}{m} \sqrt{\frac{(N-1)!\,(k\,E + \gamma\,m)}{\Gamma(2\gamma + N + 1)(k+\gamma)\alpha}} \,, \tag{A14}$$

$$N_2 = \frac{2\kappa^2}{m} \sqrt{\frac{N!\,(k\,E - \gamma\,m)}{\Gamma(2\gamma + N)(k+\gamma)\,\alpha}} \,, \tag{A15}$$

and for $N = 0$

$$N_1 = 0 \,, \tag{A16}$$

$$N_2 = 4\gamma \left(\frac{m\alpha}{|k|} \right)^{3/2} \frac{1}{\sqrt{\Gamma(2\gamma+1)k(k+\gamma)}} \,. \tag{A17}$$

For the S–states $k = -1$, (let u denotes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) and

$$\psi(\vec{r}) = \begin{pmatrix} \psi_u(\vec{r}) \\ \psi_d(\vec{r}) \end{pmatrix} = \begin{pmatrix} f_1(r)u \\ -if_2(r)\vec{\sigma}\cdot\hat{r} \end{pmatrix} \frac{1}{\sqrt{4\pi}} \,. \tag{A18}$$

I introduce also a nonrelativistic function ψ_n,

$$\psi_n = \begin{pmatrix} \psi_{nu} \\ \psi_{nd} \end{pmatrix} \tag{A19}$$

and relativistic corrections to them ψ_r

$$\psi_r = \begin{pmatrix} \psi_{ru} \\ \psi_{rd} \end{pmatrix} \tag{A20}$$

For these I have the expressions:
the 1S state

$$\psi_{nu}(\vec{r}) = \phi(r)u = \sqrt{\frac{\lambda^3}{\pi}} e^{-\lambda r} u \,, \tag{A21}$$

$$\psi_{nd}(\vec{r}) = \frac{\vec{\sigma}\cdot\vec{p}}{2m} \phi(r) u \,, \tag{A22}$$

$$\psi_{ru}(\vec{r}) = \alpha^2 \sqrt{\frac{\lambda^3}{\pi}} e^{-\lambda r} \left(\frac{5}{8} - \frac{1}{2}\Big(C + \ln(2\lambda r)\Big) \right) u \,, \tag{A23}$$

$$\psi_{rd}(\vec{r}) = i\,\frac{\alpha^3}{2} \sqrt{\frac{\lambda^3}{\pi}} e^{-\lambda r} \left(\frac{7}{8} - \frac{1}{2}\Big(C + \ln(2\lambda r)\Big) \right) \vec{\sigma}\cdot\hat{r} u \,, \tag{A24}$$

the 2S state ($l = \frac{\lambda}{2} = \frac{m\alpha}{2}$)

$$\psi_{nu}(\vec{r}) = \phi(r)u = \sqrt{\frac{l^3}{\pi}} e^{-lr} (1 - l\,r) u \,, \tag{A25}$$

$$\psi_{nd}(\vec{r}) = \frac{\vec{\sigma}\vec{p}}{2m}\phi(r)u\,, \tag{A26}$$

$$\psi_{ru}(\vec{r}) = \alpha^2\sqrt{\frac{l^3}{\pi}}e^{-lr}\,\frac{1}{32}\Big(23 - 43l\,r + 4l^2r^2 - 16(1-l\,r)(C + \ln(2l\,r))\Big)u, \tag{A27}$$

$$\psi_{rd}(\vec{r}) = i\frac{\alpha^3}{2}\sqrt{\frac{l^3}{\pi}}e^{-lr}\,\frac{1}{128}\Big(64 - 53l\,r + 4l^2r^2 - 16(2-l\,r)(C + \ln(2l\,r))\Big)\vec{\sigma}\cdot\hat{r}u. \tag{A28}$$

The function ψ_{ru} for the 1S state in momentum representation is

$$\psi_{ru}(\vec{p}) = \frac{\alpha^2}{8}\frac{16\pi\lambda^{5/2}}{(p^2+\lambda^2)^2}\frac{u}{\sqrt{4\pi}}$$
$$+\frac{\alpha^2}{2}\frac{4\pi\lambda^{3/2}}{i\,p}\left(\frac{1}{(\lambda - i\,p)^2}\ln\Big(\frac{\lambda - i\,p}{2\lambda}\Big) - \frac{1}{(\lambda + i\,p)^2}\ln\Big(\frac{\lambda + i\,p}{2\lambda}\Big)\right)\frac{u}{\sqrt{4\pi}}\,. \tag{A29}$$

