Comprehensive theory of μH hyperfine splitting

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This work attempts to present a complete theory of the μH hyperfine splitting, including all contributions above 1 ppm. Quantum electrodynamic and recoil corrections are calculated directly, while the proton structure correction is obtained with the help of the H hyperfine splitting. The resulting theoretical prediction for the ground state of μH is $E_{\rm hfs} = 182\,626(5)\ \mu$eV.

I. INTRODUCTION

The ground state hyperfine splitting (HFS) in regular hydrogen has long served as a low-energy test of the Standard Model of fundamental interactions [1–3]. Despite the apparent simplicity of the hydrogen atom, the high precision calculation of two-body effects and estimation of proton structure corrections remain a challenge. In fact, we observe a 2σ discrepancy between the most recent theoretical predictions [4] and extremely accurate HFS measurements [5–7]. This recent work [4] improved theoretical predictions by recalculation of the leading relativistic recoil correction, and the not well known proton structure is most probably the source of this remaining 2σ discrepancy. At present, there is no straightforward way to improve the theoretical estimates for the Zemach radius and the proton polarizability. Lattice QCD is not yet able to predict proton properties at the 1% level. The only viable approach to improving the hydrogen test of the Standard Model is through the measurement of HFS in another hydrogenic system, namely in μH, where the electron is replaced by a muon [8, 9].

In this work, we study QED contributions to the μH HFS, with emphasis on nuclear recoil corrections. These corrections become highly significant here due to the muon-proton mass ratio being approximately 0.1, which is much larger than the 0.0005 ratio in regular hydrogen. We aim to identify all contributions larger than 1ppm. Most of them are subsequently calculated, while a few remaining ones are only estimated and left for future investigation. Our work is probably the first attempt of a comprehensive calculation of μH HFS.

II. LEADING ORDER HFS

Let us introduce the notation before proceeding to the calculations. The spin averaged expectation value of an operator Q will be denoted by $\langle Q\rangle$. The expectation value involving the nuclear spin $\vec I$ depends on the total angular momentum F, so we denote it by $\langle Q\rangle_F$. Finally, $\langle Q\rangle_{\rm hfs}$ denotes the difference

$$\langle Q\rangle_{\rm hfs} = \langle Q\rangle_{J+1/2} - \langle Q\rangle_{J-1/2}\,. \tag{1}$$

If Q does not involve the nuclear spin, then $\langle Q\rangle_{\rm hfs} = 0$. Therefore, we will consistently drop the subscript "hfs" in $\langle Q\rangle_{\rm hfs}$ for operators Q that involve the nuclear spin.

The nuclear magnetic moment

$$\vec\mu_I = \frac{q}{2\,M}\,g\,\vec I\,, \tag{2}$$

where $q = -Z\,e$ and e is the electron charge, can be expressed in terms of the nuclear g-factor and the magnetic moment anomaly κ with $g = 2\,(1+\kappa)$. In a nonrelativistic framework, the interaction between the point-like nuclear magnetic moment and that of the muon leads to a hyperfine splitting of the atomic energy levels given by the expectation value of the magnetic interaction

$$\begin{aligned} V_{\rm hfs} =& -\frac{2}{3}\,\vec\mu_I\cdot\vec\mu_\mu\,\delta^3(r) \\ =& \frac{8}{3}\,\frac{Z\,\alpha}{m_\mu\,M}\,(1+\kappa)\,(1+a_\mu)\,\vec I\cdot\vec s\,\pi\,\delta^3(r)\,, \end{aligned} \tag{3}$$

evaluated with the nonrelativistic wave function ϕ

$$\begin{aligned} E_{\rm hfs} =& \langle\phi|V_{\rm hfs}|\phi\rangle \\ =& \frac{8}{3}\,\frac{(Z\,\alpha)^4}{n^3}\,\frac{\mu^3}{m_\mu\,M}\,(1+\kappa)\,(1+a_\mu)\,\langle\vec I\cdot\vec s\rangle \\ \equiv& E_F\,(1+a_\mu)\,, \end{aligned} \tag{4}$$

where μ is the reduced mass

$$\frac{1}{\mu} = \frac{1}{m_\mu} + \frac{1}{M}\,. \tag{5}$$

Below we investigate all corrections to HFS and express them in terms of δ defined by $E_{\rm hfs} = E_F(1+\delta)$.

III. EVP IN μH FOR A POINT NUCLEUS

For a point-like proton, the largest QED correction arises from electron vacuum polarization (EVP). This is because the spatial size of EVP is on the order of the (muonic) Bohr radius. In general, vacuum polarization modifies the photon propagator as follows [10]

$$\frac{1}{k^2} \to \frac{1}{k^2\,\big(1+\bar\omega(k^2/m_e^2)\big)} \approx -\frac{\bar\omega(k^2/m_e^2)}{k^2}\,, \tag{6}$$

where $k^2 = \omega^2 - \vec k^{\,2}$. The Coulomb interaction is modified accordingly [11]

$$\left[\frac{1}{r}\right]_{\rm vp} = \int\frac{d^3k}{(2\,\pi)^3}\,\frac{4\,\pi}{\vec k^{\,2}}\,[-\bar\omega(-\vec k^{\,2}/m_e^2)]\,e^{i\,\vec k\cdot\vec r}\,. \tag{7}$$

