Simple derivation of helium Lamb shift

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Abstract. A simple derivation of Lamb shift in the two–electron helium atom, in the spirit of nonrelativistic quantum electrodynamics, is presented. Moreover, a complete expression for the leading recoil correction is derived for the first time.

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1. Introduction

The quantum electrodynamical description of few electron systems is quite complicated. For high Z-systems the $1/Z$ expansion leads to the perturbative treatment of the electron–electron interaction. While this expansion works well for helium–like uranium, led, it has slow convergence for the helium atom. Another approach has been developed over the years to study QED effects in light few electron atoms. In this approach an electron–electron interaction is treated on the same footing as the stronger electron–nucleus interaction, and one expands only in the fine structure constant α. This expansion corresponds to the perturbative treatment of both relativistic and QED effects. Araki [1] in 1957 and Sucher [2] in 1958 have obtained for the first time a complete expression for the Lamb shift in the helium atom using the Bethe-Salpeter formalism [3]. However, there are several difficulties in comparison to the hydrogen atom. Firstly, the Shrödinger equation cannot be solved analytically, but only numerically. A large, sophisticated basis set is required for the actual calculation of matrix element of operators involved in the Lamb shift formula. Another difficulty is the numerical evaluation of the hydrogenic analog of Bethe logarithms. Up to date the most precise numbers for low lying states have been obtained Baker and collaborators in [4]. The derivation of the next order contribution ($m\,\alpha^6$) to the binding energy has been achieved by Douglas and Kroll [5] but only for the splitting between triplet P-levels. Using the Bethe-Salpeter equation they have derived corresponding nonrelativistic operators, which can be treated in a standard perturbation calculus with Schrödinger wave functions. These operators are the extension of Breit Hamiltonian [6] to the order of $m\,\alpha^6$. Usually, such an nonrelativistic expansion gives a very singular operators which could not for example be evaluated in the second order of

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perturbation calculus. However, triplet P-state wave functions are much more regular at small distances, making matrix elements of these operators finite at order $m\,\alpha^6$ for the difference between P-levels. The $m\,\alpha^6$ contribution for S–states has recently been derived in [7]. It also involves the calculation of Breit–like operators, however for S-states they have to be regularized to prevent divergences in the intermediate steps of the calculations. These divergences cancel out in the complete expression. The problem in the actual numerical calculations lies in the complexity of these regularized operators, which matrix elements is to be evaluated.

It is derived in this paper in a simple and pedagogical way the leading contribution to the Lamb shift in the helium atom and a complete recoil correction, that comes from the finite mass of the nucleus. Such a simple derivation may help nonspecialists to calculate Lamb shift for other systems like antiprotonic helium or chemical molecules. In chapter II I outline the derivation of pure recoil correction to the Lamb shift for a system of two-particles with arbitrary masses, so called Salpeter formula. In chapter III, I rederive the Lamb shift for helium atom. The derivation of leading recoil correction $m^2/M\,\alpha^5$ is presented in chapter IV. Short concluding remarks are left to the chapter V.

2. Pure recoil contribution to the Lamb shift of the two–body system

Any two–body system in quantum electrodynamics is affected by relativistic and radiative effects. A commonly known electron self–energy and a vacuum polarization leads to the Lamb shift. However, when both masses m and M are comparable, another type of diagrams including photon exchange only, are equally significant. Salpeter in 1952 has derived a closed form expression for the leading $m\,\alpha^5$ term for 1S state [8]. Later Ericson generalized this result to any state [9]. It reads

$$\Delta E = \frac{\mu^3}{m_1\,m_2}\frac{\alpha^5}{\pi\,n^3}\left\{\frac{2}{3}\,\delta_{l0}\,\ln\Big(\frac{1}{\alpha}\Big) - \frac{8}{3}\,\ln k_0(n,l) - \frac{1}{9}\,\delta_{l0} - \frac{7}{3}\,a_n \right.$$
$$\left. - \frac{2}{m_2^2-m_1^2}\,\delta_{l0}\,\left[m_2^2\,\ln\Big(\frac{m_1}{\mu}\Big) - m_1^2\,\ln\Big(\frac{m_2}{\mu}\Big)\right]\right\}, \quad (1)$$

where

$$a_n = -2\,\left[\ln\Big(\frac{2}{n}\Big) + \Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big) + 1 - \frac{1}{2\,n}\right]\,\delta_{l0} + \frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}, \quad (2)$$

μ denotes the reduced mass and $\ln k_0(n,l)$ is the Bethe logarithm. Here, I wish to present a simple derivation of “Salpeter” correction, which will be used later in the Lamb shift calculation of helium atom. Lets divide this correction into four parts

$$\Delta E = E_L + E_M + E_S + E_H\,, \quad (3)$$

that depend on the magnitude of a photon momentum: E_L is the low, E_M middle, E_S seagull (middle) and E_H high energy parts. Their precise definition is given later in the following. When the photon and particle momenta are small one can use a nonrelativistic

Hamiltonian:

$$
\begin{aligned}
H = {} & \frac{(\boldsymbol{p}_1 - e\,\boldsymbol{A}(\boldsymbol{r}_1))^2}{2\,m_1} + \frac{(\boldsymbol{p}_2 + e\,\boldsymbol{A}(\boldsymbol{r}_2))^2}{2\,m_2} - \frac{\alpha}{r} \\
& - \frac{e}{m_1}\,\boldsymbol{s}_1\cdot\boldsymbol{B}(\boldsymbol{r}_1) + \frac{e}{m_2}\,\boldsymbol{s}_2\cdot\boldsymbol{B}(\boldsymbol{r}_2) + \sum_{\lambda=1,2}\int \mathrm{d}^3k\,k\,a^+_{k,\lambda}\,a_{k,\lambda}\,,
\end{aligned}
\tag{4}
$$

where $k = |\boldsymbol{k}|$, $e^2 = 4\,\pi\,\alpha$,

$$
\boldsymbol{A}(\boldsymbol{r}) = \int \frac{\mathrm{d}^3k}{\sqrt{(2\,\pi)^3\,2\,k}}\,\boldsymbol{\epsilon}_\lambda\,\left(a^+_{k,\lambda}\,\mathrm{e}^{-i\,k\cdot r} + a_{k,\lambda}\,\mathrm{e}^{i\,k\cdot r}\right)\,,
\tag{5}
$$

