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Forbidden transitions in the helium atom

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Abstract

Nonrelativistically forbidden, single-photon transition rates between low lying states of the helium atom are rigorously derived within quantum electrodynamics theory. Equivalence of velocity and length gauges, including relativistic corrections is explicitly demonstrated. Numerical calculations of matrix elements are performed with the use of high precision variational wave functions and compared to former results.

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The existence of nonrelativistically forbidden transitions in helium, for example between the singlet and triplet states, indicates the presence of relativistic effects. The calculation of these effects in atoms or ions is a highly nontrivial task. Depending on the magnitude of nuclear charge Z one performs various approximations. Here we study light atoms, so the expansion in the small parameter $Z\,\alpha$ is the most appropriate. Forbidden transitions have already been studied for many light atoms and especially for helium (for a review see [1]). Historically, the first but approximate calculations of S-P forbidden transitions were performed by Elton in [2]. Since the dominant part comes from 2^3P_1 and 2^1P_1 mixing, he included in the calculation only these states. Drake and Dalgarno in [3] were the first to include higher excited states, which led to much higher precision. Moreover, Drake later [4] accounted for corrections to S-state wave functions. Although these calculations were correct, there was no proof that they are complete. As an example may serve the $2^3S_1-1^1S_0$ M1 transition. Feinberg and Sucher [5] derived an effective operator for this transition and showed the cancellation of electron-electron terms. However, the calculations of Drake in [6] were performed earlier with the implicit assumption, that these terms are absent. In a completely different approach based on relativistic many body perturbation theory Johnson *et al.* [1] and Derevianko *et al* [7] studied forbidden transition in both velocity and length gauge. They pointed out the significance of negative energy states. However, not all results were in agreement with the nonrelativistic approach based on the Breit hamiltonian. It is the purpose of this work to systematically derive matrix elements for forbidden transitions in helium within quantum electrodynamics theory. The equivalence of length and velocity gauges for E1 transitions, including relativistic corrections, is explicitly shown. With the use of optimized numerical wave functions, the amplitudes and transition probabilities for $2^3P_2-1^1S_0$, $2^3P_1-1^1S_0$, $2^1P_1-2^3S_1$, $2^3S_1-1^1S_0$, and $3^3S_1-2^3S_1$ are calculated with high precision and compared to former results.

The nonrelativistic helium atom interacting with the electromagnetic field is described by the Schrödinger-Pauli hamiltonian:

$$H=\frac{(\vec{p}_1-e\,\vec{A})^2}{2\,m}+\frac{(\vec{p}_2-e\,\vec{A})^2}{2\,m}+\frac{\alpha}{r}-\frac{Z\,\alpha}{r_1}-\frac{Z\,\alpha}{r_2}\,. \tag{1}$$

The single photon transition amplitude T between two eigenstates ϕ and ψ, in the electric dipole approximation is

$$T^i=\langle\phi|\frac{(p_1+p_2)^i}{m}|\psi\rangle=i\,(E_\phi-E_\psi)\,\langle\phi|(r_1+r_2)^i|\psi\rangle\,, \tag{2}$$

and the transition probability $\mathcal{A}$ is

$$\mathcal{A}=2\,\alpha\,|E_\phi-E_\psi|\,T^i\,T^{*j}\left(\delta^{ij}-\frac{k^i\,k^j}{k^2}\right). \tag{3}$$

In the effective Hamiltonian approach relativistic corrections enter in two ways, as corrections to the wave functions ϕ and ψ and the correction $\delta\vec{\jmath}$ to the current $\vec{p}/m$

$$\vec{T}=\langle\phi|\delta\vec{\jmath}|\psi\rangle+\langle\phi|\frac{\vec{p}_1+\vec{p}_2}{m}\,\frac{1}{(E-H)'}\,\delta H|\psi\rangle+\langle\phi|\delta H\,\frac{1}{(E-H)'}\,\frac{\vec{p}_1+\vec{p}_2}{m}|\psi\rangle\,. \tag{4}$$