Using the integral form

$$\bar\omega(k^2) = \frac{\alpha}{\pi}\,k^2\int_4^\infty d(q^2)\,\frac{1}{q^2\,(q^2-k^2)}\,u(q^2)\,, \tag{8}$$

one obtains

$$\left[\frac{1}{r}\right]_{\rm vp} = \frac{\alpha}{\pi}\int_4^\infty d(q^2)\,\frac{u(q^2)}{q^2}\int\frac{d^3k}{(2\,\pi)^3}\,\frac{4\,\pi\,e^{i\,\vec{k}\cdot\vec{r}}}{(k^2+m_e^2\,q^2)} = \frac{\alpha}{\pi}\int_4^\infty d(q^2)\,\frac{u(q^2)}{q^2}\,\frac{e^{-m_e q\,r}}{r}\,, \tag{9}$$

which is a convenient representation of the EVP potential. Thus, the calculation of the EVP corrections to the HFS is performed in two steps. In the first step, matrix elements with the Coulomb potential $1/r$ replaced by a Yukawa potential $\exp(-\rho\,r)/r$ are obtained analytically and denoted by $E(\rho)$. In the second step, one numerically evaluates the integral

$$E = \frac{\alpha}{\pi}\int_4^\infty d(q^2)\,\frac{u(q^2)}{q^2}\,E(m_e\,q)\,, \tag{10}$$

where

$$u(q^2) = \frac{1}{3}\sqrt{1-\frac{4}{q^2}}\,\left(1+\frac{2}{q^2}\right)\,. \tag{11}$$

A. $\delta^{(1)}_{\rm evp}$

Let us begin the calculation with the one-loop EVP. For a massive photon, the spin-spin interaction takes the form

$$V_{\rm hfs}(\rho) = \frac{8}{3}\,\frac{Z\,\alpha}{m_\mu\,m_p}\,(1+\kappa)\,(1+a_\mu)\,\vec{I}\cdot\vec{s} \times\left(\pi\,\delta^3(r) - \frac{1}{4}\,\frac{e^{-\rho\,r}}{r^3}\,(\rho\,r)^2\right)\,. \tag{12}$$

The correction to the HFS can be expressed as

$$\delta E_{\rm hfs}(\rho) = \langle V_{\rm hfs}(\rho)\rangle + 2\left\langle V_{\rm hfs}\,\frac{1}{(E-H)'}\,(-Z\,\alpha)\,\frac{e^{-\rho\,r}}{r}\right\rangle = E_F\,(1+a_\mu)\left[\frac{8}{\kappa+2}+\frac{8+8\ln(1+\kappa/2)}{(\kappa+2)^2}-\frac{16}{(\kappa+2)^3}\right]\,, \tag{13}$$

where $\kappa = \rho/(\mu\,Z\alpha)$. The result of numerical integration

$$\delta E_{\rm hfs} = \frac{\alpha}{\pi}\int_4^\infty\frac{d(\rho^2)}{\rho^2}\,u\left(\frac{\rho^2}{m_e^2}\right)\,\delta E_{\rm hfs}(\rho) = E_F\,(1+a_\mu)\,\delta^{(1)}_{\rm evp}\,, \tag{14}$$

for the 1S state is

$$\delta^{(1)}_{\rm evp} = 0.006\,075\,29\,. \tag{15}$$

B. $\delta^{(2)}_{\rm evp}$

$\delta^{(2)}_{\rm evp}$ represents the corresponding two-loop EVP corrections, which we obtain using PbarSpectr code [12]. Namely, we numerically solve the Schrödinger equation for a point-like nucleus with one- and two-loop EVP

$$\delta^{(1)}_{\rm evp,npert} = 0.006\,089\,78\,, \tag{16}$$

$$\delta^{(2)}_{\rm evp,npert} = 0.000\,046\,76\,, \tag{17}$$

and extract $\delta^{(2)}_{\rm evp}$ by subtracting $\delta^{(1)}_{\rm evp}$

$$\delta^{(2)}_{\rm evp} = \delta^{(1)}_{\rm evp,npert} + \delta^{(2)}_{\rm evp,npert} - \delta^{(1)}_{\rm evp} = 0.000\,061\,25\,. \tag{18}$$

This partially includes three-loop corrections, but they are expected to be negligibly small.

C. $\delta^{(3)}_{\rm rel,evp}$

This is the one-loop EVP calculated using the Dirac wave function, or more precisely, the relativistic correction to $\delta^{(1)}_{\rm evp}$. The relativistic form of the hyperfine interaction

$$V_{\rm hfs}(\vec{r}) = \frac{e}{4\,\pi}\,\vec{\mu}_I\cdot\vec{\alpha}\times\frac{\vec{r}}{r^3} \tag{19}$$

for a massive photon is

$$V_{\rm vp}(\vec{r},\rho) = \frac{e}{4\pi}\vec{\mu}\cdot\left(\vec{\alpha}\times\frac{\vec{r}}{r^3}\right)e^{-\rho r}(1+\rho r)\,. \tag{20}$$

$\delta E_{\rm hfs}(\rho)$ takes an expectation value form similar to the non-relativistic case

$$\delta E_{\rm hfs}(\rho) = \langle V_{\rm hfs}(\rho)\rangle + 2\left\langle V_{\rm hfs}\,\frac{1}{(E-H)'}\,(-Z\,\alpha)\,\frac{e^{-\rho\,r}}{r}\right\rangle = \delta E_{\rm hfs1}(\rho) + \delta E_{\rm hfs2}(\rho) \tag{21}$$

with the relativistic wave function ψ, the energy E, and the Hamiltonian H. The unperturbed ground state wave function ψ is described by the spherical Dirac spinor,

$$\psi(\vec{r}) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} g(r)\chi_m \\ -if(r)(\vec{\sigma}\cdot\hat{r})\chi_m\end{pmatrix}\,, \tag{22}$$

$$g(r) = Nr^{\gamma-1}e^{-Z\alpha r}\,,\quad f(r) = -\sqrt{\frac{1-\gamma}{1+\gamma}}\,g(r)\,, \tag{23}$$

where we adopt atomic units ($m=1$) and $\gamma = \sqrt{1-(Z\alpha)^2}$. The first part $\delta E_{\rm hfs1}(\rho)$ is

$$\delta E_{\rm hfs1}(\rho) = \langle\psi|V_{\rm hfs}(\rho)|\psi\rangle = E_{F\infty}\,\delta^{(1+)}_{\rm evp1}\,, \tag{24}$$