B. Basic integrals

In this apppendix I present the derivation and give the result for the integrals used throughout this work. The generic integral

$$I = \int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}\,\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\,\frac{1}{(\vec{p}\,'+\vec{k}')^2+\mu^2}\,G(\vec{p},\vec{p}\,',\Omega) \tag{B1}$$

was contained implicitly in the Gavrila and Costescu paper [40]. I follow here their derivation. Using the Schwinger representation of the Coulomb propagator $G = 1/(H-\Omega)$

$$G(\vec{p},\vec{p}\,',\Omega) = 4\pi m X^3\left(\frac{ie^{i\pi\tau}}{2\sin\pi\tau}\right)\int_1^{0^+} d\rho\rho^{-\tau}\,\frac{d}{d\rho}$$
$$\cdot\frac{1-\rho^2}{\rho}\,\frac{1}{\left[X^2(\vec{p}-\vec{p}\,')^2 + (p^2+X^2)(p'^2+X^2)\frac{(1-\rho)^2}{4\rho}\right]^2}\,, \tag{B2}$$

where $X = \sqrt{-2m\Omega}$, $\tau = m\alpha/X$, one obtains

$$I = \frac{4\pi m X^3}{(2\pi)^6}\int_0^1 \rho^{-\tau}\,\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J(\lambda,\mu)\right) \tag{B3}$$

$$J(\lambda,\mu) = \int d^3\vec{p}\int d^3\vec{p}\,'\;\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\;\frac{1}{(\vec{p}\,'+\vec{k}')^2+\mu^2}$$
$$\cdot\frac{1}{[X^2(\vec{p}-\vec{p}\,')^2 + (p^2+X^2)(p'^2+X^2)\frac{(1-\rho)^2}{4\rho}]^2}\,. \tag{B4}$$

The evaluation of $J(\lambda,\mu)$ is given in [40], with the following result

$$J(\lambda,\mu) = \frac{\pi^4}{X^2}\frac{4\rho}{1-\rho^2}\frac{1}{\sqrt{b}}\ln\left(\frac{a+\sqrt{b}}{a-\sqrt{b}}\right), \tag{B5}$$

where

$$a \;=\; a_1 + \frac{1-\rho}{2\rho}a_2\,, \tag{B6}$$

$$a_1 \;=\; X^2\Big[(\vec{k}-\vec{k}')^2 + (\lambda-\mu)^2\Big] + \mu X\Big[(X+\lambda)^2 + k^2\Big] + \lambda X\Big[(X+\mu)^2 + k'^2\Big],$$

$$(B7)$$

$$a_2 = \Big[(X+\lambda)^2+k^2\Big]\Big[(X+\mu)^2+k'^2\Big]\,, \tag{B8}$$

$$b = -X^2\Big\{\Big[(k^2+\lambda^2-X^2)\vec{k}'-(k'^2+\mu^2-X^2)\vec{k}\Big]^2+4X^2\Big[k^2k'^2-(\vec{k}\cdot\vec{k}')^2\Big]\Big\} \tag{B9}$$

After differentiation with respect to ρ

$$\frac{d}{d\rho}\left(\frac{1-\rho^2}{\rho}J(\lambda,\mu)\right) = -\frac{8\pi^4}{X^2}\frac{1}{a^2-b}\frac{da}{d\rho} = \frac{8\pi^4}{X^2}\frac{1}{a^2-b}\frac{a_2}{2\rho^2}\,, \tag{B10}$$

I becomes

$$I = \frac{mX}{2\pi}\int_0^1 d\rho\,\rho^{-\tau}\,\frac{1}{a^2-b}\,\frac{a_2}{2\rho^2}\,. \tag{B11}$$

Since

$$a^2-b = \left(\frac{a_2}{2\rho}\right)^2(1-x_1\rho)\,(1-x_2\rho)\,, \tag{B12}$$

where

$$x_1 = 1-\frac{2}{a_2}(a_1+\sqrt{b})\,,$$
$$x_2 = 1-\frac{2}{a_2}(a_1-\sqrt{b})\,, \tag{B13}$$

this integral can be transformed to

$$\begin{aligned} I &= \frac{mX}{\pi a_2}\int_0^1 d\rho\,\rho^{-\tau}\frac{1}{(x_1-x_2)\rho}\left(\frac{1}{1-x_1\rho}-\frac{1}{1-x_2\rho}\right) \\ &= -\frac{mX}{4\pi\sqrt{b}}\int_0^1 d\rho\,\rho^{-\tau-1}\left(\frac{1}{1-x_1\rho}-\frac{1}{1-x_2\rho}\right). \end{aligned} \tag{B14}$$

