# Quantum electrodynamics effects on helium fine structure

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**Abstract.** It is studied quantum electrodynamical effects on helium fine structure splitting. The Douglass and Kroll result at order $m\,\alpha^6$ is obtained in a simple way using the time ordered perturbation technique. The logarithmic correction at order $m\,\alpha^7\,\ln\alpha$ as calculated by Tao Zhang is rederived and verified.

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## 1. Introduction

The calculation of quantum electrodynamics (QED) effects in few electron atoms is a much more complex task than in the case of a hydrogenic system. The presence of the electron–electron interaction disallow one to write a simple analog of a Dirac equation for two electrons in the external field. Several approaches have been developed since the beginning of QED, mostly based on the Bethe-Salpeter equation. They have been applied up to the order $m\,\alpha^5$ [1] and at order $m\,\alpha^6$ [2] for helium fine structure splitting (fs). During the last 10 years precise spectroscopic measurements on hydrogen [3], positronium [4] and helium [5] atoms have stimulated an intensive development of bound state QED. Almost all corrections through order $m\,\alpha^7$ have been calculated for hydrogen and muonium, and recently to order $m\,\alpha^6$ [6] for positronium. However, theoretical predictions for the helium energy levels are less precise. Although one solved Schrödinger equation with the 15 digits precision [7], there was no method to incorporate higher order relativistic effects. Using an effective Hamiltonian approach we have derived complete set of operators for the $m\,\alpha^6$ correction of helium S-levels [8]. It still remains the problem, of the calculation of their matrix elements with the nonrelativistic wave function. Such a calculation has only been performed for helium fine structure [9]. Recently Zhang has derived [10] with the adapted Bethe–Salpeter formalism the operators for the $m\,\alpha^7$ helium fine structure, however the question remains on the completeness of this set of operators. The purpose of this paper is the development of the time ordered perturbation technique for the independent calculation of $m\,\alpha^7$ contribution to helium *fs* and the verification of $m\,\alpha^7\,\ln\alpha$ terms as calculated by Tao Zhang in [10, 11].

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In section 2 we summarize lower order terms for helium fine structure splitting. The section 3 is devoted to the description of time ordered perturbation technique. In section 4 we rederive helium fine structure splitting at the order $m\,\alpha^6$. The hydrogen fine structure splitting at order $m\,\alpha^7$ is recalculated in a simple way in section 5. The purpose of this section is the demonstration that the radiative correction could be incorporated by the on–mass shell electron formfactors. The logarithmic $m\,\alpha^7\ln\alpha$ correction in helium is obtained in section 6. The summary with comments on the calculation of $m\,\alpha^7$ term is presented in section 7. In this paper we heavily base on former calculations on hydrogen [12], positronium [6] and helium atom [8].

## 2. Leading orders contributions

The helium fine structure splitting is the difference between $n^3P_J$ energy levels with different values of total angular momentum $J$. It is a purely relativistic effects and comes from the interaction of the spin with angular momenta. The leading contribution to fine structure splitting of order $m\,\alpha^4$ could be obtained from the Breit Hamiltonian [13] which we denote here by $H^{(4)}$.

$$H^{(4)} = H_{SI} + H_{SL} + H_{SS}\,, \tag{1}$$

$$\begin{aligned} H_{SI} &= -\frac{1}{8\,m^3}\,(p_1^4+p_2^4) + \frac{Z\,\alpha\,\pi}{2\,m^2}\,(\delta^3(r_1)+\delta^3(r_2)) \\ &\quad -\frac{\alpha}{2\,m^2}\,p_1^i\left(\frac{\delta^{ij}}{r}+\frac{r^i\,r^j}{r^3}\right)p_2^j - \frac{\pi\,\alpha}{m^2}\,\delta^3(r) - \frac{2\,\pi\,\alpha}{3\,m^2}\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,\delta^3(r)\,, \end{aligned} \tag{2}$$

$$\begin{aligned} H_{SL} &= \frac{Z\,\alpha}{4\,m^2}\left[\frac{1}{r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \frac{1}{r_2^3}\,\boldsymbol{r}_2\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right] \\ &\quad + \frac{\alpha}{4\,m^2\,r^3}\,\left[(\boldsymbol{\sigma}_2+2\,\boldsymbol{\sigma}_1)\cdot\boldsymbol{r}\times\boldsymbol{p}_2 - (\boldsymbol{\sigma}_1+2\,\boldsymbol{\sigma}_2)\cdot\boldsymbol{r}\times\boldsymbol{p}_1\right]\,, \end{aligned} \tag{3}$$

$$H_{SS} = \frac{\alpha}{4\,m^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3} - 3\,\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{r}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{r}}{r^5}\right)\,. \tag{4}$$

Although not all these terms contribute to the P-splitting, we will use complete $H^{(4)}$ later for the derivation of higher order corrections. Only $H_{SL}$ and $H_{SS}$ contribute, so the main contribution to P-splitting is given by

$$E(^3P_J) = \langle H^{(4)}\rangle = \langle H_{SL}+H_{SS}\rangle\,. \tag{5}$$

The leading correction comes from the electron anomalous magnetic moment only, and could be easily incorporated in the Breit Hamiltonian:

$$\begin{aligned} H^{(5)} &= \frac{\alpha}{\pi}\left\{\frac{Z\,\alpha}{4\,m^2}\left(\frac{1}{r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \frac{1}{r_2^3}\,\boldsymbol{r}_2\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right)\right. \\ &\quad - \frac{\alpha}{4\,m^2\,r^3}\,\boldsymbol{r}\times(\boldsymbol{p}_1-\boldsymbol{p}_2)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) \\ &\quad \left. + \frac{\alpha}{4\,m^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{r^3} - 3\,\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{r}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{r}}{r^5}\right)\right\}. \end{aligned} \tag{6}$$

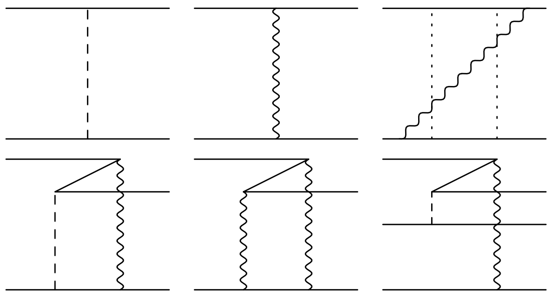

**Figure 1.** Time ordered diagrams contributing to fine structure splitting in order $m\,\alpha^6$. The middle horizontal line corresponds to helium, two outer horizontal lines correspond to two electrons. The wavy line is a transverse photon, the dashed line is a Coulomb photon, and the dotted line denotes retardation.

Recoil corrections coming from finite nuclear mass could also been incorporated in the Breit Hamiltonian (up to the order $m\,\alpha^5$ for the fine structure) and have been calculated in [7]. We will not consider them through this paper.

## 3. Time–ordered perturbation technique

For the calculation of the effective Hamiltonian we use the time–ordered perturbation technique. For example the $m\,\alpha^6$ correction is given by diagrams in Fig.1. There are rules to derive an expression from the corresponding time ordered diagram. We use the Coulomb gauge, so have a Coulomb photon (the dashed line) and the transverse one (the wavy line), the dotted line denotes something else, to be explained later. The vertex for the Coulomb exchange is $e$ ($-e$ for the positron) which is multiplied by $(2\,\pi)^3\,\delta^3(\Sigma_i \boldsymbol{p}_i)$ to conserve three-dimensional momentum. The dashed line is $1/q^2$, the Fourier transform of the Coulomb potential. The vertex for the transverse photon is $e\,\alpha^i$ ($-e\,\alpha^i$ for the positron) times the corresponding $\delta$-function to preserve momentum. The wavy line is $1/(2\,q)\,(\delta^{ij} - q^i\,q^j/q^2)$. There are projection operators between the vertices on electron lines onto positive or negative energy states, depending on the time arrow. The additional terms are energy denominators. Each denominator is equal to $E - \Sigma_i E_i$, where the sum goes over all intermediate state. Such a factor is between two Coulomb exchanges, between emission and absorption of the transverse photon, and between the transverse vertex and the Coulomb exchange. There is also a factor -1 for any electron positron pair creation. When the transverse photon is involved, the leading term in the denominator is the photon energy, and for the most cases one can neglect other terms in the denominator, and we denote it by a vertical wavy line. Since there are two possibilities, any of the electrons could first emit a photon and the second absorbs, the overall factor is $-2/q$. We have introduced a dotted line which denotes both the Coulomb exchange *and* the expansion of the energy denominator with respect to the kinetic energy. The whole expression is enclosed by $u^{+}(p')$ and $u(p)$, the free fermion bispinors, which correspond to outer lines.