where

$$E_{F\infty} = \frac{8}{3}(Z\alpha)^4\frac{m^2}{M}\frac{g}{2}\langle\vec{I}\cdot\vec{s}\rangle \tag{25}$$

is the nonrecoil limit of the Fermi splitting. Using Eqs. (22,23), one obtains

$$\delta^{(1+)}_{\rm evp1}(\rho) = \frac{1}{2(Z\alpha)^3}\int_0^\infty dr\,(-2)g(r)f(r)\,e^{-\rho r}(1+\rho r)$$

$$= \frac{1}{\gamma(2\gamma-1)}\left(\frac{2}{\tau+2}\right)^{2\gamma}(1+\gamma\,\tau)\,, \tag{26}$$

where $\tau = \rho/(m_\mu\, Z\alpha)$. Considering the second part, $\delta E_{\rm hfs2}(\rho)$, we note that within the subspace of the Dirac quantum number $\kappa = -1$, $V_{\rm hfs}$ can be simplified to

$$\langle V_{\rm hfs}\rangle = \frac{2}{3}\frac{e}{4\,\pi}\,\langle\vec{\mu}_I\cdot\vec{\sigma}\rangle_{\rm hfs}\left\langle i\,\frac{\vec{\gamma}\cdot\vec{r}}{r^3}\right\rangle. \tag{27}$$

Therefore, $\delta E_{\rm hfs2}(\rho)$ can be written as

$$\delta E_{\rm hfs2}(\rho) = 2\left\langle\psi\right| -\frac{Z\alpha}{r}e^{-\rho r}\left|\delta\psi\right\rangle\frac{2}{3}\frac{e}{4\,\pi}\,\langle\vec{\mu}_I\cdot\vec{\sigma}\rangle_{\rm hfs} = E_{F\infty}\,\delta^{(1+)}_{\rm evp2}\,, \tag{28}$$

where

$$\delta\psi(\vec{r}) = \frac{1}{(E-H)'}\,i\,\frac{\vec{\gamma}\cdot\vec{r}}{r^3}\,\psi(\vec{r}) = \begin{pmatrix} W(r)\chi_m \\ -iZ(r)(\vec{\sigma}\cdot\hat{r})\chi_m\end{pmatrix}. \tag{29}$$

Using the Shabaev method [13, 14], one obtains

$$W(r) = C\left[\frac{2(Z\alpha)^3}{\gamma}g - 3(1+\gamma)f - \frac{1+2\gamma}{r}g - 2Z\alpha(2\gamma+1)\left(\frac{\Psi(2\gamma+1)}{\gamma} + \gamma + 1 - Z\alpha r - \frac{\ln(2Z\alpha r)}{\gamma} - \frac{1}{2\gamma}\right)g\right], \tag{30}$$

$$Z(r) = C\left[\frac{2(Z\alpha)^3}{\gamma}f - 3(1-\gamma)g - \frac{3(1+2\gamma)}{r}f - 2Z\alpha(2\gamma+1)\left(\frac{\Psi(2\gamma+1)}{\gamma} + \gamma + 1 - Z\alpha r - \frac{\ln(2Z\alpha r)}{\gamma} + \frac{1}{2\gamma}\right)f\right], \tag{31}$$

where $C = 1/[4(Z\alpha)^2 - 3]$. The corresponding dimensionless contribution is

$$\begin{aligned}\delta^{(1+)}_{\rm evp2} &= \frac{1}{(Z\alpha)^2}\int_0^\infty dr\,\big[W(r)g(r) + Z(r)f(r)\big]re^{-\rho r}\\ &= \frac{\beta^{-2\gamma}}{\gamma(2\gamma-1)}\left[\frac{2(2-\gamma)}{2\gamma-1}\beta + \frac{2\gamma^3+2\gamma^2-2\gamma+1}{\gamma^2} + \frac{2}{\gamma}\ln\beta - \frac{2\gamma}{\beta}\right],\end{aligned} \tag{32}$$

where $\beta = 1 + \tau/2$. The sum of both parts $\delta^{(1+)}_{\rm evp1} + \delta^{(1+)}_{\rm evp2} = \delta^{(1)}_{\rm evp} + \delta^{(3)}_{\rm rel,evp} + \ldots$ after expansion in $Z\,\alpha$ is

$$\begin{aligned}\delta^{(3)}_{\rm rel,evp} =& \frac{\alpha}{\pi}\,(Z\alpha)^2\int_4^\infty\frac{d(\rho^2)}{\rho^2}\,u\left(\frac{\rho^2}{m_e^2}\right)\left[\frac{8+4\ln\beta}{\beta} + \frac{3+6\ln\beta+2\ln^2\beta}{\beta^2} - \frac{2+2\ln\beta}{\beta^3}\right]\\ =&\ 1.15\ \text{ppm}\,.\end{aligned} \tag{33}$$

In Sec. V, we will calculate the EVP combined with nuclear recoil or finite nuclear size effects, and in Sec. VII we will consider it combined with μVP and μSE, but before this we must briefly describe an approach for nuclear recoil corrections.