The correction to the wave function is given by the Breit hamiltonian. The part responsible for singlet-triplet transition is

$$\delta H = \left[\frac{Z\,\alpha}{4\,m^2}\left(\frac{\vec r_1}{r_1^3}\times\vec p_1 - \frac{\vec r_2}{r_2^3}\times\vec p_2\right) + \frac{\alpha}{4\,m^2}\,\frac{\vec r}{r^3}\times(\vec p_1+\vec p_2)\right]\frac{\vec\sigma_1-\vec\sigma_2}{2} \equiv \vec h\cdot\frac{\vec\sigma_1-\vec\sigma_2}{2}\,. \qquad (5)$$

Corrections to the current are given by several time ordered diagrams, shown in Fig. 1. The corresponding expression is calculated as follows. The first diagram is

$$\begin{aligned}\delta\vec\jmath_1 = u^+(p')\,\vec\alpha\,u(p) = {}& \frac{1}{2\,m}\,(\vec p\,'+\vec p) - \frac{i}{2\,m}\,[(\vec p\,'-\vec p)\times\vec\sigma] - \\ & \frac{1}{16\,m^3}\,(p'^2+3\,p^2)(\vec p+i\,\vec p\times\vec\sigma) - \frac{1}{16\,m^3}\,(p^2+3\,p'^2)(\vec p\,'-i\,\vec p\,'\times\vec\sigma)\,, \end{aligned} \qquad (6)$$

where $u(p)$ is a normalized plane wave solution of the free Dirac equation. For considered transitions one may leave spin dependent terms only. In position representation it takes a form

$$\begin{aligned}\delta\vec\jmath_1 = {}& \frac{i}{2\,m}\,\vec\sigma\times[\vec p, e^{i\,\vec k\cdot\vec r}] \\ & + \frac{i}{16\,m^3}\left\{\vec p\times\vec\sigma\,e^{i\,\vec k\cdot\vec r}\,p^2 + 3\,\vec p\times\vec\sigma\,p^2\,e^{i\,\vec k\cdot\vec r} - p^2\,e^{i\,\vec k\cdot\vec r}\,\vec p\times\vec\sigma - 3\,e^{i\,\vec k\cdot\vec r}\,\vec p\times\vec\sigma\,p^2\right\}\end{aligned} \qquad (7)$$

The photon momentum k is of order $m\,(Z\,\alpha)^2$, while r is of order $(m\,Z\,\alpha)^{-1}$. This means that $e^{i\,\vec k\cdot\vec r}$ can be expanded in powers of $\vec k\cdot\vec r$. After adding contributions from both electrons the $(Z\,\alpha)^2$ correction takes the form

$$\delta\vec\jmath_1 = \frac{1}{2\,m}\,(\vec k\cdot\vec r_1)\,\vec k\times\vec\sigma_1 + \frac{1}{2\,m}\,(\vec k\cdot\vec r_2)\,\vec k\times\vec\sigma_2\,. \qquad (8)$$

The next diagram involves one electron-positron pair and the corresponding expression is

$$\begin{aligned}\delta\vec\jmath_2 &= -\frac{Z\,e^2}{q^2}\,\frac{1}{2\,m}\,u^+(p')\left[\vec\alpha\,\Lambda_-(p+q) + \Lambda_-(p'-q)\,\vec\alpha\right]u(p) \\ &= \frac{i}{2\,m^2}\,\frac{Z\,e^2}{q^2}\,\vec q\times\vec\sigma \to -\frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r^3}\,\vec r\times\vec\sigma\,e^{i\,\vec k\cdot\vec r}\,,\end{aligned} \qquad (9)$$

where Λ_- is a projection operator into the negative energy subspace and q is a momentum exchange between electron and the nucleus. The $(Z\,\alpha)^2$ correction from both electrons becomes

$$\delta\vec\jmath_2 = -\frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_1^3}\,\vec r_1\times\vec\sigma_1 - \frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_2^3}\,\vec r_2\times\vec\sigma_2\,. \qquad (10)$$