## 4. Fine structure splitting at order $m\,\alpha^6$

This correction has been first derived by Douglas and Kroll [2] using the Bethe–Salpeter formalism and recently checked by Zhang in [10]. We rederive here their result in a much simpler way, using time ordered perturbation technique. The correction to the energy at order $m\,\alpha^6$ is the sum of two terms

$$E^{(6)} = E^{(6)}_1 + E^{(6)}_2\,. \tag{7}$$

The first one $E^{(6)}_1$ is

$$E^{(6)}_1 = \left\langle H^{(4)}\,\frac{1}{(E-H)'}\,H^{(4)}\right\rangle\,, \tag{8}$$

where $E$ and $H$ are nonrelativistic energy and Hamiltonian respectively. One calculates in (8) only the spin dependent terms, which contributes to the fine structure splitting, other terms are divergent. The second term is an analogous of the Breit Hamiltonian at order $m\,\alpha^6$.

$$E^{(6)}_2 = \langle H^{(6)}\rangle\,. \tag{9}$$

This Hamiltonian $H^{(6)}$ could be derived using time ordered perturbation diagrams, which are presented on Fig. 1. In the derivation of the effective Hamiltonian we use the fact that the wave function has spin equal to 1. It implies that all expectation values with $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ vanish. Moreover one takes advantage that the wave function vanishes when $\boldsymbol{r}_1 = \boldsymbol{r}_2$, it causes that matrix elements of all operators obtained by expansion at order $m\,\alpha^6$ are finite. However, they will be no longer finite at $m\,\alpha^7$ order and this problem would require a separate treatment. We denote by $V$ the algebraic expression that corresponds to diagram, namely the effective interaction potential. The correction to energy is obtained by taking the matrix element $\Delta E = \langle\phi|V|\phi\rangle$.

The electron–nucleus part $V_0$ could be derived from the Dirac equation using the Fouldy–Wouthuysen transformation, but it could also be obtained from the first diagram in Fig. 1. We may assume for a while that a lower line denotes a static nucleus, while the upper one denotes one of the electrons, then

$$V_0 = -\frac{Z\,e^2}{q^2}\,\bar{u}(p')\gamma^0 u(p)\,, \tag{10}$$

where $\boldsymbol{q}$ is a momentum exchange from $\boldsymbol{p}$ to $\boldsymbol{p}'$, $\boldsymbol{q} = \boldsymbol{p}' = \boldsymbol{p}$ and $u(p)$ is a positive energy solution of the free Dirac equation.

$$u(p) = \begin{pmatrix} \sqrt{\frac{E(p)+m}{2\,E(p)}} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2\,E(p)\,(E(p)+m)}} \end{pmatrix}, \tag{11}$$

where $E(p) = \sqrt{p^2+m^2}$. The scalar product after expansion in momenta takes a form

$$\begin{aligned} u^+(p')u(p) &= 1 - \frac{q^2}{8\,m^2} + \frac{i}{4\,m^2}\,\boldsymbol{p}'\times\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ &+ \frac{5}{128\,m^4}\,(p^2-p'^2)^2 + \frac{3}{64\,m^4}\,(q^2 - 2\,i\,\boldsymbol{p}'\times\boldsymbol{p}\cdot\boldsymbol{\sigma})\,(p'^2+p^2) \end{aligned} \tag{12}$$

We need only the $m\,\alpha^6$ part, so after the expansion in momentum and neglection of spin independent terms $V_0$ becomes

$$V_0 = \frac{Z\,e^2}{q^2}\,\frac{3\,i}{32\,m^4}(p^2+p'^2)\,\boldsymbol{p}'\times\boldsymbol{p}\cdot\boldsymbol{\sigma}\,. \tag{13}$$

In position representation it has a form

$$V_0 = -\frac{3}{32\,m^4}\left(p^2\,\frac{Z\,\alpha}{r^3}\,\boldsymbol{r}\times\boldsymbol{p}\cdot\boldsymbol{\sigma} + \frac{Z\,\alpha}{r^3}\,p^2\,\boldsymbol{r}\times\boldsymbol{p}\cdot\boldsymbol{\sigma}\right). \tag{14}$$

Together with the lower order Hamiltonian

$$H^{(4)} = -\frac{p^4}{8\,m^3} - \frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{r}\times\boldsymbol{p}\cdot\boldsymbol{\sigma}\,, \tag{15}$$

it gives the $m\,\alpha^6$ fine structure splitting in hydrogen in accordance with that obtained from the Dirac equation. For the case of helium one should include both electrons. However their matrix element is the same. As a final expression we have

$$V_0 = -\frac{3\,Z\,\alpha}{8\,m^4}\,p_1^2\,\frac{1}{r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1\,. \tag{16}$$

We start the calculation of electron–electron terms from the single Coulomb exchange, *i.e.* diagram 1 in Fig. 1. The initial expression is

$$V_1 = \frac{e^2}{q^2}\,u^+(p_1')u(p_1)\;u^+(p_2')u(p_2)\,, \tag{17}$$

where

$$\boldsymbol{q} = \boldsymbol{p}_1' - \boldsymbol{p}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_2'\,. \tag{18}$$

$V_1$ is expanded in momenta up to $p^4$, and we will keep only the terms that give $\alpha^6$ correction. Moreover, we calculate only the fine structure, so all the spin independent and the terms proportional to $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ are neglected. The single Coulomb exchange $V_1$ after $p$–expansion is

$$\begin{aligned} V_1 = &-\frac{1}{16\,m^4}\,\frac{e^2}{q^2} \\ &\left\{\frac{i}{2}\,q^2\,(\boldsymbol{p}_1'\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \boldsymbol{p}_2'\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2) + \boldsymbol{p}_1'\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1\,\boldsymbol{p}_2'\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right. \\ &\left.+\frac{3}{2}\,\left(p_1^2+p_1'^2\right)\,i\,\boldsymbol{p}_1'\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \frac{3}{2}\,\left(p_2^2+p_2'^2\right)\,i\,\boldsymbol{p}_2'\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right\}, \end{aligned} \tag{19}$$

where we have neglected spin independent terms. In the further calculation we compress an expression by changing the index 2 to 1, whenever it is equivalent for the matrix element. We will call it a D-K simplification. The contribution to energy in the coordinate representation becomes

$$V_1 = \frac{3\,\alpha}{8\,m^4}\,\frac{\boldsymbol{r}}{r^5}\,(\boldsymbol{r}\cdot\boldsymbol{p_2})\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \frac{3\,\alpha}{8\,m^4}\,\boldsymbol{p}_1^2\,\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 - \frac{3\,\alpha}{16\,m^4\,r^5}\,\boldsymbol{r}(\boldsymbol{r}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1)\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2 - \frac{\alpha}{16\,m^4\,r^3}\,(\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_2)\,(\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_1) \tag{20}$$

where $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2$.