$\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $[a^+_{k,\lambda}, a_{k',\lambda'}] = \delta^3(k-k')\,\delta_{\lambda,\lambda'}$. This Hamiltonian is conventionally split into a noninteracting part

$$
H_0 = \frac{\boldsymbol{p}_1^2}{2\,m_1} + \frac{\boldsymbol{p}_2^2}{2\,m_2} - \frac{\alpha}{r} + \sum_{\lambda=1,2}\int \mathrm{d}^3k\,k\,a^+_{k,\lambda}\,a_{k,\lambda}\,,
\tag{6}
$$

and a interaction

$$
\begin{aligned}
H_I = {} & -\frac{e}{m_1}\,\boldsymbol{p}_1\cdot\boldsymbol{A}(\boldsymbol{r}_1) + \frac{e}{m_2}\,\boldsymbol{p}_2\cdot\boldsymbol{A}(\boldsymbol{r}_2) + \frac{e^2}{2\,m_1}\,\boldsymbol{A}(\boldsymbol{r}_1)^2 + \frac{e^2}{2\,m_2}\,\boldsymbol{A}(\boldsymbol{r}_2)^2 \\
& - \frac{e}{m_1}\,\boldsymbol{s}_1\cdot\boldsymbol{B}(\boldsymbol{r}_1) + \frac{e}{m_2}\,\boldsymbol{s}_2\cdot\boldsymbol{B}(\boldsymbol{r}_2)\,.
\end{aligned}
\tag{7}
$$

At the rest mass system ($\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$) eigenstates of H_0 are solutions of Coulomb-Schrödinger equation with the reduced mass $1/\mu = 1/m_1 + 1/m_2$. Corrections to energy could be obtained in the standard perturbation theory. The exchange of the single transverse photon gives

$$
\begin{aligned}
\Delta E = {} & -\frac{e^2}{m_1\,m_2}\,\langle\phi|\,(\boldsymbol{p}_1\cdot\boldsymbol{A}(\boldsymbol{r}_1) - \boldsymbol{s}_1\cdot\boldsymbol{B}(\boldsymbol{r}_1))\,\frac{1}{E_0 - H_0} \\
& (\boldsymbol{p}_2\cdot\boldsymbol{A}(\boldsymbol{r}_2) - \boldsymbol{s}_2\cdot\boldsymbol{B}(\boldsymbol{r}_2))\,|\phi\rangle + \text{h.c.}\,.
\end{aligned}
\tag{8}
$$

The photon degrees of freedom could be simple traced out, leading to

$$
\begin{aligned}
\Delta E = {} & -\frac{e^2}{m_1\,m_2}\int\frac{\mathrm{d}^3k}{(2\,\pi)^3\,2\,k}\left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right) \\
& \langle\phi|\left(p_1^i + i\,(\boldsymbol{s}_1\times\boldsymbol{k})^i\right)\mathrm{e}^{i\,k\,r_1}\,\frac{1}{E_0 - k - H_0}\left(p_2^j - i\,(\boldsymbol{s}_2\times\boldsymbol{k})^j\right)\mathrm{e}^{-i\,k\,r_2}|\phi\rangle + \text{h.c.}\,,
\end{aligned}
\tag{9}
$$

where H_0 here is the atomic part of H_0 in Eq. (6). There are two characteristic energy scales for this transverse photon: binding energy $k \approx \mu\,\alpha^2$ and the inverse of Bohr radius $k \approx \mu\,\alpha$. One can split both scales by introducing a cut–off parameter ϵ. The low energy part E_L is defined by $k < \epsilon\,\mu\,\alpha$. The key point of this method [10] is the expansion in small ϵ after the expansion in α up to the required order α^5. It allows to perform the electric dipole approximation, namely $\mathrm{e}^{i\,k\,r} \to 1$ and $\boldsymbol{\sigma}\times\boldsymbol{k} \to 0$. The low energy part becomes

$$
E_L = -\frac{2\,e^2}{3\,m_1\,m_2}\int^{\epsilon\,\mu\,\alpha}\frac{\mathrm{d}^3k}{(2\,\pi)^3\,2\,k}\,\langle\phi|\,\boldsymbol{p}_1\,\frac{1}{E_0 - k - H_0}\,\boldsymbol{p}_2\,|\phi\rangle + \text{h.c.}\,.
\tag{10}
$$

In the rest frame $p_2 = -p_1 \equiv -p$. After angle integration E_L is

$$E_L = \frac{4\,\alpha}{3\,\pi\,m_1\,m_2}\int^{\epsilon\,\mu\,\alpha} \mathrm{d}k\,\langle\phi|\,\boldsymbol{p}\,\frac{k}{E_0-k-H_0}\,\boldsymbol{p}\,|\phi\rangle \tag{11}$$

$$= \frac{4\,\alpha}{3\,\pi\,m_1\,m_2}\,\langle\phi|\,\boldsymbol{p}\,(H_0-E_0)\,\ln\left(\frac{\epsilon\,\mu\,\alpha}{H_0-E_0}\right)\,\boldsymbol{p}\,|\phi\rangle\,. \tag{12}$$

It has the same form as the low energy part in the standard Lamb shift calculations, except for an additional factor 2. The further calculation goes in the same way, so one has

$$E_L = \frac{\mu^3}{m_1\,m_2}\,\frac{\alpha^5}{\pi\,n^3}\left[\frac{8}{3}\,\ln\left(\frac{2\,\epsilon}{\alpha}\right)\,\delta_{l0} - \frac{8}{3}\,\ln k_0(n,l)\right]\,, \tag{13}$$

where $\ln k_0(n,l)$ is a Bethe logarithm

$$\frac{2\,\mu^3\,\alpha^4}{n^3}\,\ln k_0(n,l) \equiv \langle\phi|\,\boldsymbol{p}\,(H_0-E_0)\,\ln\left(\frac{2\,(H_0-E_0)}{\mu\,\alpha^2}\right)\,\boldsymbol{p}\,|\phi\rangle\,. \tag{14}$$