IV. TWO-PHOTON EXCHANGE FORWARD SCATTERING AMPLITUDE

To calculate corrections beyond the point-like, static nucleus approximation, we first consider the two-photon exchange correction to the HFS. We closely follow our previous work in Ref. [15] and use the temporal gauge

$$E^{(5)}_{\rm hfs} = \frac{i}{2}\,\phi^2(0)\int\frac{d^4k}{(2\,\pi)^4}\,\frac{1}{k^4}\left(\delta^{ik} - \frac{k^i\,k^k}{\omega^2}\right)\left(\delta^{jl} - \frac{k^j\,k^l}{\omega^2}\right)t^{ji}\,T^{kl}\,. \tag{34}$$

For a point-like spin-$1/2$ particle

$$t^{ji} = e^2\left[\langle\bar{u}|\gamma^j\,\frac{1}{\not{t} - \not{k} - m}\,\gamma^i|u\rangle + \langle\bar{u}|\gamma^i\,\frac{1}{\not{t} + \not{k} - m}\,\gamma^j|u\rangle\right], \tag{35}$$

and for a finite size spin-$1/2$ particle

$$T^{kl} = (Z\,e)^2\left[\langle\bar{u}|\Gamma^k(k)\,\frac{1}{\not{t} - \not{k} - m}\,\Gamma^l(-k)|u\rangle + \langle\bar{u}|\Gamma^l(-k)\,\frac{1}{\not{t} + \not{k} - m}\,\Gamma^k(k)|u\rangle\right], \tag{36}$$

where t is the four-momentum at rest, $t = (m, \vec{0})$, and

$$\Gamma^\mu(k) = \gamma^\mu\,F_1 + \frac{i}{2\,M}\,\sigma^{\mu\nu}\,k_\nu\,F_2\,. \tag{37}$$

Using the decomposition in terms of scalar functions t_i and T_i with $i = 1, 2$

$$t^{ji} = i\,\epsilon^{ijk}\,e^2\,\omega\left(t_1\,s^k + t_2\,\frac{k^k}{\vec{k}^2}\,\vec{k}\cdot\vec{s}\right), \tag{38}$$

$$T^{ji} = i\,\epsilon^{ijk}(Ze)^2\omega\left(T_1 I^k + T_2\frac{k^k}{\vec{k}^{\,2}}\vec{k}\cdot\vec{I}\right)\,, \tag{39}$$

one obtains for the lepton

$$t_1 = \frac{4\,k^2}{(k^2-2\,m\,\omega)\,(k^2+2\,m\,\omega)}\,, \tag{40}$$

$$t_2 = 0\,, \tag{41}$$

and for the nucleus

$$T_1(-k^2,\omega) = \frac{4F_1\,M^2\,[F_1\,k^2 + F_2\,(k^2+\omega^2)] - F_2^2\,k^4}{(k^2-2M\omega)(k^2+2M\omega)M^2}\,, \tag{42}$$

$$T_2(-k^2,\omega) = \frac{4\,(k^2-\omega^2)\,F_2(F_1+F_2)}{(k^2-2M\omega)(k^2+2M\omega)}\,. \tag{43}$$

The two-photon exchange correction takes the form

$$E^{(5)}_{\rm hfs} = i\,\phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{\vec{I}\cdot\vec{s}}{3}\int\frac{d^4k}{(2\,\pi)^4}\times\left[\frac{2}{k^2}\,t_1\,T_1 + \frac{\omega^2}{k^4}\,(t_1+t_2)\,(T_1+T_2)\right]. \tag{44}$$

Since the form factors are functions of $-k^2$, one performs a Wick rotation $\omega = i\,k_0$, and $k^2 \to -k^2$, and averages over the three-dimensional sphere in Euclidean space

$$A[f] \equiv \int\frac{d\,\Omega_k}{2\,\pi^2}\,f(k,k_0) = \frac{2}{\pi}\int_0^\pi d\phi\,(\sin\phi)^2\,f\big(k,k\,\cos\phi\big)\,. \tag{45}$$

For instance,

$$A\left[\frac{1}{k^4+4\,M^2\,k_0^2}\right] = \frac{2}{k^4}\,\frac{1}{1+\sqrt{1+4\,M^2/k^2}}\,. \tag{46}$$

We thus obtain

$$E^{(5)}_{\rm hfs} = \frac{16}{3}\,\phi^2(0)\,(Z\,\alpha)^2\,\frac{\vec{I}\cdot\vec{s}}{3}\int\frac{d^4k}{(2\,\pi)^4}\times A\left[\frac{2}{k^2}\,t_1\,T_1 + \frac{k_0^2}{k^4}\,(t_1+t_2)\,(T_1+T_2)\right]. \tag{47}$$

The low k asymptotic behavior of the integrand

$$A[t_1\,T_1] \approx \frac{16\,(1+\kappa)}{M+m}\,\frac{1}{k^3} \tag{48}$$

should be subtracted out [16, 17], yielding

$$E^{(5)}_{\rm hfs} = -\frac{16}{3}\,(Z\,\alpha)^2\frac{\phi^2(0)}{M\,m}\,\vec{I}\cdot\vec{s}\int\frac{dk}{k}\left[-4\,\frac{m}{k}\,G_E(k^2)G_M(k^2) + 4\,(1+\kappa)\,\frac{m}{k} + T_{\rm rec}(k) - 4\,(1+\kappa)\,\frac{m}{k}\,\frac{m}{M+m}\right], \tag{49}$$

where the recoil part $T_{\rm rec}$ is

$$\begin{aligned} T_{\rm rec}(k) = {}& \frac{4\,m}{k}G_E(k^2)G_M(k^2) + \frac{m}{M}\left[\frac{k^2}{8\,m^2} - 2\left(1+\sqrt{1+\frac{4m^2}{k^2}}\right)^{-1}\left(\frac{k^2}{8\,m^2}-1\right)\right]F_2^2(k^2) \\ & + \frac{M\,m}{M^2-m^2}\left(1+\sqrt{1+\frac{4\,M^2}{k^2}}\right)^{-1}\left[\left(1-\frac{8M^2}{k^2}\right)F_1(k^2) + 3F_2(k^2)\right]G_M(k^2) \\ & - \frac{M\,m}{M^2-m^2}\left(1+\sqrt{1+\frac{4\,m^2}{k^2}}\right)^{-1}\left[\left(1-\frac{8m^2}{k^2}\right)F_1(k^2) + 3F_2(k^2)\right]G_M(k^2)\,. \end{aligned} \tag{50}$$