The next contribution is the exchange of a single transverse photon in the no retardation limit, diagram 2 in Fig. 1.

$$V_2 = -\frac{e^2}{q^2}\,[u^+(p_1')\,\boldsymbol{\alpha}^i\,u(p_1)]\;[u^+(p_2')\,\boldsymbol{\alpha}^j\,u(p_2)]\,\left(\delta^{ij} - \frac{q^i\,q^j}{q^2}\right). \tag{21}$$

As in the case of single Coulomb exchange one expands current in momenta:

$$u^+(p')\,\boldsymbol{\alpha}^i\,u(p) = \frac{1}{2\,m}\,(p^i + p'^i) + \frac{i}{2\,m}(\boldsymbol{\sigma}\times\boldsymbol{q})^i - \frac{1}{16\,m^3}\,[p^i\,(p'^2 + 3\,p^2) + p'^i\,(p^2 + 3\,p'^2)] + \frac{i}{16\,m^3}[(\boldsymbol{p}'\times\boldsymbol{\sigma})^i\,(p^2 + 3\,p'^2) - (\boldsymbol{p}\times\boldsymbol{\sigma})^i\,(p'^2 + 3\,p^2)] \tag{22}$$

When this current is replaced by the expanded form, after $D-K$ simplification $V_2$ becomes:

$$V_2 = -\frac{e^2}{8\,m^4\,q^2}\Big\{4\,i\,p_1'^2\,\boldsymbol{\sigma}_1\times\boldsymbol{q}\cdot\boldsymbol{p}_1 + p_1'^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\,\boldsymbol{\sigma}_2\cdot(\boldsymbol{p}_1' + \boldsymbol{p}_1) + 2\,p_1'^2\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{q}) + 2\,i\,p_1'^2\left(\delta_{ij} - \frac{q_i\,g_j}{q^2}\right)\,p_2^j\,[2\,\boldsymbol{p}_1\times\boldsymbol{\sigma}_1 + 3\,\boldsymbol{q}\times\boldsymbol{\sigma}_1]^i\,. \tag{23}$$

In configuration space $V_2$ takes the form

$$V_2 = \frac{\alpha}{2\,m^4}\,p_1^2\,\boldsymbol{\sigma}_1\cdot\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{p}_1 + \frac{9\,\alpha}{8\,m^4}\,p_1^2\,\frac{1}{r^5}\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}) - \frac{i\,\alpha}{4\,m^4}\,p_1^2\,\frac{1}{r^3}\,(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{p}_1) + \frac{i\,\alpha}{4\,m^2}\,p_1^2\,\frac{1}{r}\,\boldsymbol{p}_1\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_1 + \frac{i\,\alpha}{4\,m^2}\,p_1^2\,\frac{\boldsymbol{r}}{r^3}\,(\boldsymbol{r}\times\boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1)\boldsymbol{p}_2 - \frac{3\,\alpha}{4\,m^2}\boldsymbol{\sigma}_1\times\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{p}_2\,. \tag{24}$$

The most tricky is the calculation of the retardation part in the single transverse photon exchange, diagram 3 in Fig 1. The initial expression is

$$V_3 = e^2\int\frac{d^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right)\left(\frac{p_1^i}{m} + \frac{i\,(\boldsymbol{\sigma}_1\times\boldsymbol{k})^i}{2\,m}\right) \left\{e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\left(\frac{1}{E_0 - k - H_0} + \frac{1}{k}\right)e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2}\right\}\left(\frac{p_2^j}{m} - \frac{i\,(\boldsymbol{\sigma}_2\times\boldsymbol{k})^j}{2\,m}\right) + (1\leftrightarrow 2)\,. \tag{25}$$

Only the second term in the retardation expansion contribute at order $m\,\alpha^6$ (notice two dotted lines in the corresponding diagram):

$$\frac{1}{E_0 - k - H_0} + \frac{1}{k} = \frac{(H_0 - E_0)}{k^2} - \frac{(H_0 - E_0)^2}{k^3} + \ldots\,, \tag{26}$$

and the retardation part $V_3$ becomes

$$
\begin{aligned}
V_3 = & -e^2 \int \frac{d^3k}{(2\,\pi)^3\,2\,k} \left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right) \left(\frac{p_1^i}{m} + \frac{i\,(\boldsymbol{\sigma}_1 \times \boldsymbol{k})^i}{2\,m}\right) \\
& \left\{ e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1} \frac{(H_0 - E_0)^2}{k^3} e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2} \right\} \left(\frac{p_2^j}{m} - \frac{i\,(\boldsymbol{\sigma}_2 \times \boldsymbol{k})^j}{2\,m}\right) + (1 \leftrightarrow 2) \\
= & \; V_3^A + V_3^B \,.
\end{aligned} \tag{27}
$$

Let's consider first the spin–spin part $V_3^A$,

$$
V_3^A = \frac{e^2}{8\,m^2} \int \frac{d^3k}{(2\,\pi)^3\,k^4}\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{k}\, \boldsymbol{\sigma}_2 \cdot \boldsymbol{k}\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,(H_0 - E_0)^2 e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2} + (1 \leftrightarrow 2)\,. \tag{28}
$$

Whenever $(H_0 - E_0)$ appears on the left or on the right side, it will give zero for the matrix element. Therefore, we can use

$$
\begin{aligned}
& e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,(H_0 - E_0)^2 e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2} + (1 \leftrightarrow 2) \\
\to & \left[e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,;\, \left[(H_0 - E_0)^2\,;\; e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2}\right]\right] \\
= & -\frac{1}{2\,m^2} \left[p_1^2\,;\, \left[p_2^2\,;\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\right]\right]\,,
\end{aligned} \tag{29}
$$

and the spin–spin part $V_3^A$ becomes

$$
V_3^A = -\frac{e^2}{16\,m^4}\, \sigma_1^i\, \sigma_2^j \left[p_1^2\,;\, \left[p_2^2\,;\, \int \frac{d^3k}{(2\,\pi)^3} \frac{k^i\,k^j}{k^4}\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\right]\right]\,. \tag{30}
$$

The $k$–integral could be performed using the formula:

$$
\int \frac{d^3k}{(2\,\pi)^3} \frac{1}{k^2} \left(\frac{\delta^{ij}}{3} - \frac{k^i\,k^j}{k^2}\right) e^{i\,\boldsymbol{k}\,\boldsymbol{r}} = \frac{1}{8\,\pi\,r} \left(\frac{r^i\,r^j}{r^2} - \frac{\delta^{ij}}{3}\right)\,. \tag{31}
$$

After simple transformations and D–K simplifications $V_3^A$ is

$$
V_3^A = \frac{\alpha}{16\,m^4}\, p_1^2 \left[p_2^2; \frac{r^i\,r^j}{r^3}\right] \sigma^i\,\sigma^j \tag{32}
$$

$$
\begin{aligned}
= & \frac{3\,\alpha}{8\,m^4}\, p_1^2\, \frac{\boldsymbol{\sigma}_1 \cdot \boldsymbol{r}\, \boldsymbol{\sigma}_2 \cdot \boldsymbol{r}}{r^5} \\
& + \frac{\alpha}{8\,m^4}\, p_1^2\, \frac{i}{r^2} \left[(\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}_2)\, \boldsymbol{\sigma}_1 + (\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}_1)\, \boldsymbol{\sigma}_2 - 3\,(\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}_1)\,(\hat{\boldsymbol{r}} \cdot \boldsymbol{\sigma}_2)\,\hat{\boldsymbol{r}}\right] \boldsymbol{p}_2\,.
\end{aligned} \tag{33}
$$

The spin–orbit part $V_3^B$ is obtained from (27)

$$
\begin{aligned}
V_3^B = & -\frac{i\,e^2}{4\,m^2} \int \frac{d^3k}{(2\,\pi)^3} \frac{1}{k^4} \Big[ e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,(H_0 - E_0)^2 e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2}\, \boldsymbol{\sigma}_1 \times \boldsymbol{k} \cdot \boldsymbol{p}_2 \\
& - \boldsymbol{\sigma}_2 \times \boldsymbol{k} \cdot \boldsymbol{p}_1\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,(H_0 - E_0)^2 e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2} \Big] + (1 \leftrightarrow 2)\,.
\end{aligned} \tag{34}
$$

With the use of identity

$$
\begin{aligned}
& e^{i\,\boldsymbol{k}\,\boldsymbol{r}_1}\,(H_0 - E_0)^2 e^{-i\,\boldsymbol{k}\,\boldsymbol{r}_2} \\
= & (H_0 - E_0)\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,(H_0 - E_0) + (H_0 - E_0) \left[\frac{p_2^2}{2\,m}\,;\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\right] \\
& + \left[e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,;\, \frac{p_1^2}{2\,m}\right] (H_0 - E_0) + \left[\frac{p_2^2}{2\,m}\,;\, \left[e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,;\, \frac{p_1^2}{2\,m}\right]\right]\,,
\end{aligned} \tag{35}
$$

it could be rewritten to

$$V_3^B = -\frac{i\,e^2}{m^2}\int\frac{d^3k}{(2\,\pi)^3}\,\frac{1}{k^4}\biggl\{\left[e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,;\,\frac{p_1^2}{2\,m}\right]\,[H_0\,;\,\boldsymbol{\sigma}_1\times\boldsymbol{k}\cdot\boldsymbol{p}_2] + \left[\frac{p_2^2}{2\,m}\,;\,\left[e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,;\,\frac{p_1^2}{2\,m}\right]\right]\,\boldsymbol{\sigma}_1\times\boldsymbol{k}\cdot\boldsymbol{p}_2\biggr\}. \tag{36}$$