The remaining diagrams involve electron-electron terms. The last two are of higher order, so they will not be considered here. The expression for diagram 3 can be obtained from Eq.(9) by the replacements $-Z\,\alpha\to\alpha$. In this way one obtains

$$\delta\vec\jmath_3 = \frac{1}{2\,m^2}\,\frac{\alpha}{r^3}\,\vec r\times\vec\sigma_1\,e^{i\,\vec k\cdot\vec r_1} + (1\leftrightarrow 2)\,, \qquad (11)$$

where $\vec{r}$ denotes here $\vec{r} \equiv \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$. The $(Z\,\alpha)^2$ correction is

$$\delta\vec{\jmath}_3 = \frac{1}{2\,m^2}\,\frac{\alpha}{r^3}\,\vec{r}\times(\vec{\sigma}_1 - \vec{\sigma}_2)\,. \tag{12}$$

The expression for diagram 4 is

$$\begin{aligned}\delta j_4^i = {} & -\frac{1}{2\,m}\,\frac{e^2}{q^2}\left(\delta^{jk} - \frac{q^j\,q^k}{q^2}\right)u^+(p_2')\,\alpha^k\,u(p_2)\\ & u^+(p_1')\Big[\alpha^i\,\Lambda_-(p_1+q)\,\alpha^j + \alpha^j\,\Lambda_-(p_1'-q)\,\alpha^i\Big]\,u(p_1) + (1\leftrightarrow 2)\,.\end{aligned} \tag{13}$$

The term in the second line equals $2\,\delta^{ij}$ and that in the first line has already appeared in Eq. (6), so it becomes

$$\delta\vec{\jmath}_4 = \frac{i}{2\,m}\,\frac{e^2}{q^2}\,\vec{q}\times\vec{\sigma}_2 + (1\leftrightarrow 2) \to \frac{1}{2\,m^2}\,\frac{\alpha}{r^3}\,\vec{r}\times\vec{\sigma}_2\,e^{i\,\vec{k}\cdot\vec{r}_1} + (1\leftrightarrow 2)\,. \tag{14}$$

The $(Z\,\alpha)^2$ correction is

$$\delta\vec{\jmath}_4 = -\frac{1}{2\,m^2}\,\frac{\alpha}{r^3}\,\vec{r}\times(\vec{\sigma}_1 - \vec{\sigma}_2) \tag{15}$$

and cancels out with that from diagram 3, Eq. (12). The final expression for the relativistic correction to the current of order $O(Z\,\alpha)^2$ is the sum of (8) and (10)

$$\delta\vec{\jmath} = \frac{1}{2\,m}\,(\vec{k}\cdot\vec{r}_1)\,\vec{k}\times\vec{\sigma}_1 + \frac{1}{2\,m}\,(\vec{k}\cdot\vec{r}_2)\,\vec{k}\times\vec{\sigma}_2 - \frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_1^3}\,\vec{r}_1\times\vec{\sigma}_1 - \frac{1}{2\,m^2}\,\frac{Z\,\alpha}{r_2^3}\,\vec{r}_2\times\vec{\sigma}_2\,. \tag{16}$$

This δj could be also derived through the Fouldy-Wouythusen transformation of $\alpha^i\,e^{i\,\vec{k}\cdot\vec{r}}$, however in this way possible electron-electron terms are omitted, which happens to be correct for just this case. Having δj and δH, the transition amplitude T^i in (4) will be transformed to the length gauge with the use of identity

$$\frac{\vec{p}_1 + \vec{p}_2}{m} = i\,[H, \vec{r}_1 + \vec{r}_2] \tag{17}$$

and the fact the terms in T^i proportional to k^i do not contribute to the transition rate, as it can be seen from Eq. (3). After performing simple algebraic transformations the result is

$$\begin{aligned}T^i = {} & i\,(E_\phi - E_\psi)\left\{\langle\phi|(r_1^i + r_2^i)\,\frac{1}{(E_\psi - H)'}\,\delta H|\psi\rangle + \langle\phi|\delta H\,\frac{1}{(E_\psi - H)'}\,(r_1^i + r_2^i)|\psi\rangle\right\}\\ & + \frac{1}{2\,m}\,\epsilon^{ijk}\,\langle\phi|k^j\,T^{kl}\,k^l|\psi\rangle\,,\end{aligned} \tag{18}$$