The middle energy part is obtained from Eq. (9) with the condition $k > \epsilon\,\mu\,\alpha$. This condition allows to perform the following expansion of the denominator in Eq. (9)

$$\frac{1}{E-k-H} = -\frac{1}{k} + \frac{(H-E)}{k^2} + \cdots \tag{15}$$

The first term gives energy correction of order $\mu\,\alpha^4$ and is already included in the Breit Hamiltonian, next term contributes at $\mu\,\alpha^5$. Following terms denoted by dots are of higher order. After this expansion the middle energy part becomes

$$E_M = -\,\frac{e^2}{m_1\,m_2}\int_{\epsilon\,\mu\,\alpha}\frac{\mathrm{d}^3k}{(2\,\pi)^3\,2\,k^3}\,\left(\delta^{ij}-\frac{k^i\,k^j}{k^2}\right)$$

$$\langle\phi|\left(p_1^i + i\,(\boldsymbol{s}_1\times\boldsymbol{k})^i\right)\,\mathrm{e}^{i\,k\,r_1}\,(H_0-E_0)\,\left(p_2^j - i\,(\boldsymbol{s}_2\times\boldsymbol{k})^j\right)\,\mathrm{e}^{-i\,k\,r_2}|\phi\rangle + \mathrm{h.c.}\,. \tag{16}$$

The matrix element between ϕ could be further rewritten to $(\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2)$,

$$\begin{aligned}\langle\phi|\cdots|\phi\rangle + \mathrm{h.c.} &= \langle\phi|\Big[\left(p_1^i + i\,(\boldsymbol{s}_1\times\boldsymbol{k})^i\right)\,\mathrm{e}^{i\,k\,r_1}\,, \\ &\qquad \Big[H_0-E_0\,,\left(p_2^j - i\,(\boldsymbol{s}_2\times\boldsymbol{k})^j\right)\,\mathrm{e}^{-i\,k\,r_2}\Big]\Big]|\phi\rangle \\ &= \langle\phi|\mathrm{e}^{i\,k\,r}\,\left[p_1^i\,,\left[-\frac{\alpha}{r}\,,p_2^j\right]\right]\,|\phi\rangle\end{aligned} \tag{17}$$

and the middle energy part becomes

$$E_M = \frac{e^2}{m_1\,m_2}\int_{\epsilon\,\mu\,\alpha}\frac{\mathrm{d}^3k}{(2\,\pi)^3\,2\,k^3}\,\left(\delta^{ij}-\frac{k^i\,k^j}{k^2}\right)\,\langle\phi|\mathrm{e}^{i\,k\,r}\left[p^i\,,\left[p^j\,,\frac{\alpha}{r}\right]\right]\,|\phi\rangle\,. \tag{18}$$

It is worth to notice, that all the spin dependence cancels out in E_M. If one assumes $r > 0$, then E_M will be equal to

$$E_M \approx -\frac{2\,\alpha^2}{3\,\pi\,m_1\,m_2}\,\langle\phi|\frac{1}{r^3}|\phi\rangle\,. \tag{19}$$

This matrix element is divergent for S-states at low r, however it is finite and therefore correct for states with $l \neq 0$ and also for the difference $\langle\cdots\rangle_{nS} - \langle\cdots\rangle_{1S}/n^3$. Following Araki [1] and Sucher [2] one introduces a distribution $P(1/r^3)$ defined as

$$\langle\phi|P\left(\frac{1}{r^3}\right)|\psi\rangle = \lim_{a\to 0}\int \mathrm{d}^3r\,\phi^*(\boldsymbol{r})\left[\frac{1}{r^3}\,\Theta(r-a) + 4\,\pi\,\delta^3(r)\,(\gamma+\ln a)\right]\,\psi(\boldsymbol{r})\,. \tag{20}$$

It is finite for S-states and for the general hydrogenic state

$$\left\langle P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right)\right\rangle_{nl} = \frac{4}{n^3}\left(\ln\frac{n}{2}-\Psi(n)-\gamma+\frac{1}{2}-\frac{1}{2\,n}\right)\delta_{l0} + \frac{2}{n^3}\frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}\,. \tag{21}$$

The remaining part is proportional to $\delta^3(r)$ and could obtained by direct integration in Eq. (18) for the $1S$ state

$$E_M(1S) = \frac{8}{3\,\pi}\,\alpha^5\,\frac{\mu^3}{m_1\,m_2}\,\left(\frac{4}{3}-\ln\frac{\epsilon}{2}\right)\,. \tag{22}$$

From Eqs. (19, 21, 22) one gets complete E_M

$$E_M = \frac{8}{3\,\pi}\,\alpha^5\,\frac{\mu^3}{m_1\,m_2}\,\left[-\frac{1}{4}\left\langle P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right)\right\rangle_{nl}+\left(\frac{4}{3}-\ln\epsilon\right)\frac{\delta_{l0}}{n^3}\right]\,. \tag{23}$$

This expression finishes the treatment of the single transverse photon exchange. The ϵ dependence in E_M and E_L cancels out, as it should be, if the method is correct.