Using the integral representation of the hypergeometric function

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{B15}$$

one obtains the final expression for I

$$I = \frac{mX}{4\pi\sqrt{b}\,\tau}\Big({}_2F_1(1,-\tau,-\tau+1;x_1)-{}_2F_1(1,-\tau,-\tau+1;x_2)\Big)\,. \tag{B16}$$

Having calculated the generic integral I I pass to the evaluation of R given in (5.4).

$$\begin{aligned} R = &\left(\prod_{i=1}^{4}\int\frac{d^3\vec{p}_i}{(2\pi)^3}\right)\frac{1}{((\vec{p}_4+\vec{k})^2+\lambda^2)^2}\,\frac{1}{X^2+\vec{p}_4^{\,2}}\,\frac{4\pi\alpha}{(\vec{p}_4-\vec{p}_3)^2} \\ &\cdot\frac{1}{X^2+\vec{p}_3^{\,2}}\,\frac{4\pi\alpha}{(\vec{p}_3-\vec{p}_2)^2}\,\frac{1}{X^2+\vec{p}_2^{\,2}}\,\frac{4\pi\alpha}{(\vec{p}_2-\vec{p}_1)^2}\,\frac{1}{X^2+\vec{p}_1^{\,2}}\,\frac{1}{((\vec{p}_1+\vec{k})^2+\lambda^2)^2} \\ = &\frac{1}{(2m)^4}\int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p}\,'}{(2\pi)^3}G^{(3)}(\vec{p},\vec{p}\,',\Omega)\,\frac{1}{((\vec{p}+\vec{k})^2+\lambda^2)^2}\,\frac{1}{((\vec{p}\,'+\vec{k})^2+\lambda^2)^2} \\ = &\frac{1}{(2m)^4}\,\frac{\partial}{\partial\mu^2}\,\frac{\partial}{\partial\lambda^2}\bigg|_{\mu=\lambda} I(\vec{k}=\vec{k}') \\ = &\frac{1}{(2m)^4}\,\frac{mX}{2\pi}\tau^3\int_0^1 d\rho\,\frac{(-\ln\rho)^3}{3!}\,\frac{\partial}{\partial\mu^2}\,\frac{\partial}{\partial\lambda^2}\bigg|_{\mu=\lambda}\frac{1}{a^2-b}\,\frac{a_2}{2\rho^2}\,. \end{aligned} \tag{B17}$$

For my purposes I calculate R in the lowest order in λ. Since

$$b = -X^2 k^2 (\lambda^2 - \mu^2)^2\,,$$

I can put $b = 0$ in (B17). So that,

$$\int_0^1 d\rho \, \frac{(-\ln\rho)^3}{3!} \, \frac{1}{a^2} \, \frac{a_2}{2\rho^2} = \frac{2}{a_2} \int_0^1 d\rho \, \frac{(-\ln\rho)^3}{3!} \, \frac{1}{(1-\chi\rho)^2} = \frac{2}{a_2\chi} \, L_3(\chi)\,, \quad \text{(B18)}$$

where

$$L_3(\chi) = \sum_{i=1}^{\infty} \frac{\chi^i}{i^3} \quad \text{and} \quad \chi = 1 - \left. \frac{2a_1}{a_2} \right|_{\vec{k}=\vec{k}'}.$$

R becomes

$$R = \frac{1}{(2m)^4} \, \frac{mX}{2\pi} \tau^3 \, \frac{1}{4\lambda\mu} \, \frac{\partial}{\partial\lambda} \, \frac{\partial}{\partial\mu} \bigg|_{\mu=\lambda} \frac{2}{a_2\chi} \, L_3(\chi)\,. \quad \text{(B19)}$$

The lowest order appears when the differentiation acts on L_i function. Performing the differentiation I obtain

$$R = -\frac{\alpha}{16\pi m^2} \, \frac{1}{(X^2+k^2)^4} \ln\left(\frac{4\lambda X}{X^2+k^2}\right) + o(\lambda)^2\,. \quad \text{(B20)}$$