The $k$–integral is performed with the use of

$$\int\frac{d^3k}{(2\,\pi)^3}\,\frac{4\,\pi}{k^4}\,\left(e^{i\,\boldsymbol{k}\,\boldsymbol{r}}-1\right) = -\frac{r}{2}. \tag{37}$$

After evaluation of comutators spin–orbit term becomes

$$V_3^B = \frac{\alpha^2}{2\,m^3}\,\boldsymbol{\sigma}_1\times\frac{\boldsymbol{r}}{r^4}\cdot\boldsymbol{p}_1 + \frac{Z\,\alpha^2}{2\,m^3\,r}\,\boldsymbol{\sigma}_1\times\frac{\boldsymbol{r}_1}{r_1^3}\cdot\boldsymbol{p}_2 + \frac{Z\,\alpha^2}{2\,m^3}\,\boldsymbol{\sigma}_1\times\frac{\boldsymbol{r}}{r^3}\cdot\frac{\boldsymbol{r}_1}{r_1^3}\,\boldsymbol{r}\cdot\boldsymbol{p}_2 - \frac{\alpha}{2\,m^4}\,p_1^2\,\boldsymbol{\sigma}_1\times\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{p}_2 + \frac{\alpha}{2\,m^2}\,p_1^2\,\boldsymbol{\sigma}_1\times\frac{i\,\boldsymbol{r}}{r^3}\,(\boldsymbol{r}\cdot\boldsymbol{p}_2)\cdot\boldsymbol{p}_2\,. \tag{38}$$

The next contribution is a group of diagrams No. 4 in Fig. 1. Let's first consider the upper electron line. By $q_1$ and $q_2$ we denote momenta transfered by transverse and Coulomb photon respectively,

$$u^+(p_1')\,\alpha^j\,\Lambda_-(p_1+q_2)\,u(p_1) + u^+(p_1')\,\Lambda_-(p_1'-q_2)\,\alpha^j\,u(p_1) = -\frac{i}{m}\,(\boldsymbol{q}_2\times\boldsymbol{\sigma}_1)^j\,. \tag{39}$$

The lower electron line is

$$u^+(p_2')\,\alpha^i\,\Lambda_+(p_2'+q_1)\,u(p_2) + u^+(p_2')\,\Lambda_+(p_2-q_1)\,\alpha^i\,u(p_2) = \frac{1}{m}(p_2^i+p_2'^i) + \frac{i}{m}\,(\boldsymbol{q}_1\times\boldsymbol{\sigma}_2)^i\,. \tag{40}$$

So, the effective interaction $V_4$ is:

$$V_4 = 2\int\frac{d^3q_1}{(2\,\pi)^3}\int\frac{d^3q_2}{(2\,\pi)^3}\,(2\,\pi)^3\,\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2-\boldsymbol{q})\,\frac{e^2}{q_2^2}\,\frac{e^2}{2\,q_1^2}\left(\delta^{ij}-\frac{q_1^i\,q_1^j}{q_1^2}\right)\frac{1}{2\,m}\,\frac{i}{m}\,(\boldsymbol{q}_2\times\boldsymbol{\sigma}_1)^j\left[\frac{1}{m}(p_2^i+p_2'^i)+\frac{i}{m}\,(\boldsymbol{q}_1\times\boldsymbol{\sigma}_2)^i\right] \tag{41}$$

$$= \frac{1}{2\,m^3}\int\frac{d^3q_1}{(2\,\pi)^3}\int\frac{d^3q_2}{(2\,\pi)^3}\,(2\,\pi)^3\,\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2-\boldsymbol{q})\,\frac{e^2}{q_1^2}\,\frac{e^2}{q_2^2}\left[(\boldsymbol{\sigma}_1\cdot\boldsymbol{q}_1)\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{q}_2) - i\left(\delta^{ij}-\frac{q^i\,q^j}{q^2}\right)\,(\boldsymbol{\sigma}_1\times\boldsymbol{q}_2)^i\,(\boldsymbol{p}_2+\boldsymbol{p}_2')^j\right]. \tag{42}$$

The factor 2 in front of the first integral comes from the fact that $Z$ is on upper or on the lower electron line. In configuration space $V_4$ is

$$V_4 = \frac{\alpha^2}{2\,m^3\,r^4}\,\left(\boldsymbol{\sigma}_1\times\boldsymbol{r}\cdot\boldsymbol{p}_2 - \boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{r}}\,\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{r}}\right)\,. \tag{43}$$

The next contribution is double transverse photon exchange with single $Z$, diagram 5 in Fig. 1. This contribution consists of twelve diagrams. They reduce to two diagrams upon the non retardation limit.

$$V_5 = \int \frac{d^3 q_1}{(2\,\pi)^3} \int \frac{d^3 q_2}{(2\,\pi)^3}\,(2\,\pi)^3\,\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2-\boldsymbol{q})$$
$$\frac{e^2}{q_1^2}\left(\delta^{ik}-\frac{q_1^i\,q_1^k}{q_1^2}\right)\frac{e^2}{q_2^2}\left(\delta^{jk}-\frac{q_2^j\,q_2^k}{q_2^2}\right)\frac{1}{2\,m}$$
$$\Big[\frac{(2\,\boldsymbol{p}_2+i\,\boldsymbol{q}_1\times\boldsymbol{\sigma}_2)^i}{2\,m}\,\frac{(2\,\boldsymbol{p}_2'+i\,\boldsymbol{q}_2\times\boldsymbol{\sigma}_2)^j}{2\,m}$$
$$+\frac{(2\,\boldsymbol{p}_1-i\,\boldsymbol{q}_1\times\boldsymbol{\sigma}_1)^i}{2\,m}\,\frac{(2\,\boldsymbol{p}_1'-i\,\boldsymbol{q}_2\times\boldsymbol{\sigma}_1)^j}{2\,m}\Big]\,. \tag{44}$$

In the configuration space the fine structure part using D-K simplification becomes:

$$V_5 = -\frac{\alpha^2}{2\,m^2}\,\frac{\boldsymbol{r}}{r^4}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1\,. \tag{45}$$

The last diagram No. 6 is a three–body single Coulomb–single transverse exchange. It is similar to the diagram No. 4.

$$V_6 = \frac{1}{4\,m^3}\int \frac{d^3 q_1}{(2\,\pi)^3}\int \frac{d^3 q_2}{(2\,\pi)^3}\,(2\,\pi)^3\,\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2-\boldsymbol{q})\,\frac{e^2}{q_1^2}\,\frac{Z\,e^2}{q_2^2}$$
$$\left[-\boldsymbol{\sigma}_1\cdot\boldsymbol{q}_1\,\boldsymbol{\sigma}_2\cdot\boldsymbol{q}_2+i\left(\delta^{ij}-\frac{q_1^i\,q_1^j}{q_1^2}\right)(\boldsymbol{\sigma}_1\times\boldsymbol{q}_2)^i\,(\boldsymbol{p}_2+\boldsymbol{p}_2')^j\right]$$
$$+\,(1\leftrightarrow 2)\,. \tag{46}$$

In the configuration space after D-K simplification it becomes

$$V_6 = \frac{Z\,\alpha^2}{2\,m^3}\Big[\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{r}}{r^3}\,\frac{\boldsymbol{\sigma}_2\cdot\boldsymbol{r}_1}{r_1^3}-\frac{1}{r\,r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_1-\frac{1}{r^3\,r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{r}\cdot\boldsymbol{\sigma}_1\,(\boldsymbol{r}\cdot\boldsymbol{p}_2)\Big] \tag{47}$$