The next contribution, denoted by E_S is a double transverse photon exchange. It is obtained from the nonrelativistic Hamiltonian in Eq. (7), namely

$$E_S = \frac{e^4}{4\,m_1\,m_2}\,\langle\phi|\boldsymbol{A}(\boldsymbol{r}_1)^2\,\frac{1}{E_0-H_0}\,\boldsymbol{A}(\boldsymbol{r}_2)^2\,|\phi\rangle + \text{h.c.} \tag{24}$$

This expression, what would be clear in a moment is ultraviolet divergent. It will be regularized by some photon momentum cut-off. The photon states are traced out, what leads to the additional factor 2. The contraction of both $\boldsymbol{A}$ in the term $\boldsymbol{A}(\boldsymbol{r})^2$ leads to the electron self–energy. It is not included here since we consider only photon exchange diagrams. The kinetic energy of both particles and the binding energy is order in α smaller than that of photon in H_0, therefore it could neglected. After these transformations one obtains the following expression in a momentum space

$$E_S = -\frac{e^4}{4\,m_1\,m_2}\int\frac{d^3p}{(2\,\pi)^3}\int\frac{d^3q_1}{(2\,\pi)^3}\int\frac{d^3q_2}{(2\,\pi)^3}\,\phi(\boldsymbol{p}-\boldsymbol{q}_1)\,\phi(\boldsymbol{p}+\boldsymbol{q}_2)\,\frac{1}{q_1\,q_2\,(q_1+q_2)}\left(\delta^{ij}-\frac{q_1^i\,q_1^j}{q_1^2}\right)\left(\delta^{ij}-\frac{q_2^i\,q_2^j}{q_2^2}\right)\,. \tag{25}$$

Without any cut-off these integrals leads to the expression

$$E_S \approx -\frac{\alpha^2}{2\,\pi\,m_1\,m_2}\,\langle\phi|\frac{1}{r^3}|\phi\rangle\,, \tag{26}$$

which is again divergent at small r for S-states. It is the reason for introducing an upper cut-off λ. We achieve this by multiplying photon propagators by $\Lambda^2/(\boldsymbol{q}^2+\Lambda^2)$ with $\Lambda = \mu\,\alpha\,\lambda$, so E_S becomes

$$E_S = -\frac{e^4}{4\,m_1\,m_2}\int\frac{d^3p}{(2\,\pi)^3}\int\frac{d^3q_1}{(2\,\pi)^3}\int\frac{d^3q_2}{(2\,\pi)^3}\,\phi(\boldsymbol{p}-\boldsymbol{q}_1)\,\phi(\boldsymbol{p}+\boldsymbol{q}_2)\,\frac{1}{q_1\,q_2\,(q_1+q_2)}\left(\delta^{ij}-\frac{q_1^i\,q_1^j}{q_1^2}\right)\left(\delta^{ij}-\frac{q_2^i\,q_2^j}{q_2^2}\right)\frac{\Lambda^2}{\boldsymbol{q}_1^2+\Lambda^2}\,\frac{\Lambda^2}{\boldsymbol{q}_2^2+\Lambda^2}\,. \tag{27}$$

One can express E_S in terms of $P(1/r^3)$ as in Eq. (26) and the remaining will be proportional to $\delta^3(r)$, so it could be obtained from the result for the 1S state

$$E_S(1S) = -\frac{2}{\pi}\,\alpha^5\,\frac{\mu^3}{m_1\,m_2}\,\left(\ln\lambda + \frac{4}{3}\,\ln 2 - \frac{17}{6}\right)\,, \tag{28}$$

where it is assumed an expansion in large λ. The result for arbitrary state is obtained from Eqs. (21, 26, 28)

$$E_S = -\frac{2}{\pi}\,\alpha^5\,\frac{\mu^3}{m_1\,m_2}\,\left[\frac{1}{4}\left\langle P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right)\right\rangle_{nl} + \left(\ln\lambda + \frac{7}{3}\,\ln 2 - \frac{17}{6}\right)\,\frac{\delta_{l0}}{n^3}\right] \tag{29}$$

However it depends on the cut-off λ. This dependence is canceled out by the high energy part E_H. This part is obtained from the forward scattering amplitude involving an exchange of two–photons. This amplitude can calculated in the Feynman gauge, since it is gauge independent,

$$\begin{aligned} E_H = & -e^4\,\phi(0)^2\,\int_s \frac{d^4k}{(2\,\pi)^4\,i}\,\frac{1}{k^4}\,\mathrm{Tr}\left[\gamma^\mu\,\frac{1}{\not t - \not k - m_1}\,\gamma^\nu\,\frac{\gamma^0+I}{4}\right] \\ & \left\{\mathrm{Tr}\left[\gamma^\mu\,\frac{1}{\not t + \not k - m_2}\,\gamma^\nu\,\frac{\gamma^0+I}{4}\right] + \mathrm{Tr}\left[\gamma^\nu\,\frac{1}{\not t - \not k - m_2}\,\gamma^\mu\,\frac{\gamma^0+I}{4}\right]\right\}. \end{aligned} \tag{30}$$

$(\gamma^0 + I)/2$ is a projection operator, for particle at rest and with averaged spin, t is a four-momentum $t = (m, \mathbf{0})$, and the subscript S denotes a subtraction which will be defined later. There is also an amplitude proportional to $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ which will be considered separately. After performing traces the spin independent amplitude has the form

$$\begin{aligned} E_H = & -\frac{128\,\pi\,\alpha^5\,m_1\,m_2\,\mu^3}{n^3}\,\delta_{l0}\,\int_s \frac{\mathrm{d}^4k}{(2\,\pi)^4\,i}\,(k^4 + 2\,\omega^4)\,k^{-4}\,(k^2 - 2\,m_1\,\omega)^{-1} \\ & (k^2 + 2\,m_1\,\omega)^{-1}\,(k^2 - 2\,m_2\,\omega)^{-1}\,(k^2 + 2\,m_2\,\omega)^{-1}\,, \end{aligned} \tag{31}$$

where $\omega \equiv k^0$. The ω-integration leads to

$$\begin{aligned} E_H = & -\frac{4\,\pi\,\alpha^5\,m_1\,m_2\,\mu^3}{n^3}\,\delta_{l0}\,\int_s \frac{\mathrm{d}^3k}{(2\,\pi)^3}\,\left[\frac{1}{k^3\,m_1^2\,m_2^2}\right. \\ & \left. + \frac{k^4 + 6\,k^2\,m_1^2 + 8\,m_1^4}{k^6\,m_1^2\,\sqrt{k^2+m_1^2}\,(m_1^2 - m_2^2)} + \frac{k^4 + 6\,k^2\,m_2^2 + 8\,m_2^4}{k^6\,m_2^2\,\sqrt{k^2+m_2^2}\,(m_2^2 - m_1^2)}\right]. \end{aligned} \tag{32}$$