The basic integral

$$J = \frac{1}{(\pi)^4} \int d^3\vec{p} \int d^3\vec{p}\,'$$

$$\cdot \frac{1}{(\vec{p}+\vec{k})^2+\lambda^2} \, \frac{1}{X^2+\vec{p}^{\,2}} \, \frac{1}{(\vec{p}-\vec{p}\,')^2} \, \frac{1}{X^2+\vec{p}\,'^2} \, \frac{1}{(\vec{p}\,'+\vec{k}')^2+\mu^2} \quad \text{(B21)}$$

is calculated by expressing it in terms of I integral

$$J = \frac{1}{\pi^4} \frac{(2\pi)^6}{4\pi\alpha} \frac{1}{(2m)^2}$$

$$\cdot \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}\,'}{(2\pi)^3} \, \frac{1}{((\vec{p}+\vec{k})^2+\lambda^2)} \, \frac{1}{((\vec{p}\,'+\vec{k})^2+\mu^2)} \, G^{(1)}(\vec{p},\vec{p}\,',\Omega). \quad \text{(B22)}$$

Using (B14) one obtains

$$\begin{aligned} J &= -\frac{1}{\sqrt{b}} \int_0^1 d\rho \frac{(-\ln(\rho))}{\rho} \left(\frac{1}{1-x_1\rho} - \frac{1}{1-x_2\rho} \right) \\ &= \frac{1}{\sqrt{b}} \Big(L_2(x_2) - L_2(x_1) \Big) \equiv J(\lambda,\mu,X,\vec{k},\vec{k}')\,. \quad \text{(B23)} \end{aligned}$$

where x_1 and x_2 are defined in (B13). The basic J–integral allows calculation of all other J_i integrals by differentiation over $\lambda, \mu, \vec{k}$ and expansion in λ. So that,

$$\begin{aligned} J_1(X) &= \frac{1}{(\pi)^4} \int d^3\vec{p} \int d^3\vec{p}\,' \\ & \quad \frac{1}{(\vec{p}+\vec{k})^2+\lambda^2} \, \frac{1}{X^2+\vec{p}^{\,2}} \, \frac{1}{(\vec{p}-\vec{p}\,')^2} \, \frac{1}{X^2+\vec{p}\,'^2} \, \frac{1}{(\vec{p}\,'+\vec{k})^2+\lambda^2} \\ &= J(\lambda,\lambda,X,\vec{k},\vec{k}') = -\frac{4}{(X^2+k^2)^2} \ln\left(\frac{4\lambda X}{X^2+k^2}\right) + o(\lambda) \quad \text{(B24)} \end{aligned}$$

$$J_2(X) = \frac{1}{(\pi)^4} \int d^3\vec{p} \int d^3\vec{p}\,'$$

$$
\begin{aligned}
&\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\,\frac{1}{X^2+\vec{p}^{\,2}}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{(\vec{p}^{\,\prime}+\vec{k})^2+\lambda^2}\\
&= \lim_{X'\to\infty} X'^2 J(X,X',\lambda,\vec{k},\vec{k}') = \frac{4}{X^2+k^2}\\
&-\frac{2}{i\,k}\left(\frac{1}{X+i\,k}\ln\Big(\frac{X+i\,k}{2\lambda}\Big)-\frac{1}{X-i\,k}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\right)+o(\lambda) \qquad \text{(B25)}
\end{aligned}
$$

$$
\begin{aligned}
J_3(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\frac{1}{(\vec{p}+\vec{k})^2+\lambda^2}\,\frac{1}{X^2+\vec{p}^{\,2}}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{X^2+\vec{p}^{\,\prime 2}}\\
&= \lim_{\mu\to\infty}\mu^2 J(\lambda,\mu,X,\vec{k},\vec{k}')\\
&= \frac{1}{i\,kX}\left(L_2\Big(-\frac{X-i\,k}{X+i\,k}\Big)-L_2\Big(-\frac{X+i\,k}{X-i\,k}\Big)\right)+o(\lambda) \qquad \text{(B26)}
\end{aligned}
$$

$$
J_4(X) = \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\frac{1}{p^2+\lambda^2}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{(p'^2+\lambda^2)^2} = \frac{1}{\lambda^2} \qquad \text{(B27)}
$$