where

$$T^{kl} = \frac{1}{2}\left[r^k\,\frac{(\sigma_1 - \sigma_2)^l}{2} + r^l\,\frac{(\sigma_1 - \sigma_2)^k}{2} - \frac{2}{3}\,\delta^{kl}\,\vec{r}\cdot\frac{(\vec{\sigma}_1 - \vec{\sigma}_2)}{2}\right]. \tag{19}$$

The first term in Eq. (18) corresponds to electric dipole, and the second one to magnetic quadrupole transitions. It is worth noting that for electric dipole transitions, as given in length gauge, relativistic corrections enter only through corrections to the hamiltonian δH.

So far, we have considered only forbidden transitions with spin change between S and P states, namely $2^3P_2 \to 1^1S_0$, $2^3P_1 \to 1^1S_0$ and $2^1P_1 \to 2^3S_1$. However, even more forbidden $M1$ transitions $2^3S_1 \to 1^1S_0$ and $3^3S_1 \to 2^3S_1$ arrive at the order $O(Z\,\alpha)^3$, so they are not described by the expression in Eq. (16). No second order type of terms contribute and in the calculation of δj_M one takes the next corresponding term in the expansion of $e^{i\,\vec{k}\cdot\vec{r}}$ in Eqs.(6,9,12,16)

$$\delta\vec{\jmath}_M = \frac{i}{2\,m}\,\frac{(\vec{k}\cdot\vec{r}_1)^2}{2}\,\vec{k}\times\vec{\sigma}_1 + \frac{i}{4\,m^3}\,p_1^2\,\vec{k}\times\vec{\sigma}_1 + \frac{i}{4\,m^3}\,(\vec{k}\cdot\vec{p}_1)\,\vec{p}_1\times\vec{\sigma}_1 - \frac{i}{2\,m^2}\,(\vec{k}\cdot\vec{r}_1)\,\frac{Z\,\alpha}{r_1^3}\,\vec{r}_1\times\vec{\sigma}_1 + \frac{i}{2\,m^2}\,(\vec{k}\cdot\vec{r})\,\frac{\alpha}{r}\,\vec{r}\times\vec{\sigma}_1 + (1\leftrightarrow 2)\,. \tag{20}$$

This result agrees with the former one, obtained by Feinberg and Sucher in [5]. For $M1$ transition between 2^3S_1 and 1^1S_0 it could be further simplified to

$$\delta\vec{\jmath}_M = \frac{i}{m}\,\vec{k}\times\frac{(\vec{\sigma}_1-\vec{\sigma}_2)}{2}\left[\frac{k^2}{12}\,(r_1^2-r_2^2) + \frac{1}{3\,m^2}\,(p_1^2-p_2^2) - \frac{1}{6\,m}\left(\frac{Z\,\alpha}{r_1}-\frac{Z\,\alpha}{r_2}\right)\right]. \tag{21}$$

k^2 in the above can be replaced by

$$k^2\,r_1^2 \to [H,[H,r_1^2]] = \frac{2}{m}\,\frac{Z\,\alpha}{r_1} - \frac{2}{m^2}\,p_1^2 - \frac{2}{m}\,\frac{\alpha}{r^3}\,\vec{r}\cdot\vec{r}_1\,, \tag{22}$$

in this way one obtains for δj_M another simple expression

$$\delta\vec{\jmath}_M = \frac{i}{m}\,\vec{k}\times\frac{(\vec{\sigma}_1-\vec{\sigma}_2)}{2}\left[\frac{1}{6\,m^2}\,(p_1^2-p_2^2) - \frac{1}{6\,m}\,\frac{\alpha}{r^3}\,(r_1^2-r_2^2)\,.\right] \tag{23}$$