The radiative correction $V_7$ to fine structure at order $m\,\alpha^6$ consists of the anomalous magnetic moment $A_e$ modification of the Breit Hamiltonian. It could be derived from the single Coulomb and the transverse photon exchange upon the replacement for the vertex

$$\gamma^\mu \rightarrow \gamma^\mu + \frac{i}{2\,m}\,\sigma^{\mu\nu}\,q^\nu\,A_e\,. \tag{48}$$

The simple derivation gives

$$V_7 = \frac{\alpha^2}{\pi^2}\Big\{-\frac{\alpha}{4\,m^2\,r^3}\,\boldsymbol{r}\times(\boldsymbol{p}_1-\boldsymbol{p}_2)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\,2\,a_e^{(2)}$$
$$+\frac{Z\,\alpha}{4\,m^2}\left(\frac{\boldsymbol{r_1}}{r_1^3}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1+\frac{\boldsymbol{r_2}}{r_2^3}\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right)\,2\,a_e^{(2)}$$
$$-\frac{3\,\alpha}{4\,m^2\,r^5}\,\boldsymbol{\sigma}_1\cdot\boldsymbol{r}\,\boldsymbol{\sigma}_2\cdot\boldsymbol{r}\,\left[(a_e^{(1)})^2+2\,a_e^{(2)}\right]\Big\}\,. \tag{49}$$

where

$$A_e = \frac{\alpha}{\pi}\,a_e^{(1)}+\left(\frac{\alpha}{\pi}\right)^2\,a_e^{(2)}+\ldots\,, \tag{50}$$
$$a_e^{(1)} = 0.5\,, \tag{51}$$
$$a_e^{(2)} = -\,0.328\,478\,965\,. \tag{52}$$

The effective Hamiltonians $H^{(6)}$ is expressed in terms of $V_i$

$$H^{(6)} = \Sigma_{i=0}^{7} V_i\,. \tag{53}$$

This complete result agrees with the former calculation of Douglass and Kroll in [2]. It confirms the correctness of the present calculation and puts the ground for calculations of higher order terms.

## 5. Electron self–energy effect on fine structure splitting in hydrogen

We describe here the calculation of one loop electron self–energy effect on the hydrogen fine structure at order $m\,\alpha^7$. It is a preparatory calculation for the much more complex work on the same correction in the helium atom. The results obtained here, will be used for the derivation of helium $m\,\alpha^7\ln\alpha$ and $m\,\alpha^7$ corrections.

This calculation consists of two parts. In the low energy part $E_L$ the photon energy $\omega$ is bounded by some arbitrary parameter $\epsilon$. In the high energy part $\omega > \epsilon$. The purpose is to demonstrate that in this simple hydrogenic case high energy part of the radiative correction could be effectively accounted for by using one loop electron formfactors $F_1$ and $F_2$.

$$\gamma^\mu \to \gamma^\mu\,F_1(q^2) + \frac{i}{2\,m}\,\sigma^{\mu\nu}\,q_\nu\,F_2(q^2)\,, \tag{54}$$

where

$$F_1(q^2) = 1 + \frac{\alpha}{3\,\pi}\,\frac{q^2}{m^2}\left[\ln\left(\frac{m}{2\,\epsilon}\right) + \frac{11}{24}\right]\,, \tag{55}$$

$$F_2(q^2) = \frac{\alpha}{2\,\pi}\left(1 + \frac{q^2}{6\,m^2}\right)\,. \tag{56}$$

In the low energy part one can use nonrelativistic formula for the energy shift

$$\Delta E = e^2\int_0^\epsilon \frac{d^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right)\left\langle\phi\left|\frac{p^i}{m}\,\frac{1}{E-H-k}\,\frac{p^j}{m}\right|\phi\right\rangle\,, \tag{57}$$

to obtain a correction due to $LS$ coupling,

$$\delta H = \frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\,, \tag{58}$$

which affects $H$, $E$ and $\phi$. So, this correction is

$$\begin{aligned} E_L^1 = \frac{2\,\alpha}{3\,\pi}\int_0^\epsilon k\,dk\,\Biggl\{\;2\left\langle\phi\left|\delta H\,\frac{1}{(E-H)'}\,\frac{\boldsymbol{p}}{m}\,\frac{1}{E-H-k}\,\frac{\boldsymbol{p}}{m}\right|\phi\right\rangle \\ + \left\langle\phi\left|\frac{\boldsymbol{p}}{m}\,\frac{1}{E-H-k}\,\delta H\,\frac{1}{E-H-k}\,\frac{\boldsymbol{p}}{m}\right|\phi\right\rangle \\ - \langle\phi|\delta H|\phi\rangle\left\langle\phi\left|\frac{\boldsymbol{p}}{m}\,\frac{1}{(E-H-k)^2}\,\frac{\boldsymbol{p}}{m}\right|\phi\right\rangle\Biggr\}\,. \end{aligned} \tag{59}$$

The second effect is due to correction to the electron current. It has been calculated by Jentschura and me in [12] using Fouldy-Wouthuysen transformation. The spin dependent part is

$$\boldsymbol{j} = \frac{\boldsymbol{p}}{m} + \delta\boldsymbol{j}\,, \tag{60}$$

$$\delta\boldsymbol{j} = -\frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r^3}\,\boldsymbol{r}\times\boldsymbol{\sigma} + \frac{1}{2\,m}\,(\boldsymbol{k}\,\boldsymbol{r})\,\boldsymbol{k}\times\boldsymbol{\sigma}\,. \tag{61}$$

The energy shift is given by

$$E_L^2 = e^2\int_0^\epsilon \frac{d^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right)\,2\left\langle\phi\left|\delta j^i\,\frac{1}{E-H-k}\,\frac{p^j}{m}\right|\phi\right\rangle\,, \tag{62}$$

and could be further simplified to

$$E_L^2 = \frac{2\,\alpha}{3\,\pi}\int_0^\epsilon k\,dk\left\langle\phi\left|-\frac{Z\,\alpha}{2\,m^2}\frac{\boldsymbol{r}\times\boldsymbol{\sigma}}{r^3}\frac{1}{E-H-k}\,\frac{\boldsymbol{p}}{m}\right|\phi\right\rangle\,. \tag{63}$$

Both corrections

$$E_L = E_L^1 + E_L^2\,, \tag{64}$$

could be calculated only numerically. Such a calculation was done by Jentschura *et al.* for $2P$, $3P$ and $4P$ states in [12] and [14]. The results are the following:

$$E_L = m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^6}{n^3}\left[-\frac{1}{3}\left(1-\frac{1}{n^2}\right)\,\ln\!\left(\frac{2\,\epsilon}{(Z\,\alpha)^2}\right) + k(n)\right]\,, \tag{65}$$

$$k(2) = 0.38442(1)\,, \tag{66}$$

$$k(3) = 0.45560(1)\,, \tag{67}$$

$$k(4) = 0.48384(1)\,. \tag{68}$$

The high energy contribution $E_H$ consists of few terms. The $q^2$ term from $F_1$ gives

$$E_H^1 = \left\langle\psi\left|-\frac{4\,\pi\,Z\,\alpha}{\boldsymbol{q}^2}\,F_1'(0)\,(-\boldsymbol{q}^2)\right|\psi\right\rangle\,. \tag{69}$$

One performs nonrelativistic approximation and leaves only spin dependent terms

$$\begin{aligned} E_H^1 &= \frac{1}{4\,m^2}\,F_1'(0)\,\langle\phi|i\,\sigma^k\,\epsilon^{ijk}\,p^i\,4\,\pi\,Z\,\alpha\,\delta^3(r)\,p^j\,|\phi\rangle \\ &= -\frac{9}{4\,m^2}\,4\,\pi\,Z\,\alpha\,\phi'^2(0)\,F_1'(0)\,. \end{aligned} \tag{70}$$

Using the formula which is valid for $nP$ states

$$\phi'^2(0) = \frac{4}{9\,n^3}\left(1-\frac{1}{n^2}\right)\,(m\,Z\,\alpha)^5\,\frac{1}{4\,\pi}\,, \tag{71}$$

one derives for $E_H^1$

$$E_H^1 = -\frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\frac{1}{3}\left(1-\frac{1}{n^2}\right)\left[\ln\!\left(\frac{m}{2\,\epsilon}\right) + \frac{11}{24}\right]\,. \tag{72}$$