$$
\begin{aligned}
J_5(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\\
&\frac{1}{p^2+\lambda^2}\,\frac{1}{X^2+(\vec{p}-\vec{k})^2}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{X^2+(\vec{p}^{\,\prime}-\vec{k})^2}\,\frac{1}{(p'^2+\lambda^2)^2}\\
&= -\frac{1}{2}\frac{\partial}{\partial\lambda^2}J(\lambda,\lambda,X,\vec{k},\vec{k}) = \frac{1}{\lambda^2}\frac{1}{(X^2+k^2)^2}-\frac{1}{\lambda}\frac{2X}{(X^2+k^2)^3}\\
&+4\frac{(X^2-k^2)}{(X^2+k^2)^4}\left(1+\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\right)+o(\lambda) \qquad \text{(B28)}
\end{aligned}
$$

$$
\begin{aligned}
J_6(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\\
&\frac{1}{p^2+\lambda^2}\,\frac{1}{X^2+(\vec{p}-\vec{k})^2}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{\vec{k}\cdot\vec{p}^{\,\prime}}{X^2+(\vec{p}^{\,\prime}-\vec{k})^2}\,\frac{1}{(p'^2+\lambda^2)^2}\\
&= -\frac{1}{4}\vec{k}\frac{\partial}{\partial\vec{k}}J_1(X)\\
&= -\frac{4k^2}{(X^2+k^2)^3}\left(\frac{1}{2}+\ln\Big(\frac{4\lambda X}{X^2+k^2}\Big)\right)+o(\lambda) \qquad \text{(B29)}
\end{aligned}
$$

$$
\begin{aligned}
J_7(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\\
&\frac{1}{X^2+(\vec{p}-\vec{k})^2}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{X^2+(\vec{p}^{\,\prime}-\vec{k})^2}\,\frac{1}{(p'^2+\lambda^2)^2}\\
&= -\frac{\partial}{\partial(\lambda^2)}\lim_{\mu\to\infty}\mu^2 J(\lambda,\mu,X,\vec{k},\vec{k}') = \frac{i}{k\lambda}\,\frac{1}{X^2+k^2}\,\ln\Big(\frac{X-i\,k}{X+i\,k}\Big)\\
&-\frac{2}{(X^2+k^2)^2}\left(1+\ln\Big(\frac{4X^2}{X^2+k^2}\Big)\right)+o(\lambda) \qquad \text{(B30)}
\end{aligned}
$$

$$
\begin{aligned}
J_8(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}^{\,\prime}\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{(\vec{p}-\vec{p}^{\,\prime})^2}\,\frac{1}{X^2+(\vec{p}^{\,\prime}-\vec{k})^2}\,\frac{1}{p'^2+\lambda^2}\\
&= \frac{1}{\lambda^2}\frac{1}{(X+\lambda)^2+k^2}+o(\lambda) \qquad \text{(B31)}
\end{aligned}
$$

$$
\begin{aligned}
J_9(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}\,'\frac{1}{p^2+\lambda^2}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{X^2+(\vec{p}\,'-\vec{k})^2}\,\frac{1}{(p'^2+\lambda^2)^2} \\
&= -\frac{\partial}{\partial\lambda^2}J_2(X) - J_8(X) \\
&= \frac{1}{\lambda^2}\frac{1}{X^2+k^2} + \frac{1}{9}\frac{1}{(X^2+k^2)^2} - \frac{4}{9}\frac{X^2}{(X^2+k^2)^3} \\
-&\frac{2}{3}\frac{i}{k}\Big(\frac{1}{(X+i\,k)^3}\ln\Big(\frac{X+i\,k}{2\lambda}\Big) - \frac{1}{(X-i\,k)^3}\ln\Big(\frac{X-i\,k}{2\lambda}\Big)\Big) + o(\lambda)
\end{aligned}
\tag{B32}
$$

$$
\begin{aligned}
J_{10}(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}\,'\frac{1}{X^2+(\vec{p}-\vec{k})^2}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{(p'^2+\lambda^2)^2} \\
&= \frac{i}{k\lambda}\ln\Big(\frac{X+\lambda-i\,k}{X+\lambda+i\,k}\Big) = \frac{i}{k\lambda}\ln\Big(\frac{X-i\,k}{X+i\,k}\Big) - \frac{2}{X^2+k^2} + o(\lambda)
\end{aligned}
\tag{B33}
$$