The analogous expression for the $3^3S_1 - 2^3S_1$ transition reads

$$\delta\vec{\jmath}_M = \frac{i}{m}\,\vec{k}\times\frac{(\vec{\sigma}_1+\vec{\sigma}_2)}{2}\left[\frac{1}{3\,m}\left(\frac{Z\,\alpha}{r_1}+\frac{Z\,\alpha}{r_2}\right) - \frac{1}{6\,m}\,\frac{\alpha}{r}\right]. \tag{24}$$

We now consider the spin algebra in the calculation of the transition probability, as given by Eqs. (3) and (18). One summs up over final states and averages out over initial states. The appropriate formulas are:

$$|^1S_0\rangle\langle^1S_0| = |^1S\rangle\langle^1S|\left(1-\frac{s^2}{2}\right), \tag{25}$$

$$\frac{1}{3}\sum_m |^3S_1,m\rangle\langle^3S_1,m| = |^3S\rangle\langle^3S|\,\frac{s^2}{6}\,, \tag{26}$$

$$|^3P_0\rangle\langle^3P_0| = |^3P^i\rangle\langle^3P^j|\left(\delta^{ij}\,\frac{s^2}{2} - s^j\,s^i\right), \tag{27}$$

$$\frac{1}{3}\sum_m |^3P_1,m\rangle\langle^3P_1,m| = |^3P^i\rangle\langle^3P^j|\,\frac{1}{2}\,s^i\,s^j\,, \tag{28}$$

$$\frac{1}{5}\sum_m |^3P_2,m\rangle\langle^3P_2,m| = |^3P^i\rangle\langle^3P^j|\,\frac{1}{10}\left(2\,s^2\,\delta^{ij} - 3s^i\,s^j + 2\,s^j\,s^i\right), \tag{29}$$

$$\frac{1}{3}\sum_m |^1P_1,m\rangle\langle^1P_1,m| = |^1P^i\rangle\langle^3P^j|\,\delta^{ij}\left(1-\frac{s^2}{2}\right), \tag{30}$$

where $s = \sigma_1/2 + \sigma_2/2$ and the following normalization is utilized: $\langle P^i|P^j\rangle = \delta^{ij}/3$. Moreover, for this calculations one needs two formulas for spin product

$$(\sigma_1 - \sigma_2)^i \left(1 - \frac{s^2}{2}\right) (\sigma_1 - \sigma_2)^j = 2\,\delta^{ij} s^2 - 4\, s^j\, s^i\,, \tag{31}$$

$$(\sigma_1 - \sigma_2)^i\, s^2\, (\sigma_1 - \sigma_2)^j = 8\,\delta^{ij} \left(1 - \frac{s^2}{2}\right), \tag{32}$$

and the following set of formulas for spin traces

$$\mathrm{Tr}\, s^i = 0\,, \tag{33}$$

$$\mathrm{Tr}\, s^i\, s^j = 2\,\delta^{ij}\,, \tag{34}$$

$$\mathrm{Tr}\, s^i\, s^j\, s^k = i\,\epsilon^{ijk}\,, \tag{35}$$

$$\mathrm{Tr}\, s^i\, s^j\, s^k\, s^l = \delta^{ij}\,\delta^{kl} + \delta^{jk}\,\delta^{il}\,. \tag{36}$$

With the help of the above formulas one obtains for transition probabilities (for simplicity we put $m = 1$) the following expressions

$$\mathcal{A}(^3P_1 \to {}^1S_0) = \frac{2}{9}\,\alpha\, k^3 \left|\epsilon^{ijk}\,\langle^3P^k|h^i\,\frac{1}{E_P - H}(r_1 + r_2)^j + (r_1 + r_2)^j\,\frac{1}{E_S - H}\,h^i|^1S\rangle\right|^2, \tag{37}$$

$$\mathcal{A}(^3P_2 \to {}^1S_0) = \frac{1}{30}\,\alpha\, k^5\, |\langle^3P^i|r^i|^1S\rangle|^2\,, \tag{38}$$