The second term is due to $q^2$ in $F_2$

$$E_H^2 = \left\langle\psi\left|-\frac{4\,\pi\,Z\,\alpha}{\boldsymbol{q}^2}\,\frac{i}{2\,m}\,\gamma^0\,\sigma^{0i}\,q_i\,F_2'(0)\,(-\boldsymbol{q}^2)\right|\psi\right\rangle\,. \tag{73}$$

In the nonrelativistic limit it becomes

$$E_H^2 = \frac{1}{2\,m^2}\,F_2'(0)\,\langle\phi|i\,\sigma^k\,\epsilon^{ijk}\,p^i\,4\,\pi\,Z\,\alpha\,\delta^3(r)\,p^j\,|\phi\rangle$$
$$= -\frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\frac{1}{6}\left(1-\frac{1}{n^2}\right)\,. \tag{74}$$

The remaining term is due to anomalous magnetic moment. It could be calculated with the Dirac wave functions. However we want to use the effective operator approach, which is to be applied in the case of helium, therefore we perform calculation in the following way. The second order term $E_H^3$ comes from the radiative correction to spin–orbit interaction

$$\frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L} \to (1+2\,F_2(0))\,\frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\,, \tag{75}$$

and the relativistic correction to Hamiltonian

$$\delta H = -\frac{\boldsymbol{p}^4}{8\,m^3} + \frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\,, \tag{76}$$

It has the form

$$E_H^3 = 2\,\left\langle\phi\left|\left(-\frac{\boldsymbol{p}^4}{8\,m^3} + \frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\right)\,\frac{1}{(E-H)'}\,\frac{\alpha}{\pi}\,\frac{Z\,\alpha}{4\,m^2\,r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\right|\phi\right\rangle$$
$$= \frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\left(\frac{919}{1440} + \frac{5}{16\,n} - \frac{9}{10\,n^2}\right)\,. \tag{77}$$

The fourth and the last term comes from the further relativistic correction to spin–orbit interaction. It is derived from the matrix element of the anomalous magnetic moment interaction by expanding in external momenta

$$-\frac{F_2(0)}{2\,m}\,u^+(p')\gamma^i\,q^i\,\frac{4\,\pi\,Z\,\alpha}{\boldsymbol{q}^2}u(p)$$
$$= \frac{F_2(0)}{8\,m^4}\,i\,\frac{4\,\pi\,Z\,\alpha}{\boldsymbol{q}^2}\,\boldsymbol{q}\times\boldsymbol{p}\cdot\boldsymbol{\sigma}\,\left(\boldsymbol{p}'^2+\boldsymbol{p}^2\right)\,. \tag{78}$$

The Fourier transform of this expression forms an effective interaction

$$E_H^4 = -\frac{F_2(0)}{8\,m^4}\,\left\langle p^2\,\frac{Z\,\alpha}{r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L} + \frac{Z\,\alpha}{r^3}\,\boldsymbol{\sigma}\,\boldsymbol{L}\,p^2\right\rangle$$
$$= \frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\left(\frac{13}{40\,n^2} - \frac{3}{10}\right)\,. \tag{79}$$

Another possible correction, with so called one pair creation is spin independent and therefore does not contribute to fine structure. The high energy contribution is the sum

$$E_H = E_H^1 + E_H^2 + E_H^3 + E_H^4$$
$$= \frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\left(\frac{3}{160} - \frac{23}{90\,n^2} + \frac{5}{16\,n} + \frac{1}{3}\left(1-\frac{1}{n^2}\right)\,\ln(2\,\epsilon)\right)\,. \tag{80}$$

The complete radiative correction to the fine structure splitting $E_{FS}$ in hydrogen at order $m\,\alpha^6$ is the sum of $E_L + E_H$.

$$E_{FS}(n) = \frac{\alpha}{\pi}\,\frac{m\,(Z\,\alpha)^6}{n^3}\,\left\{\frac{3}{160} - \frac{23}{90\,n^2} + \frac{5}{16\,n}\right.$$
$$\left. + \frac{1}{3}\left(1-\frac{1}{n^2}\right)\,\ln[(Z\,\alpha)^2] + k(n)\right\}\,. \tag{81}$$

The agreement with the full relativistic treatment by Jentschura in [12, 14] shows that this simplified approach using on shell formfactors works at least in the hydrogen case.

## 6. Logarithmic $m\,\alpha^7\ln\alpha$ correction to helium fine structure splitting

The purpose of this section is the verification of the Tao Zhang result in [10, 11]. This result plays an important role in deriving theoretical predictions for helium fine structure, from which one hopes to derive a new value for the fine structure constant $\alpha$.

The expression for the helium fine structure splitting at order $m\,\alpha^7$ could be split into three: low, middle and high energy parts depending on the photon momentum involved. Lets introduce two cut–offs $\epsilon$ and $\lambda$ In the low energy part $k<\epsilon\,m\,\alpha$, in the middle one $\epsilon\,m\,\alpha<k<\lambda\,m\,\alpha$ and in the high one $\lambda\,m\,\alpha<k$. It is understood that the limits $\epsilon\to 0$ and $\lambda\to\infty$ are obtained after $\alpha$ expansion and integrals are performed. $\ln\alpha$ corrections are usually identified as logarithmic divergences in these integrals. In his calculations, Tao Zhang derived logarithmic correction from the middle and high energy parts. In our calculations, we will use the low and the high energy parts, what we think, simplifies the whole treatment.

The low energy part could be obtained using a nonrelativistic expansion of the QED Hamiltonian $H$ (see for example [12]). It consists of three terms. The first one is

$$E_{L1} = -\frac{e^2}{m^2}\int^{\epsilon\,m\,\alpha}\frac{d^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij}-\frac{k^i\,k^j}{k^2}\right)\langle\phi|(p_1^i+p_2^i)\,\frac{1}{H-E+k}\,(p_1^j+p_2^j)|\phi\rangle\,, \tag{82}$$

$\phi$, $H$, and $E$ contain corrections due to Breit interaction $H_{SL}+H_{SS}$. The logarithmic correction is obtained by expansion of $1/(H-E+k)$ in large $k$ and selection of the second term only. After angle average it reads

$$\begin{aligned}E_{L1} &= \frac{2\,\alpha}{3\,\pi\,m^2}\int_{m\,(Z\,\alpha)^2}^{\epsilon\,m\,\alpha}\frac{dk}{k}\,\langle\phi|(p_1^i+p_2^i)\,(H-E)\,(p_1^i+p_2^i)|\phi\rangle \\ &= \frac{\alpha}{3\,\pi\,m^2}\,\ln\Big(\frac{\epsilon}{Z^2\,\alpha}\Big)\,\langle\phi|[[p_1^i+p_2^i\,;\,H-E];\,p_1^i+p_2^i]|\phi\rangle\,.\end{aligned} \tag{83}$$

When $\phi$ contains correction from $H_{SL}+H_{SS}$, then $H$ is simple $H_0$ and the commutator leads to

$$[[p_1^i+p_2^i\,;\,H_0-E_0];\,p_1^i+p_2^i] = 4\,\pi\,Z\,\alpha\,(\delta^3(r_1)+\delta^3(r_2))\,. \tag{84}$$

When $H$ contains correction from $H_{SL}+H_{SS}$, then

$$\begin{aligned}&[[p_1^i+p_2^i\,;\,H_{SL}+H_{SS}];\,p_1^i+p_2^i] \\ &= \frac{Z\,\alpha}{4\,m^2}\left[\left[p_1^i+p_1^i\,;\,\frac{\boldsymbol{r}_1\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1}{r_1^3}+\frac{\boldsymbol{r}_2\times\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2}{r_2^3}\right];\,p_1^i+p_2^i\right] \\ &= \frac{Z\,\alpha}{4\,m^2}\,4\,\pi\,i\,\left[\boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1+\boldsymbol{p}_2\times\delta^3(r_2)\,\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2\right]\,.\end{aligned} \tag{85}$$

The first low energy term becomes:

$$\begin{aligned}E_{L1} = \frac{8\,Z\,\alpha^2}{3\,m^2}\,\ln\Big(\frac{\epsilon}{Z^2\,\alpha}\Big)\,\Big\{&\langle\phi|H_{SL}+H_{SS})\,\frac{1}{(E_0-H_0)'}\,[\delta^3(r_1)+\delta^3(r_2)]|\phi\rangle \\ &+\frac{i}{8\,m^2}\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1+\boldsymbol{p}_2\times\delta^3(r_2)\,\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2|\phi\rangle\Big\}\,.\end{aligned} \tag{86}$$