$$
\begin{aligned}
J_{11}(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}\,' \\
&\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{X^2+(\vec{p}-\vec{k})^2}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{X^2+(\vec{p}\,'-k)^{\vec{2}}}\,\frac{1}{(p'^2+\lambda^2)^2} \\
&= \frac{\partial}{\partial(\mu^2)}\bigg|_{\mu=\lambda}\frac{\partial}{\partial(\lambda^2)}J(\lambda,\mu,X,\vec{k},\vec{k}) \\
&= \frac{1}{4\lambda^4}\frac{1}{(X^2+k^2)^2} + \frac{1}{\lambda^2}\Big(-\frac{7}{3}\frac{X^2}{(X^2+k^2)^4} + \frac{5}{6}\frac{1}{(X^2+k^2)^3}\Big) \\
&+ \frac{1}{\lambda}\Big(\frac{8X^3}{(X^2+k^2)^5} - \frac{4X}{(X^2+k^2)^4}\Big) \\
&+ \Big(-\frac{116}{3}\frac{X^4}{(X^2+k^2)^6} + \frac{36X^2}{(X^2+k^2)^5} - \frac{79}{12}\frac{1}{(X^2+k^2)^4}\Big) \\
&+ \ln\left(\frac{4\lambda X}{k^2+X^2}\right)\Big(-\frac{64}{3}\frac{X^4}{(X^2+k^2)^6} + \frac{64}{3}\frac{X^2}{(X^2+k^2)^5} \\
&- \frac{4}{(X^2+k^2)^4}\Big) + o(\lambda)
\end{aligned}
\tag{B34}
$$

$$
\begin{aligned}
J_{12}(X) &= \frac{1}{(\pi)^4}\int d^3\vec{p}\int d^3\vec{p}\,'\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{(\vec{p}-\vec{p}\,')^2}\,\frac{1}{X^2+(\vec{p}\,'-\vec{k})^2}\,\frac{1}{(p'^2+\lambda^2)^2} \\
&= \frac{1}{\lambda}K_3(X)\,.
\end{aligned}
\tag{B35}
$$

I proceed now to the definition of K-integrals. All of them can be derived from the first one

$$
K_1(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{p^2+\lambda^2}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = \frac{i}{k}\ln\Big(\frac{X+\lambda-i\,k}{X+\lambda+i\,k}\Big) \tag{B36}
$$

$$
K_2(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = \frac{1}{\lambda}\,\frac{1}{(X+\lambda)^2+k^2} \tag{B37}
$$

$$
K_3(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^3}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = -\frac{1}{2}\,\frac{\partial}{\partial(\lambda^2)}\,K_2(X) \tag{B38}
$$

$$
K_4(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^4}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = \frac{1}{6}\,\frac{\partial^2}{\partial(\lambda^2)^2}\,K_2(X) \tag{B39}
$$

$$K_5(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^2} = \frac{1}{\lambda} \tag{B40}$$

$$K_6(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{2\vec{k}\cdot\vec{p}}{(p^2+\lambda^2)^2}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = -\vec{k}\,\frac{\partial}{\partial\vec{k}}\,K_1(X) \tag{B41}$$

$$K_7(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{2\vec{k}\cdot\vec{p}}{(p^2+\lambda^2)^4}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = -\frac{1}{3}\vec{k}\,\frac{\partial}{\partial\vec{k}}\,K_3(X)\,. \tag{B42}$$

The next integrals of G-type originate from the integrals with the relativistic wave function. Each of them is calculated separately in zeroth order in λ. The first G-integral

$$G_1(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{p^2+\lambda^2}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2}$$
$$\cdot\frac{1}{i\,p}\Big(\frac{1}{(\lambda-i\,p)^2}\ln\Big(\frac{\lambda-i\,p}{2\lambda}\Big) - \frac{1}{(\lambda+i\,p)^2}\ln\Big(\frac{\lambda+i\,p}{2\lambda}\Big)\Big) \tag{B43}$$

after the angle integration

$$\int d\Omega_p\,\frac{1}{(\vec{p}-\vec{k})^2+X^2} = \frac{\pi}{p\,k}\ln\Big(\frac{X^2+(p+k)^2}{X^2+(p-k)^2}\Big) \tag{B44}$$

becomes

$$G_1(X) = \frac{1}{i\pi\,k}\int_{-\infty}^{\infty}dp\,\frac{1}{p^2+\lambda^2}\,\frac{1}{(\lambda-i\,p)^2}\ln\Big(\frac{\lambda-i\,p}{2\lambda}\Big)$$
$$\cdot\Big\{\ln\Big(X+i(p+k)\Big)+\ln\Big(X-i(p+k)\Big)$$
$$-\ln\Big(X+I(p-k)\Big)-\ln\Big(X-i(p-k)\Big)\Big\}. \tag{B45}$$