$$\begin{aligned}\mathcal{A}(^1P_1 \to {}^3S_1) = {}& \frac{2}{9}\,\alpha\, k^3 \left|\epsilon^{ijk}\,\langle^1P^k|h^i\,\frac{1}{E_P - H}(r_1 + r_2)^j + (r_1 + r_2)^j\,\frac{1}{E_S - H}\,h^i|^3S\rangle\right|^2 \\ & +\frac{1}{18}\,\alpha\, k^5\, |\langle^1P^i|r^i|^3S\rangle|^2\,,\end{aligned} \tag{39}$$

$$\mathcal{A}(^3S_1 \to {}^1S_0) = \frac{4}{3}\,\alpha\, k^3 \left|\langle^1S|\frac{1}{6}\,(p_1^2 - p_2^2) - \frac{1}{6}\,\frac{\alpha}{r^3}\,(r_1^2 - r_2^2)|^3S\rangle\right|^2, \tag{40}$$

$$\mathcal{A}(^3S_1 \to {}^3S_1) = \frac{4}{3}\,\alpha\, k^3 \left|\langle^3S|\frac{1}{3}\left(\frac{Z\,\alpha}{r_1} + \frac{Z\,\alpha}{r_2}\right) - \frac{1}{6}\,\frac{\alpha}{r}|^3S\rangle\right|^2, \tag{41}$$

where $k = |\Delta E|$, and h^i is defined by Eq. (5). It is worth noting that $^1P_1 \to {}^3S_1$ is not only $E1$ transition but also $M2$, which has not yet been recognized in the literature.

Once transition probabilities are expressed in terms of matrix elements between nonrelativistic wave functions, they can be calculated numerically with high precision. In the numerical calculation we follow an approach developed by Korobov [8]. The wave function is expressed in terms of exponentials

$$\phi_S = \sum_i c_i [e^{-\alpha_i\, r_1 - \beta_i\, r_2 - \gamma_i\, r} \mp (r_1 \leftrightarrow r_2)]\,, \tag{42}$$

$$\vec{\phi}_P = \sum_i c_i [\vec{r}_1\, e^{-\alpha_i\, r_1 - \beta_i\, r_2 - \gamma_i\, r} \mp (r_1 \leftrightarrow r_2)]\,, \tag{43}$$

$$\vec{\phi}_{P+} = \sum_i c_i\, \vec{r}_1 \times \vec{r}_2\, [e^{-\alpha_i\, r_1 - \beta_i\, r_2 - \gamma_i\, r} \mp (r_1 \leftrightarrow r_2)]\,. \tag{44}$$

The parameters $\alpha_i, \beta_i, \gamma_i$ are chosen randomly between some minimal and maximal values, which were found by minimization of energy of a specified state. The maximal dimension of this basis set was 600. Lower values were used for checking convergence. The advantage of this basis set is simplicity of matrix elements, which are expressed in terms of integral

$$\frac{1}{16\,\pi^2}\int d^3r_1\,d^3r_2\,\frac{e^{-\alpha\,r_1-\beta\,r_2-\gamma\,r}}{r_1\,r_2\,r} = \frac{1}{(\alpha+\beta)\,(\beta+\gamma)\,(\gamma+\alpha)}\,. \tag{45}$$

For some more singular matrix elements an additional integral with respect to corresponding parameters has to be performed. The disadvantage of this basis set is the necessity of using quadruple precision for $N > 100$. Moreover, the second order terms require more careful tuning of parameters due to the singularity of δH and large mixing of 2^3P_1 and 2^1P_1 states. These, which involve odd parity intermediate P-states, are much larger than those which involve even parity P-states, by approximately three orders of magnitude. It is due to the fact that energies of even parity P states lie beyond the ionization level. Most often, these small second order terms were neglected in the former calculations. However, they are not neglected here. Our numerical results for forbidden transitions between low lying states are presented in Table I.