The subtraction terms are small k singularities of the expression in square brackets, namely $1/k^6$, $1/k^4$, and $1/k^3$. They correspond to the leading Schrödinger Hamiltonian, Breit correction [6] and middle energy part E_M, respectively. The subtraction term $1/k^3$ has an additional factor $\Lambda^4/(\Lambda^2+k^2)^2$ as in E_M. After subtraction, this integral is finite and amounts to

$$\begin{aligned} E_H = & -\frac{\mu^3}{m_1\,m_2}\,\frac{\alpha^5}{\pi\,n^3}\,\delta_{l0}\,\left\{\frac{7}{3} - 2\,\ln(2\,\lambda\,\alpha)\right. \\ & \left. + \frac{2}{m_1^2 - m_2^2}\,\left[m_1^2\,\ln\Big(\frac{m_2}{\mu}\Big) - m_2^2\,\ln\Big(\frac{m_1}{\mu}\Big)\right]\right\}. \end{aligned} \tag{33}$$

The sum of all parts in Eq. (3) is finite, does not depend on cut-offs and equals to

$$\Delta E = E_L + E_M + E_S + E_H$$

$$= \frac{\alpha^5}{\pi\, n^3}\,\frac{\mu^3}{m_1\, m_2}\left\{\frac{2}{3}\,\delta_{l0}\,\ln\frac{1}{\alpha}-\frac{8}{3}\,\ln\,k_0(n,l)-\frac{7}{6}\,n^3\left\langle P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right)\right\rangle_{nl}\right.$$
$$\left.+\frac{62}{9}\,\delta_{l0}-\frac{2\,\delta_{l0}}{m_1^2-m_2^2}\left[m_1^2\,\ln\Bigl(\frac{m_2}{\mu}\Bigr)-m_2^2\,\ln\Bigl(\frac{m_1}{\mu}\Bigr)\right]\right\}. \tag{34}$$

what agrees with Eq. (1).

As it was mentioned already there is also a spin dependent part in the scattering amplitude that leads to an additional hyperfine splitting of S-states. It is calculated in the same way as the spin independent amplitude, only the projection operators are different $(\gamma^0+I)\,\gamma^i\,\gamma^5/2$.

$$E'_H = -\,e^4\,\phi(0)^2\,\frac{\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle}{3}\int_s\frac{d^4k}{(2\,\pi)^4\,i}\,\frac{1}{k^4}$$
$$\mathrm{Tr}\left[\gamma^\mu\,\frac{1}{\not t-\not k-m_1}\,\gamma^\nu\,(\gamma^0+I)\,\gamma^i\,\gamma^5/4\right]$$
$$\times\left\{\mathrm{Tr}\left[\gamma^\mu\,\frac{1}{\not t+\not k-m_2}\,\gamma^\nu\,(\gamma^0+I)\,\gamma^i\,\gamma^5/4\right]\right.$$
$$\left.+\,\mathrm{Tr}\left[\gamma^\nu\,\frac{1}{\not t-\not k-m_2}\,\gamma^\mu\,(\gamma^0+I)\,\gamma^i\,\gamma^5/4\right]\right\}$$
$$= -\,\frac{2\,\alpha^5}{\pi\,n^3}\,\delta_{l0}\,\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle\,\frac{\mu^3}{m_2^2-m_1^2}\,\ln\Bigl(\frac{m_2}{m_1}\Bigr). \tag{35}$$

This correction is finite, does not require any regularization, since there is no corresponding middle energy part. It is worth to notice that in general E_L and E_M are calculated in the Coulomb and E_H in the Feynman gauge. The correctness of a final result is preserved by the limit of infinite λ. This method is seemingly similar to the original Lamb shift calculations [11], moreover it could be extended to higher orders. In the same spirit, these calculations could be extended to the more complicated system, namely the helium atom, and next sections are devoted to them.

3. Helium Lamb shift

Calculations of bound state energies of the helium atom are nontrivial due to the presence of the electron–electron interaction, which has to be treated on the same footing as the electron–nucleus interactions. One could not write a simple analog of the Dirac equation for the two–electron system. However, for the leading quantum electrodynamical corrections one can use a simplified treatment, very similar to the one demonstrated in the previous section. In other words we present a simple derivation of the $m\,\alpha^5$ contribution to helium energy levels based on the nonrelativistic QED. There are self-energy and photon exchange diagrams to be considered. The treatment of electron self–energy in the lowest order is similar to that in hydrogen [11]. One divides this correction into the low and the high energy part using the ϵ cut–off. For the low energy part one can use the nonrelativistic Hamiltonian of two electrons in the external

Coulomb field:

$$H = \frac{(\boldsymbol{p}_1 - e\,\boldsymbol{A}(\boldsymbol{r}_1))^2}{2\,m} + \frac{(\boldsymbol{p}_2 + e\,\boldsymbol{A}(\boldsymbol{r}_2))^2}{2\,m} - \frac{Z\,\alpha}{r_1} - \frac{Z\,\alpha}{r_2} + \frac{\alpha}{r_{12}}$$
$$+ \sum_{\lambda=1,2} \int \mathrm{d}^3k\, k\, a^+_{k,\lambda}\, a_{k,\lambda}\,, \tag{36}$$

and a standard perturbation formulas. The atomic noninteracting part H_0 is

$$H_0 = \frac{p_1^2}{2\,m_1} + \frac{p_2^2}{2\,m_2} - \frac{Z\,\alpha}{r_1} - \frac{Z\,\alpha}{r_2} + \frac{\alpha}{r_{12}}\,, \tag{37}$$

and a interaction H_I

$$H_I = -\frac{e}{m}\,\boldsymbol{p}_1 \cdot \boldsymbol{A}(\boldsymbol{r}_1) - \frac{e}{m}\,\boldsymbol{p}_2 \cdot \boldsymbol{A}(\boldsymbol{r}_2) + \frac{\boldsymbol{A}^2(\boldsymbol{r}_1)}{2\,m} + \frac{\boldsymbol{A}^2(\boldsymbol{r}_2)}{2\,m}\,. \tag{38}$$