If we assume that the contour is closed in the upper half plain, the integral can be rewritten as:

$$G_1(X) = \frac{1}{i\pi\,k}\int_{-\infty}^{\infty}dp\,\frac{1}{p^2+\lambda^2}\,\frac{1}{(\lambda-i\,p)^2}\,\ln\Big(\frac{\lambda-i\,p}{2\lambda}\Big)\,\ln\Big(X+i(p+k)\Big)+c.c. \tag{B46}$$

where the contour of integration is along only the branch cut of the second logarithm in the above expression. After expanding in λ, I obtain

$$G_1(X) = -\frac{2}{3}\,\frac{i}{k(X+i\,k)^3}\Big(\frac{1}{3}+\ln\Big(\frac{X+i\,k}{2\lambda}\Big)\Big)+c.c.+o(\lambda)\,. \tag{B47}$$

The evaluation of G_2 integral proceeds in the similar way

$$G_2(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2}\,\frac{1}{i\,p}\Big(\frac{1}{(\lambda-i\,p)^2}\ln\Big(\frac{\lambda-i\,p}{2\lambda}\Big)$$
$$-\frac{1}{(\lambda+i\,p)^2}\ln\Big(\frac{\lambda+i\,p}{2\lambda}\Big)\Big)$$
$$= \frac{2i}{k}\,\frac{1}{X+i\,k}\Big(1+\ln\Big(\frac{X+i\,k}{2\lambda}\Big)\Big)+c.c.+o(\lambda)\,. \tag{B48}$$

G_3 integral is

$$G_3(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{(\vec{p}-\vec{k})^2+X^2}$$

	1S	2S
$\langle\psi_u\|\psi_u\rangle =$	$1-\frac{\alpha^2}{4}-\frac{\alpha^4}{16}-\frac{\alpha^6}{32}$	$1-\frac{\alpha^2}{16}-\frac{5\alpha^4}{256}-\frac{21\alpha^6}{2048}$
$\langle\psi_u\|\frac{\alpha}{r}\|\psi_u\rangle =$	$\alpha^2+\frac{\alpha^4}{4}+\frac{3\alpha^6}{16}$	$\frac{\alpha^2}{4}+\frac{7\alpha^4}{64}+\frac{79\alpha^6}{1024}$
$\langle\psi_d\|\psi_d\rangle =$	$\frac{\alpha^2}{4}+\frac{\alpha^4}{16}+\frac{\alpha^6}{32}$	$\frac{\alpha^2}{16}+\frac{5\alpha^4}{256}+\frac{21\alpha^6}{2048}$
$\langle\psi_d\|\frac{\alpha}{r}\|\psi_d\rangle =$	$\frac{\alpha^4}{4}+\frac{3\alpha^6}{16}$	$\frac{3\alpha^4}{64}+\frac{47\alpha^6}{1024}$
$\langle\psi_u\|\left(\frac{\alpha}{r}\right)^2\|\psi_u\rangle =$	$2\alpha^4+\frac{5\alpha^6}{2}$	$\frac{\alpha^4}{4}+\frac{29\alpha^6}{64}$
$\langle\psi_d\|\left(\frac{\alpha}{r}\right)^2\|\psi_d\rangle =$	$\frac{\alpha^6}{2}$	$\frac{5\alpha^6}{64}$
$\langle\psi_{nd}\|\frac{\alpha}{r}\|\psi_{rd}\rangle =$	$\frac{3\alpha^6}{32}$	$\frac{47\alpha^6}{2048}$
$\langle\psi_{nu}\|\frac{\alpha}{r}\|\psi_{ru}\rangle =$	$\frac{\alpha^4}{8}$	$\frac{7\alpha^4}{128}$
$\langle\psi_{nu}\|\left(\frac{\alpha}{r}\right)^2\|\psi_{ru}\rangle =$	$\frac{5\alpha^6}{4}$	$\frac{29\alpha^6}{128}$
$\langle\psi_{nd}\|p^2\|\psi_{rd}\rangle =$	$\frac{33\alpha^6}{32}$	$\frac{425\alpha^6}{2048}$
$\langle\psi_u\|p^2\|\psi_u\rangle =$	$\alpha^2+\frac{3\alpha^4}{4}+\frac{15\alpha^6}{16}$	$\frac{\alpha^2}{4}+\frac{15\alpha^4}{64}+\frac{279\alpha^6}{1024}$
$\langle\psi_d\|p^2\|\psi_d\rangle =$	$\frac{5\alpha^4}{4}+\frac{33\alpha^6}{16}$	$\frac{13\alpha^4}{64}+\frac{425\alpha^6}{1024}$
$\langle\overline{\psi}_n\|\vec{p}\cdot\vec{\gamma}\,\frac{\alpha}{r}+\frac{\alpha}{r}\vec{p}\cdot\vec{\gamma}\|\psi_r\rangle =$	$3\alpha^6$	$\frac{19\alpha^6}{32}$
$\langle\psi_{nu}\|p^2\,\frac{\alpha}{r}\|\psi_{ru}\rangle =$	$\frac{19\alpha^6}{8}$	$\frac{225\alpha^6}{512}$
$\langle\psi_{nu}\|\vec{p}\,\frac{\alpha}{r}\vec{p}\|\psi_{ru}\rangle =$	$\frac{9\alpha^6}{8}$	$\frac{121\alpha^6}{512}$