In the comparison with former work we start with the $M1$ transition $2^3S_1 \to 1^1S_0$. This transition was measured by Moos and Woodworth in [9] with the result $\mathcal{A} = 1.10(33)\,10^{-4}\,\mathrm{s}^{-1}$ and Gordon Berry from Notre Dame is currently preparing a more precise measurement. The first (correct) theoretical result obtained by Drake in [6] $1.272\cdot 10^{-4}\,\mathrm{s}^{-1}$ is in agreement with the experimental value. However, as pointed out by Feinberg and Sucher in [5], Drake has not considered electron-electron terms, which happened to cancel out for this transition. Later, Johnson *et al* [1] used RMBPT to calculate forbidden transitions for any helium-like ions and obtained a result for $Z = 2$, which is $1.266\cdot 10^{-4}\,\mathrm{s}^{-1}$. It differs slightly from the result obtained here $1.272426\cdot 10^{-4}\,\mathrm{s}^{-1}$, due to inclusion in [1] of some higher order terms, while electron correlations were not well accounted for. Moreover there are unknown radiative corrections and exchange type of diagrams of order $\alpha/(2\,\pi)$, the last two in Fig. (1), to any of these transitions. Therefore only first 3 digits are physically significant. Numerical results are presented with higher precision for the purpose of comparison with former results. Next the M1 transition $3^3S_1 \to 2^3S_1$ rate was obtained only by Derevianko *et al* in [7]. Their result $1.17\,10^{-8}\,\mathrm{s}^{-1}$, disagrees with ours, $6.484690\,10^{-9}\,\mathrm{s}^{-1}$. The reason of this discrepancy is left unexplained. It may indicate the loss of accuracy of RMBPT due to strong numerical cancellation. This discrepancy does not have experimental impact since this rate is too small for $Z = 2$ to be measured. However, calculations should be verified for higher Z, where this transition rate grows with Z^{10} and becomes measurable at some value of Z. The next considered transition is M2: $2^3P_2 \to 1^1S_0$. It was first obtained by Drake [4]: $\mathcal{A} = 0.327\,\mathrm{s}^{-1}$, and later by Johnson *et al* [1] $\mathcal{A} = 0.3271\,\mathrm{s}^{-1}$, in agreement with our result $\mathcal{A} = 0.3270326\,\mathrm{s}^{-1}$. The calculation of the intercombination E1 transition $2^3P_1 \to 1^1S_0$ was little more elaborate, since it involves infinite summation over intermediate states. In former works the second term in Eq. (37) involving even parity P-states was neglected. Indeed, calculations show it is smaller than 1%. The first complete result by Drake [4] is $\mathcal{A} = 176.4\,\mathrm{s}^{-1}$. RMBPT calculations of Johnson *et al* [1] including negative energy states is $\mathcal{A} = 175.7\,\mathrm{s}^{-1}$ and our result $\mathcal{A} = 177.5771\,\mathrm{s}^{-1}$ agrees within 1%. The last transition $2^1P_1 \to 2^3S_1$ is a sum of E1 and M2. The result $\mathcal{A} = 1.55\,\mathrm{s}^{-1}$ obtained by Drake includes only E1 transition. Our result is $\mathcal{A} = 1.548945\,\mathrm{s}^{-1}$ and the magnetic transition happened to be negligible 0.000019 due to small energy splitting.

In summary, we have presented a rigorous derivation of rates for nonrelativistically forbidden transitions. We demonstrated equivalence of length and velocity gauges including relativistic correction for forbidden transitions. We confirmed the commonly used fact that in the length gauge relativistic corrections enters only through corrections to wave function as given by Breit hamiltonian. We verified that M2 $2^1P_1 \rightarrow 2^3S_1$ transition is much smaller than E1, which was implicitly assumed in former works. Our numerical calculations using simple exponential functions confirmed former results with the exception of $3^3S_1 \rightarrow 2^3S_1$ transition, where our result is approximately twice smaller than of [7].

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REFERENCES

[1] W.R. Johnson, D.R. Plante, and J. Sapirstein, Adv. At., Mol., Opt. Phys. **35**, 255 (1995).

[2] R.C. Elton, Ap. J. **148**, 573 (1967); R.C. Elton and W.W. Koppendorfer, Phys. Rev. **160**, 194 (1967).