The second low energy term $E_{L2}$ comes from the correction to the current as in Eq. (60).

$$
\begin{aligned}
\boldsymbol{j} &= \frac{\boldsymbol{p}_1+\boldsymbol{p}_2}{m} + \delta\boldsymbol{j}\,, \qquad (87)\\
\delta\boldsymbol{j} &= -\frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{\sigma}_1 - \frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_2^3}\,\boldsymbol{r}_2\times\boldsymbol{\sigma}_2\\
&\quad + \frac{1}{2\,m}\,(\boldsymbol{k}\,\boldsymbol{r}_1)\,\boldsymbol{k}\times\boldsymbol{\sigma}_1 + \frac{1}{2\,m}\,(\boldsymbol{k}\,\boldsymbol{r}_2)\,\boldsymbol{k}\times\boldsymbol{\sigma}\\
&\quad + \frac{1}{2\,m^2}\,\frac{\alpha}{r_{12}^3}\,\boldsymbol{r}_{12}\times\boldsymbol{\sigma}_1 + \frac{1}{2\,m^2}\,\frac{\alpha}{r_{21}^3}\,\boldsymbol{r}_{21}\times\boldsymbol{\sigma}_2\,, \qquad (88)
\end{aligned}
$$

and is equal to

$$
E_{L2} = -\frac{e^2}{m^2}\int^{\epsilon\,m\,\alpha}\frac{d^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij}-\frac{k^i\,k^j}{k^2}\right)\,2\,\langle\phi|\delta j^i\,\frac{1}{H-E+k}\,(p_1^j+p_2^j)|\phi\rangle\,. \qquad (89)
$$

The simple derivation as for the previous term shows, that no $\ln\alpha$ is contained in $E_{L2}$. The third and last low energy term comes from the $\boldsymbol{A}^2/(2m)$ in the nonrelativistic Hamiltonian. It should be pointed out here, that when both transverse photons has low momentum, it contributes at higher order. So, only one photon momentum may be low $\sim m\,\alpha^2$. Coming back to $\boldsymbol{A}^2/(2m)$ term, it generates several diagrams. One exchange diagram gives the expression

$$
\begin{aligned}
E_{L3} &\sim \frac{1}{m^3}\int\frac{d^3q_1}{(2\,\pi)^3}\int\frac{d^3q_2}{(2\,\pi)^3}\,\frac{e^2}{2\,q_1}\left(\delta^{ik}-\frac{q_1^i\,q_1^k}{q_1^2}\right)\frac{e^2}{2\,q_2}\left(\delta^{kj}-\frac{q_2^k\,q_2^j}{q_2^2}\right)\\
&\quad \langle\phi|\left(p_1^i-\frac{i}{2}\,(\boldsymbol{q}_1\times\boldsymbol{\sigma}_1)^i\right)e^{i\,\boldsymbol{q}_1\cdot\boldsymbol{r}_1}\,\frac{1}{H_0-E_0+q_1}\,e^{-i\,\boldsymbol{q}_1\cdot\boldsymbol{r}_2}\\
&\quad e^{-i\,\boldsymbol{q}_2\cdot\boldsymbol{r}_2}\,\frac{1}{H_0-E_0+q_2}\left(p_1^j-\frac{i}{2}\,(\boldsymbol{q}_2\times\boldsymbol{\sigma}_1)^j\right)e^{i\,\boldsymbol{q}_2\cdot\boldsymbol{r}_1}\,|\phi\rangle\,. \qquad (90)
\end{aligned}
$$

When $q_1$ is small then $q_2>\epsilon\,m\,\alpha$ and we can neglect $H_0-E_0$ in the denominator with $q_2$. With the use of integral

$$
\int\frac{d^3q}{(2\,\pi)^3}\,\frac{4\,\pi}{q^2}\left(\delta^{ij}-\frac{q^i\,q^j}{q^2}\right)\mathrm{e}^{i\,\boldsymbol{q}\cdot\boldsymbol{r}} = \frac{1}{2\,r}\left(\delta^{ij}+\frac{r^i\,r^j}{r^2}\right). \qquad (91)
$$

$E_{L3}$ could be transformed to

$$
\begin{aligned}
E_{L3} &\sim \frac{1}{m^3}\int\frac{d^3q_1}{(2\,\pi)^3}\,\frac{e^2}{2\,q_1}\left(\delta^{ik}-\frac{q_1^i\,q_1^k}{q_1^2}\right)\langle\phi|p_1^i\,\frac{1}{H_0-E_0+q_1}\\
&\quad\left(-\frac{i}{2}\right)\left[(\boldsymbol{p}_1\times\boldsymbol{\sigma}_1)^j\,;\,\frac{\alpha}{4\,r}\left(\delta^{kj}+\frac{r^k\,r^j}{r^2}\right)\right]|\phi\rangle + h.c.\,. \qquad (92)
\end{aligned}
$$

When it is combined with other exchange and self–energy diagrams the complete expression

$$
\begin{aligned}
E_{L3} &= \frac{2}{m^3}\int\frac{d^3q_1}{(2\,\pi)^3}\,\frac{e^2}{2\,q_1}\,\frac{2}{3}\,\langle\phi|(p_1^i+p_2^i)\,\frac{1}{H_0-E_0+q_1}\\
&\quad\left(-\frac{i}{2}\right)\left[(\boldsymbol{p}_1\times\boldsymbol{\sigma}_1)^j+(\boldsymbol{p}_2\times\boldsymbol{\sigma}_2)^j\,;\,\frac{\alpha}{4\,r}\left(\delta^{ij}+\frac{r^i\,r^j}{r^2}\right)\right]|\phi\rangle + h.c. \qquad (93)
\end{aligned}
$$

vanishes identically for triplet states. Therefore, the internal cancellation of various terms causes that the logarithmic correction in the low energy part is given by Eq. (86) only.

The middle energy part does not lead to $\ln\alpha$, what is clear when all momenta are expressed in atomic units. The integrals are limited by $\epsilon$ and $\lambda$, what leads to $\ln\epsilon$ and $\ln\lambda$ only. In the high energy part at least one photon momentum is large. This part consists of 5 terms calculated as follows. When radiative photon is involved the logarithmic correction could be obtained from the electron on–mass shell formfactor $F_1$ as it is in hydrogen case, see Eq. (55). For the purposes in this section we use a slightly different $\epsilon$, so $F_1$ is given here by

$$F_1(q^2) = 1 + \frac{\alpha}{3\,\pi}\,\frac{q^2}{m^2}\left[-\ln(2\,\epsilon\,\alpha) + \frac{11}{24}\right]. \tag{94}$$

When combined with the single Coulomb exchange it leads to the interaction potential which in the logarithmic limit is

$$-\frac{4}{3}\,\ln(\epsilon\,\alpha)\,\frac{Z\,\alpha^2}{m^2}\,[\delta^3(r_1) + \delta^3(r_2)]\,. \tag{95}$$

The second order correction due to this potential and the Breit interaction $H_{SL} + H_{SS}$ is the first high energy term:

$$E_{H1} = -\frac{8}{3}\,\ln(\epsilon\,\alpha)\,\frac{Z\,\alpha^2}{m^2}\,\langle\phi|(H_{SL}+H_{SS})\,\frac{1}{(E_0 - H_0)'}\,[\delta^3(r_1)+\delta^3(r_2)]|\phi\rangle\,. \tag{96}$$

Since the triplet wave function is antisymmetric in $\boldsymbol{r}_1, \boldsymbol{r}_2$ the $\delta^3(r_{12})$ does not contribute. The second term comes from the relativistic correction to interaction potential (95) and is the same as in hydrogen case, see Eq. (70).

$$E_{H2} = -\frac{Z\,\alpha^2}{3\,m^4}\,\ln(\epsilon\,\alpha)\,i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \boldsymbol{p}_2\times\delta^3(r_2)\,\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2|\phi\rangle\,. \tag{97}$$