The essential feature of this method [10] is the fact that after the expansion in α one takes the limit ϵ goes to zero, keeping the finite and divergent terms only. It allows $\boldsymbol{A}^2$ terms to be neglected in the self–energy part. Moreover, it allows also for the dipole approximation. So, the formula one gets is:

$$E_{SL} = \frac{e^2}{m^2}\langle\phi|\boldsymbol{p}_1 \cdot \boldsymbol{A}(0)\,\frac{1}{E_0 - H_0}\,\boldsymbol{p}_1 \cdot \boldsymbol{A}(0)\,|\phi\rangle + (1 \to 2)\,. \tag{39}$$

The photon degrees of freedom could be traced out leading to

$$E_{SL} = \frac{2\,e^2}{3\,m^2} \int^{\epsilon\,m} \frac{\mathrm{d}^3k}{(2\,\pi)^3\,2\,k}\,\langle\phi|\,\boldsymbol{p}_1\,\frac{1}{E_0 - k - H_0}\,\boldsymbol{p}_1\,|\phi\rangle + (1 \to 2)\,. \tag{40}$$

This expression is linearly divergent in large ϵ. Since after expansion in α one goes with ϵ to 0, the linear term could be subtracted out, but the logarithmic term stays. After k–integration E_{SL} takes a form

$$E_{SL} = \frac{2\,\alpha}{3\,\pi\,m^2}\,\langle\phi|\,\boldsymbol{p}_1\,(H_0 - E_0)\,\ln\left(\frac{\epsilon\,m}{H_0 - E_0}\right)\boldsymbol{p}_1\,|\phi\rangle + (1 \to 2) \tag{41}$$
$$= \left[-\frac{2\,\alpha}{3\,\pi\,m^2}\langle\boldsymbol{p}_1\,(H_0 - E_0)\,\ln\left(\frac{H_0 - E_0}{Z^2\,m\,\alpha^2/2}\right)\boldsymbol{p}_1\rangle + (1 \to 2)\right]$$
$$+ \left[\frac{4\,Z\,\alpha^2}{3\,m^2}\,\langle\delta^3(r_1) + \delta^3(r_2)\rangle - \frac{8\,\alpha^2}{3\,m^2}\,\langle\delta^3(r_{12})\rangle\right]\,\ln\frac{2\,\epsilon}{Z^2\,\alpha^2}\,. \tag{42}$$

When combined with the low energy part from the 2–photon exchange, it will form a Bethe log for the helium atom. It is the peculiar feature of radiative corrections that the high energy part could be obtained from the on–shell form factors only. The spin independent terms are proportional to Dirac delta, and are known from the hydrogen case [11]:

$$E_{SH} = \frac{Z\,\alpha^2}{m^2}\left(\frac{10}{9} - \frac{4}{15} - \frac{4}{3}\,\ln 2\epsilon\right)\,\langle\delta^3(r_1) + \delta^3(r_2)\rangle$$
$$- \frac{\alpha^2}{m^2}\left(2\,\frac{10}{9} - \frac{4}{15} - 2\,\frac{4}{3}\,\ln 2\epsilon\right)\,\langle\delta^3(r_{12})\rangle\,, \tag{43}$$

where 10/9 comes from the electron self-energy and 4/15 from the vacuum polarization. The factor 2 in the second line comes from the fact that both electron self–energies contribute, while there is only one vacuum polarization diagram between them. The spin dependent terms E_B are the anomalous magnetic moment correction to the Breit interaction [12]. For the system of two particles with the same charges this correction is given by Eqs. (52–56).

Photon exchange diagrams in the helium atom are calculated in the same way as for the two–body system which was considered in the previous section. The low energy part reads:

$$E_L = \frac{2\,e^2}{3\,m^2}\int^{\epsilon\,m\,\alpha/2}\frac{d^3k}{(2\,\pi)^3\,2\,k}\,\langle\phi|\,\boldsymbol{p}_1\,\frac{1}{E_0-k-H_0}\,\boldsymbol{p}_2\,|\phi\rangle + \text{h.c.} \tag{44}$$

$$= \frac{2\,\alpha}{3\,\pi\,m^2}\,\langle\phi|\,\boldsymbol{p}_1\,(H_0-E_0)\,\ln\left(\frac{\epsilon\,m\,\alpha/2}{H_0-E_0}\right)\boldsymbol{p}_2\,|\phi\rangle + (1\leftrightarrow 2) \tag{45}$$

$$= \left[-\frac{2\,\alpha}{3\,\pi\,m^2}\,\langle\boldsymbol{p}_1\,(H_0-E_0)\,\ln\left(\frac{H_0-E_0}{Z^2\,m\,\alpha^2/2}\right)\boldsymbol{p}_2\rangle + (1\leftrightarrow 2)\right] + \frac{8}{3}\,\frac{\alpha^2}{m^2}\,\ln\frac{\epsilon}{Z^2\,\alpha}\,\langle\delta^3(r_{12})\rangle\,. \tag{46}$$

The middle energy part E_M is obtained from the corresponding part of a two–particle binding energy in the previous section, by putting $m_1 = m_2$ and $\mu = m/2$. One notices that all the terms proportional to δ_{l0} come from the expectation value of the Dirac delta function, so one replaces $\mu^3\,\alpha^3/(\pi\,n^3)$ by $\langle\delta^3(r_{12})\rangle$. Moreover P–distribution is transformed according to

$$(\mu\,\alpha)^3\,P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right) = (m\,\alpha)^3\,P\left(\frac{1}{(m\,\alpha\,r)^3}\right) + 4\,\pi\,\delta^3(r)\,\ln\frac{\mu}{m}\,. \tag{47}$$