The Sachs electric G_E and magnetic G_M form factors are related to F_1 and F_2 by

$$G_E(k^2) = F_1(k^2) - \frac{k^2}{4\,M^2}\,F_2(k^2)\,, \tag{51}$$

$$G_M(k^2) = F_1(k^2) + F_2(k^2)\,, \tag{52}$$

with the normalization $G_M(0) = 1+\kappa = g/2$. For a point-like nucleus $F_1 = 1$ and $F_2 = 0$, we obtain

$$E^{(5)}_{\rm point,hfs} = -8\,(Z\,\alpha)^2\frac{\phi^2(0)}{M^2-m^2}\,(\vec{I}\cdot\vec{s})\,\ln\frac{M}{m}, \tag{53}$$

in agreement with a well-known result [18] for the recoil correction to the hyperfine splitting. For a finite size nucleus in the nonrecoil limit, we obtain the Zemach correction [19]

$$\delta^{(1)}_{\rm fns} = \frac{2\,Z\,\alpha\,m}{\pi^2}\int\frac{d^3k}{k^4}\left[\frac{G_E(k^2)\,G_M(k^2)}{1+\kappa} - 1\right] = -\,2\,Z\,\alpha\,m\,r_Z\,. \tag{54}$$

It is convenient to express this finite nuclear size correction in terms of the Zemach radius r_Z [19], defined as

$$r_\mathrm{Z} = \int d^3r_1 \int d^3r_2\, \rho_E(r_1)\, \rho_M(r_2)\, |\vec{r}_1 - \vec{r}_2|, \tag{55}$$

where ρ_E and ρ_M are the Fourier transforms of G_E and $G_M/(1+\kappa)$, respectively. Using the dipole parametrization for the nuclear form factors $\rho_E = \rho_M = \rho$ with

$$\rho(k^2) = \frac{\Lambda^4}{(\Lambda^2 + k^2)^2}\,, \tag{56}$$

one finds $r_Z = 35/(8\,\Lambda)$. Using the dipole parametrization with Λ adjusted to the Zemach radius $r_Z = 1.054\,\mathrm{fm}$, we obtain a recoil correction of $\delta^{(1)}_\mathrm{rec} = 0.001\,667$. This is not very different from the more accurate value of $\delta^{(1)}_\mathrm{rec} = 0.001\,672(3)$ obtained using more realistic proton form factors [20]. Therefore, our subsequent calculations of VP and SE corrections to the two-photon exchange amplitude will employ the dipole approximation for the proton form factors.

V. VP COMBINED WITH RECOIL AND FNS

The radiative recoil $Z\,\alpha^2\, m/M\, E_F$ correction has previously been studied only for muonium HFS [18], but not for a finite size nucleus with an arbitrary g-factor. Here we derive formulas without expansion in m/M and Λ/M, expressing the result in terms of a one-dimensional integral. The electron vacuum polarization modifies the photon propagator according to Eq. (6). The explicit formula for $\bar{\omega}$ is given by [10]

$$\bar{\omega}(-k^2) = -\frac{\alpha}{3\,\pi}\left\{\frac{1}{3} + 2\left(1 - \frac{2}{k^2}\right) \times \left[\sqrt{1+\frac{4}{k^2}}\,\mathrm{arccoth}\sqrt{1+\frac{4}{k^2}} - 1\right]\right\}. \tag{57}$$

The VP correction due to a lepton of mass m' ($m' \neq m_e$ for μH) can be easily obtained from the two-photon exchange amplitude (without subtracting the small k asymptotic behavior)

$$\begin{aligned} E^{(6)}_\mathrm{vp} &= \frac{16}{3}\,(Z\,\alpha)^2 \frac{\phi^2(0)}{M\,m}\,\vec{I}\cdot\vec{s}\int\frac{dk}{k} \\ &\quad\times\left[4\,\frac{m}{k}\,G_E(k^2)G_M(k^2) - T_\mathrm{rec}(k)\right](-2)\,\bar{\omega}\Big(-\frac{k^2}{m'^2}\Big) \\ &= E^{(6)}_\mathrm{vp,point} + E^{(6)}_\mathrm{vp,fns} + E^{(6)}_\mathrm{vp,rec}\,. \end{aligned} \tag{58}$$

$E^{(6)}_\mathrm{vp,point}$ arises from the low k asymptotic behavior of the integrand

$$\begin{aligned} E^{(6)}_\mathrm{vp,point} &= \frac{16}{3}\,(Z\,\alpha)^2 \frac{\phi^2(0)}{M\,m}\,\vec{I}\cdot\vec{s}\int\frac{dk}{k^2} \\ &\quad\times 4\,(1+\kappa)\,\mu\,(-2)\,\bar{\omega}\Big(-\frac{k^2}{m'^2}\Big) \\ &= E_F\,\frac{3}{4}\,Z\,\alpha^2\,\frac{\mu}{m'}, \end{aligned} \tag{59}$$

where μ is the reduced mass. $E^{(6)}_\mathrm{fns,vp}$ is the finite size correction in the nonrecoil limit,

$$\begin{aligned} E^{(6)}_\mathrm{vp,fns} &= E_F\,\frac{2\,Z\,\alpha\,m}{\pi^2}\int\frac{d^3k}{k^4}\left[\frac{G_E(k^2)\,G_M(k^2)}{1+\kappa} - 1\right] \\ &\quad\times(-2)\,\bar{\omega}\Big(-\frac{k^2}{m'^2}\Big) \\ &\approx -E_F\,2\,Z\,\alpha\,m\,r_Z\,\frac{\alpha}{\pi}\left(\frac{2}{3}\,\ln\frac{\Lambda^2}{m'^2} - \frac{634}{315}\right), \end{aligned} \tag{60}$$

where the dipole parametrization of nuclear form factors is assumed, in agreement with Ref. [21]. Finally, $E^{(6)}_\mathrm{rec,vp}$ is the recoil VP correction

$$\begin{aligned} E^{(6)}_\mathrm{vp,rec} &= -\frac{16}{3}\,(Z\,\alpha)^2 \frac{\phi^2(0)}{mM}\,\vec{I}\cdot\vec{s}\int\frac{dk}{k} \\ &\quad\times\left[T_\mathrm{rec}(k) - 4\,(1+\kappa)\,\frac{m}{k}\,\frac{m}{M+m}\right](-2)\,\bar{\omega}\Big(-\frac{k^2}{m'^2}\Big). \end{aligned} \tag{61}$$