Table 1: Expectation values of some operators in order α^6, which were used in the evaluation of high energy part

$$
\cdot \frac{1}{i\,p}\left(\frac{1}{(\lambda - i\,p)^2} \ln\Big(\frac{\lambda - i\,p}{2\lambda}\Big) - \frac{1}{(\lambda + i\,p)^2} \ln\Big(\frac{\lambda + i\,p}{2\lambda}\Big)\right)
$$

$$
= \frac{1}{i\pi\,k}\int_{-\infty}^{\infty} dp\,\frac{1}{(p^2+\lambda^2)^2}\,\frac{1}{(\lambda - i\,p)^2}\ln\Big(\frac{\lambda - i\,p}{2\lambda}\Big)
$$

$$
\cdot\Big\{\ln\Big(X + i(p+k)\Big) + \ln\Big(X - i(p+k)\Big)
$$

$$
- \ln\Big(X + i(p-k)\Big) - \ln\Big(X - i(p-k)\Big)\Big\}. \tag{B49}
$$

The pole at $p = i\,\lambda$ also contributes here, thus G_3 becomes

$$
G_3(X) = \frac{2i}{k}\,\frac{1}{(2\lambda)^5}\ln\Big(\frac{X^2 + (k + i\,\lambda)^2}{X^2 + (k - i\,\lambda)^2}\Big)
$$

$$
+ \Big\{\frac{2i}{5k}\,\frac{1}{(X + i\,k)^5}\Big(\frac{1}{5} + \ln\Big(\frac{X + i\,k}{2\lambda}\Big)\Big) + c.c.\Big\} + o(\lambda)\,. \tag{B50}
$$

The evaluation of G_4 is similar to the previous one, hence

$$
G_4(X) = \frac{1}{\pi^2}\int d^3\vec{p}\,\frac{1}{(p^2+\lambda^2)^2}\,\frac{2\vec{k}\cdot\vec{p}}{(\vec{p}-\vec{k})^2 + X^2}
$$

$$
\cdot \frac{1}{i\,p}\left(\frac{1}{(\lambda - i\,p)^2} \ln\Big(\frac{\lambda - i\,p}{2\lambda}\Big) - \frac{1}{(\lambda + i\,p)^2} \ln\Big(\frac{\lambda + i\,p}{2\lambda}\Big)\right)
$$

$$
= \frac{1}{(2\lambda)^4}\left(\frac{i}{k\lambda}(X^2 + k^2 - \lambda^2)\ln\Big(\frac{X^2 + (k + i\lambda)^2}{X^2 + (k - i\lambda)^2}\Big) + 4\right)
$$

$$
+ \Big\{\frac{2i}{5k}\,\frac{X^2 + k^2}{(X + i\,k)^5}\Big(\frac{1}{5} + \ln\Big(\frac{X + i\,k}{2\lambda}\Big)\Big)
$$

$$
- \frac{2i}{3k}\,\frac{1}{(X + i\,k)^3}\Big(\frac{1}{3} + \ln\Big(\frac{X + i\,k}{2\lambda}\Big)\Big) + c.c.\Big\}. \tag{B51}
$$

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