After these replacements E_M reads:

$$E_M = -\frac{2\,m\,\alpha^5}{3\,\pi}\left\langle P\left(\frac{1}{(m\,\alpha\,r_{12})^3}\right)\right\rangle + \frac{8}{3}\,\frac{\alpha^2}{m^2}\left(\frac{4}{3}+\ln 2-\ln\epsilon\right)\langle\delta^3(r_{12})\rangle\,. \tag{48}$$

Double transverse part E_S is obtained in the same way as E_M

$$E_S = -\frac{m\,\alpha^5}{2\,\pi}\left\langle P\left(\frac{1}{(m\,\alpha\,r_{12})^3}\right)\right\rangle - \frac{2\,\alpha^2}{m^2}\left(\ln\lambda+\frac{4}{3}\,\ln 2-\frac{17}{6}\right)\langle\delta^3(r_{12})\rangle\,. \tag{49}$$

So the high energy part

$$E_H = -\frac{\alpha^2}{m^2}\left(4\,\boldsymbol{s}_1\cdot\boldsymbol{s}_2+\frac{4}{3}-2\,\ln\lambda\,\alpha\right)\langle\delta^3(r_{12})\rangle\,. \tag{50}$$

The complete helium Lamb shift is a sum of terms $E_{SL}+E_{SH}+E_B+E_L+E_M+E_S+E_H$. The explicit expression in the notation based on Sapirstein review in [13] is:

$$E^{(5)} = \frac{\alpha}{\pi}\left\langle H_{\rm so}+H'_{\rm soo}+\frac{1}{2}\,H''_{\rm soo}+H_{\rm ss}+H_{\rm ssc}\right\rangle + \left[\frac{89}{15}+\frac{14}{3}\,\ln\alpha-4\,\boldsymbol{s}_1\cdot\boldsymbol{s}_2\right]\frac{\alpha^2}{m^2}\,\langle\delta^3(r_{12})\rangle + \left[\frac{19}{30}+\ln(Z\,\alpha)^{-2}\right]\frac{4\,\alpha^2\,Z}{3\,m^2}\,\langle\delta^3(r_1)+\delta^3(r_2)\rangle - \frac{14}{3}\,m\,\alpha^5\left\langle\frac{1}{4\,\pi}\,P\left(\frac{1}{(m\,\alpha\,r_{12})^3}\right)\right\rangle - \frac{2\,\alpha}{3\,\pi\,m^2}\,\mathcal{M}\,, \tag{51}$$

where

$$H_{so} = \frac{\alpha\,Z}{2\,m^2}\left[\frac{\boldsymbol{s}_1\cdot\boldsymbol{r}_1\times\boldsymbol{p}_1}{r_1^3}+\frac{\boldsymbol{s}_2\cdot\boldsymbol{r}_2\times\boldsymbol{p}_2}{r_2^3}\right], \tag{52}$$

$$H'_{soo} = -\frac{\alpha}{2\,m^2\,r_{12}^3}[\boldsymbol{s}_1\cdot\boldsymbol{r}_{12}\times\boldsymbol{p}_1+\boldsymbol{s}_2\cdot\boldsymbol{r}_{21}\times\boldsymbol{p}_2], \tag{53}$$

$$H''_{soo} = -\frac{\alpha}{m^2\,r_{12}^3}[\boldsymbol{s}_2\cdot\boldsymbol{r}_{12}\times\boldsymbol{p}_1+\boldsymbol{s}_1\cdot\boldsymbol{r}_{21}\times\boldsymbol{p}_2], \tag{54}$$

$$H_{ss} = \frac{\alpha}{m^2\,r_{12}^3}\,[\boldsymbol{s}_1\cdot\boldsymbol{s}_2-3\,(\boldsymbol{s}_1\cdot\boldsymbol{r}_{12})(\boldsymbol{s}_2\cdot\boldsymbol{r}_{12})], \tag{55}$$

$$H_{ssc} = -\frac{8\,\pi\,\alpha}{3\,m^2}\,\boldsymbol{s}_1\cdot\boldsymbol{s}_2\,\delta^3(r_{12}), \tag{56}$$

and

$$\mathcal{M} = \left\langle(\boldsymbol{p}_1+\boldsymbol{p}_2)\,(H_0-E_0)\,\ln\frac{2\,(H_0-E_0)}{(Z\,\alpha)^2\,m}\,(\boldsymbol{p}_1+\boldsymbol{p}_2)\right\rangle$$
$$\equiv 2\,\pi\,\alpha\,Z\,\langle\delta^3(r_1)+\delta^3(r_2)\rangle\,\ln k_0\,. \tag{57}$$

The electron–electron terms in Eqs. (51, 56) could be further simplified, because $\delta^3(r_{12})$ does not vanish only for singlet states. therefore $\boldsymbol{s}_1\cdot\boldsymbol{s}_2\,\delta^3(r_{12}) = -3/4\,\delta^3(r_{12})$. Between others, the helium Bethe log is the most difficult in the numerical calculation. We refer reader to the recent works on this problem: [4] and [14]. There are various corrections beyond this leading order Lamb shift. The $m\,\alpha^6$ correction has been considered in [7]. A rather complicated, complete set of operators has been derived but the numerical calculation of their matrix elements has not yet been performed. Other corrections are connected with the finite nucleus mass. They have been partially treated by Drake in [15]. Here, in the next section, we derive a complete recoil correction of order $m/M^2\,\alpha^5$.