Using Eq. (58), we obtain for μVP in H

$$\delta_{\mu\mathrm{vp,point}}(\mathrm{H}) = 0.193 \text{ ppm}, \tag{62}$$

$$\delta_{\mu\mathrm{vp,fns}}(\mathrm{H}) = -\,0.121 \text{ ppm}, \tag{63}$$

$$\delta_{\mu\mathrm{vp,rec}}(\mathrm{H}) = -\,0.001 \text{ ppm}. \tag{64}$$

In the case of EVP in μH, $E^{(6)}_\mathrm{vp,point}$ is treated separately, because the scattering approximation is not valid here. The calculations for a point-like nucleus have already been described in Sec. III. Here we consider only $E^{(6)}_\mathrm{vp,fns}$ and $E^{(6)}_\mathrm{vp,rec}$. Using Eqs. (59) and (61), respectively, we obtain

$$\delta_\mathrm{evp,fns}(\mu\mathrm{H}) = -\,149.81 \text{ ppm}\,, \tag{65}$$

$$\delta_\mathrm{evp,rec}(\mu\mathrm{H}) = 25.24 \text{ ppm}\,. \tag{66}$$

The final cases to consider are EVP H and μVP μH, where the VP particle is the same as the one in the atom. The point-like VP contribution in the nonrecoil limit is already included in δ_QED, so we rearrange the remaining corrections as follows. FNS VP in the nonrecoil limit is given by Eq. (61) with $m' = m$, while the recoil correction is redefined and includes the recoil part from $E^{(6)}_\mathrm{vp,point}$, namely

$$E^{(6)}_\mathrm{evp,rec} = -\frac{16}{3}\,\alpha^2\,\frac{\phi^2(0)}{m\,m_p}\,\vec{I}\cdot\vec{s}\int\frac{dk}{k}\,T_\mathrm{rec}(k)\,(-2)\,\bar{\omega}\Big(-\frac{k^2}{m^2}\Big). \tag{67}$$

Our numerical results for H are

$$\delta_\mathrm{evp,fns}(\mathrm{H}) = -\,0.725 \text{ ppm}, \tag{68}$$

$$\delta_\mathrm{evp,rec}(\mathrm{H}) = -\,0.032 \text{ ppm}, \tag{69}$$

and for μH

$$\delta_{\mu\mathrm{vp,fns}}(\mu\mathrm{H}) = -\,25.10 \text{ ppm}, \tag{70}$$

$$\delta_{\mu\mathrm{vp,rec}}(\mu\mathrm{H}) = -\,1.18 \text{ ppm}. \tag{71}$$

VI. SE COMBINED WITH RECOIL AND FNS

The lepton self-energy (SE) correction is obtained by replacing t^{ji} with the self-energy corrected tensor $t^{ji}_{\rm se}$ in Eqs. (34), (38), and (44)

$$
\begin{aligned}
E^{(6)}_{\rm se} &= \frac{i}{2}\,\phi^2(0)\int\frac{d^4k}{(2\,\pi)^4}\,\frac{1}{k^4}\left(\delta^{ik}-\frac{k^i\,k^k}{\omega^2}\right)\left(\delta^{jl}-\frac{k^j\,k^l}{\omega^2}\right)t^{ji}_{\rm se}\,T^{kl}\,,\\
&= i\,\phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{\vec I\cdot\vec s}{3}\int\frac{d^4k}{(2\,\pi)^4}\left[\frac{2}{k^2}\,t_{\rm 1se}\,T_1+\frac{\omega^2}{k^4}\,(t_{\rm 1se}+t_{\rm 2se})\,(T_1+T_2)\right],
\end{aligned}
\tag{72}
$$

where the arguments of the lepton and proton factors are $(-k^2\,,\omega)$. The self-energy corrected lepton line is (for its calculation see Appendix A)

$$
\begin{aligned}
t_{\rm 1se}(-k^2,\omega) = \frac{\alpha}{2\,\pi}\Biggl\{&-\frac{4}{k^2+2\,\omega}-J\left[\frac{1}{2}+\omega+\frac{8\,(1+\omega)}{k^2+2\,\omega}+\frac{\omega\,(1+2\,\omega)\,(2+\omega)}{2\,(k^2-\omega^2)}\right]\\
&-\left[1+\frac{8}{k^2+2\,\omega}+\frac{1+\omega}{2\,(1+k^2+2\,\omega)\,(k^2-\omega^2)}+\frac{-1+3\,\omega+2\,\omega^2}{2\,(k^2-\omega^2)}\right]\ln(-k^2-2\,\omega)\\
&+\left[2+\frac{18}{k^2+2\,\omega}+\frac{16}{(k^2-4)\,(k^2+2\,\omega)}+\frac{2\,\omega^2}{k^2-\omega^2}\right]\arcsin\left(\frac{k}{2}\right)\sqrt{\frac{4}{k^2}-1}+(\omega\to-\omega)\Biggr\},
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
t_{\rm 2se}(-k^2,\omega) = \frac{\alpha}{2\,\pi}\Biggl\{&-\frac{1+\omega}{\omega\,(1+k^2+2\,\omega)}+J\left[\frac{k^2}{\omega}+\frac{8+15\,\omega+6\,\omega^2}{2\,\omega}+\frac{3\,\omega\,(2+5\,\omega+2\,\omega^2)}{2\,(k^2-\omega^2)}\right]\\
&+\left[\frac{4+3\,\omega}{\omega}-\frac{1+\omega}{\omega\,(1+k^2+2\,\omega)^2}-\frac{(1+\omega)\,(2+\omega+2\,\omega^2)}{2\,\omega\,(1+k^2+2\,\omega)\,(k^2-\omega^2)}+\frac{2-\omega+9\,\omega^2+6\,\omega^3}{2\,\omega\,(k^2-\omega^2)}\right]\ln(-k^2-2\,\omega)\\
&+\left[-6+\frac{2}{k^2+2\,\omega}-\frac{6\,\omega^2}{k^2-\omega^2}\right]\arcsin\left(\frac{k}{2}\right)\sqrt{\frac{4}{k^2}-1}+(\omega\to-\omega)\Biggr\},
\end{aligned}
\tag{74}
$$