The same type of correction but to the electron–electron Coulomb interaction leads to

$$E_{H3} = \frac{2\,\alpha^2}{3\,m^4}\,\ln(\epsilon\,\alpha)\,i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r)\,\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1 + \boldsymbol{p}_2\times\delta^3(r)\,\boldsymbol{p}_2\cdot\boldsymbol{\sigma}_2|\phi\rangle\,. \tag{98}$$

The radiative correction to the single transverse photon exchange (see Eq. (21) leads to the interaction potential

$$V = -\frac{4\,\pi\,\alpha}{q^2}\,(F^2(-q^2) - 1)[u^+(p_1')\,\boldsymbol{\alpha}^i\,u(p_1)]\;[u^+(p_2')\,\boldsymbol{\alpha}^j\,u(p_2)]\left(\delta^{ij} - \frac{q^i\,q^j}{q^2}\right). \tag{99}$$

The corresponding energy shift for triplet P-states in the nonrelativistic limit is

$$\begin{aligned} E_{H4} = \frac{4\,\alpha^2}{3\,m^4}\,\ln(\epsilon\,\alpha)\,\langle\phi|i\,\boldsymbol{p}_1\times\delta^3(r)\,\boldsymbol{p}_1\cdot\boldsymbol{s} + i\,\boldsymbol{p}_2\times\delta^3(r)\,\boldsymbol{p}_2\cdot\boldsymbol{s} \\ -(\boldsymbol{s}\cdot\nabla)\,(\boldsymbol{s}\cdot\nabla)\,\delta^3(r)|\phi\rangle\,, \end{aligned} \tag{100}$$

where $\boldsymbol{s}$ is a spin operator $\boldsymbol{s} = (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$. The last logarithmic term are present in the two–photon exchange amplitude, see Fig. 2. We assume here that photon momentum

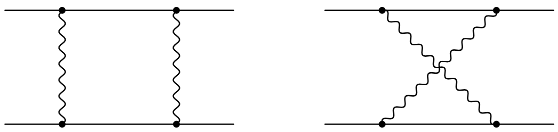

**Figure 2.** Electron–electron scattering amplitude with two–photon exchange

is large, $q > \lambda\, m\, \alpha$. The effective potential interaction, as given by this diagrams in the Feynman gauge is

$$V_H = -\frac{e^4}{2}\int_\lambda \frac{d^4q}{(2\,\pi)^4\, i}\,\frac{1}{(q+(p'-p)/2)^2}\,\frac{1}{(-q+(p'-p)/2)^2}$$
$$\Big\{u(\boldsymbol{p}')\,\gamma^\mu \frac{1}{\not{q}+(\not{p}+\not{p}')/2-m}\,\gamma^\nu\, u(\boldsymbol{p}) + u(\boldsymbol{p}')\,\gamma^\nu \frac{1}{-\not{q}+(\not{p}+\not{p}')/2-m}\,\gamma^\mu\, u(\boldsymbol{p})\Big\} \qquad (101)$$
$$\Big\{u(-\boldsymbol{p}')\,\gamma_\mu \frac{1}{-\not{q}+(\tilde{\not{p}}+\tilde{\not{p}}')/2-m}\,\gamma_\nu\, u(-\boldsymbol{p}) + u(-\boldsymbol{p}')\,\gamma_\nu \frac{1}{\not{q}+(\tilde{\not{p}}+\tilde{\not{p}}')/2-m}\,\gamma_\mu\, u(-\boldsymbol{p})\Big\}.$$

where $\tilde{p} = (p_0, -\boldsymbol{p})$. This integral is calculated as follows. First, the expansion in external momenta is performed, leaving only these terms, that do not vanish on the triplet P-state, namely that in Eq. (100). We can assume here that the time–like components $p_0 = p'_0 = m$. Next, one performs integral in respect to $q_0$, and separates out the logarithmic divergence in $\boldsymbol{q}^2$, This last integral is bounded from below by $\lambda\, m\, \alpha$ what leads to the logarithmic terms.

$$E_{H5} = \frac{\alpha^2}{m^4}\ln(\lambda\,\alpha)\,\langle\phi| - \frac{i}{2}\Big[\boldsymbol{p}_1\times\delta^3(r)\,\boldsymbol{p}_1\cdot\boldsymbol{s} + \boldsymbol{p}_2\times\delta^3(r)\,\boldsymbol{p}_2\cdot\boldsymbol{s}\Big]$$
$$+\frac{7}{3}(\boldsymbol{s}\cdot\nabla)\,(\boldsymbol{s}\cdot\nabla)\,\delta^3(r)|\phi\rangle\,. \qquad (102)$$

The total logarithmic correction is a sum of Eqs. (86,96,97,98,100,102)

$$\Delta E = \frac{8\,Z\,\alpha^2}{3\,m^2}\ln(Z\,\alpha)^{-2}\,\langle\phi|H_{SL}+H_{SS})\,\frac{1}{(E_0-H_0)'}\,[\delta^3(r_1)+\delta^3(r_2)]|\phi\rangle$$
$$+\frac{Z\,\alpha^2}{3\,m^4}\ln(Z\,\alpha)^{-2}\, i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{s}+\boldsymbol{p}_2\times\delta^3(r_2)\,\boldsymbol{p}_2\cdot\boldsymbol{s}|\phi\rangle$$
$$+\frac{3\,\alpha^2}{2\,m^4}\ln(\alpha)\, i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r)\,\boldsymbol{p}_1\cdot\boldsymbol{s}+\boldsymbol{p}_2\times\delta^3(r)\,\boldsymbol{p}_2\cdot\boldsymbol{s}|\phi\rangle$$
$$+\frac{\alpha^2}{m^4}\ln(\alpha)\,\langle\phi|(\boldsymbol{s}\cdot\nabla)\,(\boldsymbol{s}\cdot\nabla)\,\delta^3(r)|\phi\rangle \qquad (103)$$

in agreement with Tao Zhang result in [10, 11]. However his result was presented using a different but equivalent set of operators. One can prove the correspondence

$$i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r_1)\,\boldsymbol{p}_1\cdot\boldsymbol{s}|\phi\rangle \;=\; -3\,\langle\phi|\boldsymbol{r}_1\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1\,\frac{\delta^3(r_1)}{r_1^2}\,|\phi\rangle\,, \qquad (104)$$

$$i\,\langle\phi|\boldsymbol{p}_1\times\delta^3(r)\,\boldsymbol{p}_1\cdot\boldsymbol{s}|\phi\rangle \;=\; -3\,\langle\phi|\boldsymbol{r}\times\boldsymbol{p}_1\cdot\boldsymbol{\sigma}_1\,\frac{\delta^3(r)}{r^2}\,|\phi\rangle\,, \qquad (105)$$

$$\langle\phi|(\boldsymbol{s}\cdot\nabla)\,(\boldsymbol{s}\cdot\nabla)\,\delta^3(r)|\phi\rangle = \frac{15}{2}\,\langle\phi|\boldsymbol{r}\cdot\boldsymbol{\sigma}_1\,\boldsymbol{r}\cdot\boldsymbol{\sigma}_2\,\frac{\delta^3(r)}{r^4}|\phi\rangle\,, \qquad (106)$$

to show explicitly the equivalence of both results.

## 7. Summary

We have rederived and verified two significant results obtained for the helium fine structure splitting: $m\,\alpha^6$ term by Douglass and Kroll in [2] and $m\,\alpha^7\,\ln\alpha$ by Tao Zhang in [10]. The numerical value of the matrix elements of derived operators were calculated in [11, 9]. The most recent experiment with comparison to theoretical predictions may be found in [15]. The theoretical predictions are limited by the unknown yet $m\,\alpha^7$ correction. The motivation to calculate this term is a long standing project for the determination of $\alpha$ constant from the helium fine structure splitting. Tao Zhang has derived the corresponding operators in [10], however this calculations was very complex and tedious so it deserves for an independent confirmation. The additional problem in this calculation are the logarithmic divergences, which require some kind of regularizations. One has to introduce regulators without spoiling the structure of an expression. We have not found this problem well adressed in the literature. The rederivation of Douglass and Kroll result and of hydrogen fine structure verifies the time ordered perturbation technique together with the effective Hamiltonian approach. We think this is the best method to calculate the effective operators for the $m\,\alpha^7$ correction.

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