4. Recoil correction to helium Lamb shift

Recoil corrections to energy levels come from finite nuclear mass. While the leading terms are calculated in the limit of infinite nuclear mass, the first recoil correction is proportional to the electron–nuclear mass ratio. Since we are working at order α^5 it is sufficient to derive only this first in the mass ratio recoil term. It is obtained by adding the nonrelativistic nucleus kinetic energy

$$H_M = \frac{(\boldsymbol{P}+e\,\boldsymbol{A}(\boldsymbol{R}))^2}{2\,M} \tag{58}$$

to the total Hamiltonian of our system. The rigorous derivation would go along the lines in the previous sections. The results could also be obtained from the following analysis. The gross of recoil terms are obtained from the modification of nonrelativistic wave function, Hamiltonian H_0 and energy E_0 in $E^{(5)}$ Eq. (51) by

$$\frac{\boldsymbol{P}^2}{2\,M} = \frac{(\boldsymbol{p}_1+\boldsymbol{p}_2)^2}{2\,M} = \frac{\boldsymbol{p}_1^2+\boldsymbol{p}_2^2}{2\,M}+\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{M}\,. \tag{59}$$

The correction E_{R1} due to the first term could be easily incorporated by the reduced mass rescaling of r_1 and r_2. All the operators in $E^{(5)}$ go like $\mu^3/m^3 \approx 1-3m/M$, except for the Bethe log, which has an additional term

$$\delta\mathcal{M} = \ln\frac{\mu}{m}\,\langle\phi|(\boldsymbol{p}_1+\boldsymbol{p}_2)\,(H_0-E_0)\,(\boldsymbol{p}_1+\boldsymbol{p}_2)|\phi\rangle \tag{60}$$

$$\approx -\frac{m}{M}\,2\,\pi\,Z\,\alpha\,\langle\delta^3(r_1)+\delta^3(r_2)\rangle\,, \tag{61}$$

therefore E_{R1} is

$$E_{R1} = -3\,\frac{m}{M}\,E^{(5)} + \frac{4\,Z\,\alpha^2}{3\,m\,M}\,\langle\delta^3(r_1)+\delta^3(r_2)\rangle\,. \tag{62}$$

The correction E'_{R1} due to mass polarization term $\boldsymbol{p}_1\cdot\boldsymbol{p}_2/M$ is also easily incorporated in $E^{(5)}$ as a correction to ϕ, H_0 and E_0.

Further recoil corrections come from additional electron–nucleus interaction terms. There is a Breit type interaction E_{R2} of the nucleus with the anomalous magnetic moment of an electron that has single power of M in the denominator

$$E_{R2} = \frac{\alpha}{2\,\pi}\,\left\langle -\frac{Z\,\alpha}{m\,M\,r_1^3}\,\boldsymbol{r}_1\times\boldsymbol{P}\cdot\boldsymbol{s}_1 + (1\to 2)\right\rangle$$

$$= \frac{\alpha}{2\,\pi}\,\frac{Z\,\alpha}{m\,M}\,\left\langle\left(\frac{\boldsymbol{s}_1\times\boldsymbol{r}_1}{r_1^3}+\frac{\boldsymbol{s}_2\times\boldsymbol{r}_2}{r_2^3}\right)\cdot(\boldsymbol{p}_1+\boldsymbol{p}_2)\right\rangle\,. \tag{63}$$

The last recoil correction is a kind of two–photon exchange between electron and nucleus, similar to the Salpeter correction, considered in the Section 2. Lets rewrite this correction in Eq. (34) to the following form

$$\Delta E = \frac{\alpha^2}{m_1\,m_2}\,\langle\delta^3(r)\rangle\,\left(\frac{2}{3}\,\ln\alpha^{-1}+\frac{62}{9}\right) - \frac{7}{6}\,\frac{\alpha^5}{\pi}\,\frac{\mu^3}{m_1\,m_2}\,\left\langle P\left(\frac{1}{(\mu\,\alpha\,r)^3}\right)\right\rangle$$

$$+\frac{4}{3}\,\frac{\alpha}{\pi}\,\frac{1}{m_1\,m_2}\,\left\langle\boldsymbol{p}_1(H_0-E_0)\ln\left[\frac{2(H_0-E_0)}{\mu\,\alpha^2}\right]\boldsymbol{p}_2\right\rangle\,. \tag{64}$$

We have omitted here the term that goes like $1/m_2^2$, since we will assume that the particle No. 2 corresponds to the nucleus. The corresponding recoil correction E_{R3} for helium case could be easily obtained by the replacement of α by $Z\,\alpha$, r by r_1 or r_2, m_2 by M, m_1 by m, $\boldsymbol{p}_2$ by $\boldsymbol{P}=-\boldsymbol{p}_1-\boldsymbol{p}_2$;

$$E_{R3} = \frac{(Z\,\alpha)^2}{M\,m}\,\left(\frac{2}{3}\,\ln(Z\,\alpha)^{-1}+\frac{62}{9}\right)\,\langle\delta^3(r_1)+\delta^3(r_2)\rangle$$

$$-\frac{4}{3\,\pi}\,\frac{Z\,\alpha}{m\,M}\,\left\langle(\boldsymbol{p}_1+\boldsymbol{p}_2)\,(H_0-E_0)\,\ln\left[\frac{2\,(H_0-E_0)}{m\,(Z\,\alpha)^2}\right]\,(\boldsymbol{p}_1+\boldsymbol{p}_2)\right\rangle$$

$$-\frac{7}{6\,\pi}\,(Z\,\alpha)^5\,\frac{m^2}{M}\,\left\langle P\left(\frac{1}{(m\,Z\,\alpha\,r_1)^3}\right)+P\left(\frac{1}{(m\,Z\,\alpha\,r_2)^3}\right)\right\rangle\,. \tag{65}$$

The complete recoil correction is a sum

$$E_R = E_{R1} + E'_{R1} + E_{R2} + E_{R3}\,. \tag{66}$$

5. Summary

It was our goal to present a simple derivation of QED effects on energy levels for few electron systems, like the helium atom. We have rederived the Lamb shift at order

$m\,\alpha^5$ and obtained a complete expression for the leading recoil correction. Our method was inspired on idea of nonrelativistic QED, which was first introduced by Caswell and Lepage [16]. This method has been applied successfully to many problems, like higher order corrections to the Lamb shift [10] and, we believe, could be extended to any few body system. The numerical calculations of the derived matrix elements for recoil correction is as difficult as the leading Lamb shift, therefore it could be performed equally well.

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