and J is a master integral defined in Appendix A. The nuclear factors T_1 and T_2 are defined in Eqs. (42,43). After performing a Wick rotation, we average over the three-dimensional sphere. The low k asymptotic behavior of the integrand

$$
A[t_{\rm 1se}\,T_1]\approx\frac{16\,a_e\,(1+\kappa)}{M+m}\,\frac{1}{k^3}
\tag{75}
$$

is subtracted out, because it corresponds to the already included lower order term, thus

$$
\begin{aligned}
E^{(6)}_{\rm se} &= \phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{\vec I\cdot\vec s}{3}\int\frac{d^4k}{(2\,\pi)^4}\,A\left[\frac{2}{k^2}\,t_{\rm 1se}\,T_1+\frac{k_0^2}{k^4}\,(t_{\rm 1se}+t_{\rm 2se})\,(T_1+T_2)-\frac{32\,a_e\,(1+\kappa)}{M+m}\,\frac{1}{k^5}\right]\\
&= E^{(6)}_{\rm se,point}+E^{(6)}_{\rm se,fns}+E^{(6)}_{\rm se,rec}.
\end{aligned}
\tag{76}
$$

The nonrecoil contribution for a point-like nucleus is

$$
E^{(6)}_{\rm se,point} = -\,\phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{8\,(1+\kappa)}{3\,M}\,\vec I\cdot\vec s\int\frac{d^4k}{(2\,\pi)^4}\,\frac{1}{k^3}\left[t_{\rm 1se}(k^2,0)+\frac{4\,a_e}{k^2}\right],
\tag{77}
$$

where

$$
t_{\rm 1se}(k^2,0)=\frac{\alpha}{2\,\pi}\left\{\frac{8}{k^2}+\left(\frac{16}{k^2}-1\right)J(k^2,0)+4\left(1-\frac{5}{k^2}-\frac{28}{k^4}\right)\frac{\text{arcsinh}\bigl(\frac{k}{2}\bigr)}{\sqrt{1+\frac{4}{k^2}}}-\left(2-\frac{16}{k^2}+\frac{1}{k^2-1}\right)\ln(k^2)\right\}.
\tag{78}
$$

Numerical integration yields $\delta_{\rm se,point}=-136.16\,$ppm, in agreement with the known [18] analytic result $\delta_{\rm se,point}=Z\alpha^2\,(\ln 2-\frac{13}{4})$, which is included in $\delta^{(2)}$ in Eq. (91).

The nonrecoil finite nuclear size contribution, using $t_{1\rm se} = 5/k^2\,\alpha/\pi + o(k^{-4})$ is

$$\begin{aligned} E^{(6)}_{\rm se,fns} =&\ -\phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{2\,\vec I\cdot\vec s}{3\,M}\int\frac{d^4k}{(2\,\pi)^4} \\ &\times\frac{4\,\big(G_E\,G_M-(1+\kappa)\big)}{k^3}\,t_{1\rm se}(k^2,0) \qquad (79)\\ =&\ -2\,Z\,\alpha\,m\,r_Z\,E_F\,\frac{\alpha}{\pi}\Big[-\frac{5}{4}+O\Big(\frac{m^2}{\Lambda^2}\Big)\Big]\,, \qquad (80)\end{aligned}$$

in agreement with Ref. [21].

Using Eq. (79), we obtain for muonic hydrogen

$$\delta_{\rm se,fns}(\mu{\rm H}) = 16.18\ {\rm ppm}, \qquad (81)$$

which significantly differs from the result obtained by omitting the $O(m^2/\Lambda^2)$ terms $\delta_{\rm se,fns} = 23.92$ ppm. For hydrogen, we obtain

$$\delta_{\rm se,fns}({\rm H}) = 0.115\ {\rm ppm}, \qquad (82)$$

which differs very slightly from the result obtained by omitting the $O(m^2/\Lambda^2)$ terms $\delta_{\rm se,fns} = 0.116$ ppm.

The nuclear recoil contribution is

$$\begin{aligned} E^{(6)}_{\rm se,rec} =&\ \phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{\vec I\cdot\vec s}{3}\int\frac{d^4k}{(2\,\pi)^4}\,\frac{1}{k^4} \\ &\times\bigg[2\,k^2\,t_{1\rm se}\,T_1 + k_0^2\,(t_{1\rm se}+t_{2\rm se})\,(T_1+T_2) \\ &+\frac{8\,k}{M}\,t_{1\rm se}(k^2,0)\,G_E(k^2)\,G_M(k^2)+\frac{32\,a_e\,(1+\kappa)\,m}{k\,M\,(M+m)}\bigg]. \qquad (83)\end{aligned}$$

This integral is evaluated numerically, yielding the following results

$$\delta_{\rm se,rec}(\mu{\rm H}) = 16.48\ {\rm ppm}\,, \qquad (84)$$

$$\delta_{\rm se,rec}({\rm H}) = 0.104\ {\rm ppm}\,, \qquad (85)$$

where Λ has been adjusted to match the Zemach radius r_Z.