[3] G.W.F. Drake and A. Dalgarno, Ap. J. **157**, 459 (1969).

[4] G.W.F. Drake, J. Phys. B**9**, L169 (1976).

[5] G. Feinberg and J. Sucher, Phys. Rev. Lett. **26**, 681 (1971).

[6] G.W.F. Drake, Phys. Rev. A **3**, 908 (1971).

[7] A. Derevianko, I.M. Savukov, and W.R. Johnson, Phys. Rev. A **58**, 4453 (1998).

[8] V.I. Korobov, Phys. Rev. A **61**, 064503 (2000).

[9] H.W. Moos and J.R. Woodworth, Phys. Rev. A **12**, 2455 (1975).

[10] Physical constants are from [11]: $m\,\alpha^2 \to (2\,\pi)\,2\,\mathrm{R}\,c = 4.13413733 \cdot 10^{16}\,\mathrm{s}^{-1}$, $\alpha^{-1} = 137.0359996$.

[11] P.J. Mohr and B.N. Taylor, Rev. Mod. Phys. **72**, 351 (2000).

FIGURES

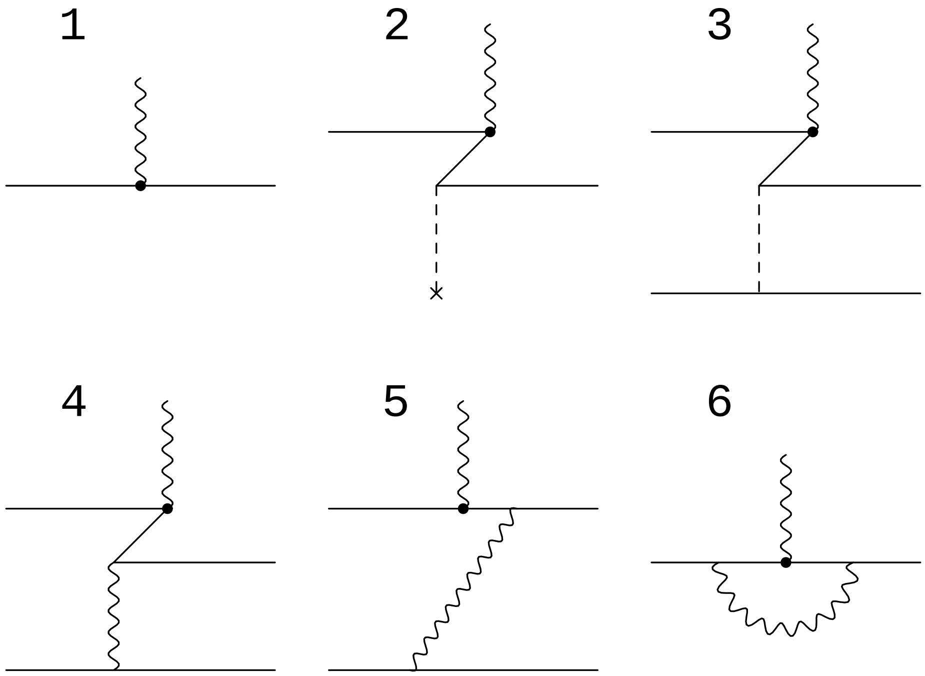

FIG. 1. Time ordered diagrams for corrections to the current. Dashed line is a Coulomb photon, the wavy line is the transverse photon.

TABLES

TABLE I. Transition rates in helium in units s^{-1}, $[n] \equiv 10^n$

transition	ΔE in atomic units	rate $\mathcal{A}$
E1+M2: $2^1P_1 \to 2^3S_1$	0.0513862917	1.548945
E1: $2^3P_1 \to 1^1S_0$	0.7705606863	1.775771[2]
M2: $2^3P_2 \to 1^1S_0$	0.7705606863	3.270326[-1]
M1: $2^3S_1 \to 1^1S_0$	0.7284949988	1.272426[-4]
M1: $3^3S_1 \to 2^3S_1$	0.1065403108	6.484690[-9]