VII. EVP COMBINED WITH μVP AND SE

The combined electronic and muonic vacuum polarizations contribution in the nonrecoil limit for a point-like nucleus is

$$E^{(7)}_{\mu{\rm vp,evp}} = E_F\,\frac{2\,Z\,\alpha\,m_\mu}{\pi^2}\,6\int\frac{d^3k}{k^4}\,\bar\omega\Big(-\frac{k^2}{m_e^2}\Big)\,\bar\omega\Big(-\frac{k^2}{m_\mu^2}\Big), \qquad (86)$$

from which we obtain

$$\delta^{(3)}_{\mu{\rm vp,evp}}\,(\mu{\rm H}) = 1.17\ {\rm ppm}. \qquad (87)$$

Including FNS, this correction decreases to 0.32 ppm, which indicates the significance of the FNS effect. Here we neglect FNS for consistency, as all α^3 corrections are calculated for a point nucleus, and estimate the unknown $\delta^{(3)}_{\rm fns} = \pm 2$ ppm, while $\delta^{(2)}_{\rm fns}$ is calculated separately for the VP, SE, and REL parts.

Another correction is the muon one-loop self-energy combined with EVP inserted into the exchanged photons between the lepton and the nucleus. In the nonrecoil limit, this correction is given by

$$\begin{aligned} E^{(7)}_{\rm se,evp1} =&\ -\phi^2(0)\,(4\,\pi\,Z\,\alpha)^2\,\frac{2\,(1+\kappa)\,\vec I\cdot\vec s}{3\,M} \\ &\times\int\frac{d^3k}{(2\,\pi)^3}\,\frac{1}{k^2}\,\Big(t_{1\rm se}(k^2,0)+\frac{4\,a_\mu}{k^2}\Big)\,(-2)\,\bar\omega\Big(-\frac{k^2}{m_e^2}\Big). \qquad (88)\end{aligned}$$

After numerical integration, we obtain

$$\delta^{(3)}_{\rm se,evp1}\,(\mu{\rm H}) = -1.70\ {\rm ppm}, \qquad (89)$$

and FNS would decrease this correction to -1.37 ppm.

VIII. SUMMARY OF HYPERFINE SPLITTING IN MUONIC HYDROGEN

The ground state hyperfine splitting of μH is conveniently represented as

$$E_{\rm hfs} = E_F\,(1+\delta)\,, \qquad (90)$$

where the dimensionless δ is the sum of various contributions listed in Table I. Let us now explain the meaning of all δ contributions. $\delta^{(1)}_{\rm evp}$ in Eq. (15) and $\delta^{(2)}_{\rm evp}$ in Eq. (18) are one- and two-loop EVP corrections to the contact Fermi (spin-spin) interaction. $\delta^{(2)}$ and $\delta^{(3)}$ are QED corrections, which are exactly the same as in the electronic case [18],

$$\delta^{(2)} = \frac{3}{2}\,(Z\,\alpha)^2+\alpha\,(Z\,\alpha)\Big(\ln(2)-\frac{5}{2}\Big)\,, \qquad (91)$$

$$\begin{aligned}\delta^{(3)} =&\ \frac{\alpha\,(Z\,\alpha)^2}{\pi}\Big[-\frac{8}{3}\ln(Z\,\alpha)\Big(\ln(Z\,\alpha)-\ln(4)+\frac{281}{480}\Big) \\ &+17.122\,338\,751\,3-\frac{8}{15}\,\ln(2)+\frac{34}{225}\Big] \\ &+\frac{\alpha^2\,(Z\,\alpha)}{\pi}\,0.770\,99(2)\,. \qquad (92)\end{aligned}$$

$\delta^{(3)}_{\rm rel,evp}$ in Eq. (33) is the relativistic correction to the one-loop EVP. $\delta^{(3)}_{\mu{\rm vp,evp}}$ in Eq. (87) is the combined μVP and EVP correction. $\delta^{(3)}_{\rm se,evp1}$ in Eq. (89) is the combined SE and EVP correction to the Coulomb interaction. $\delta^{(3)}_{\rm se,evp2}$ is the EVP correction on the SE photon. It was calculated by Krachkov and Lee in Ref. [24] with the result

$$\begin{aligned}\delta^{(3)}_{\rm se,evp2} =&\ \frac{\alpha^2\,(Z\,\alpha)}{\pi}\,\bigg[\Big(-\frac{13}{6}+\frac{2}{3}\,\ln 2\Big)\,\ln\frac{m_\mu}{m_e} \\ &+\frac{379}{72}-\frac{14}{9}\,\ln 2-\frac{\pi^2}{18}+\frac{1}{3}\,\ln^2 2\end{aligned}$$

TABLE I. Contributions to HFS in μH, constants from Ref. [22], $g_p = 5.585\,694\,6893(16)$, $\nu_F = 44\,114\,600.4(2.0)$ MHz, $E_F = 0.182\,443\,32\,8(8)$ eV, a_μ is the muon magnetic moment anomaly.

Term	Value	Reference
a_μ	$0.001\,165\,92$	Ref. [22]
$(1+a_\mu)\,\delta^{(1)}_{\rm evp}$	$0.006\,082\,37$	Eq. (15)
$(1+a_\mu)\,\delta^{(2)}_{\rm evp}$	$0.000\,061\,32$	Eq. (18)
$\delta^{(2)}$	$-0.000\,016\,34$	Eq. (91), Ref.[18]
$\delta^{(3)}$	$-0.000\,007\,10$	Eq. (92), Ref.[18, 23]
$\delta^{(3)}_{\rm rel,evp}$	$0.000\,001\,15$	Eq. (33)
$\delta^{(3)}_{\mu\rm vp,evp}$	$0.000\,001\,17$	Eq. (87)
$\delta^{(3)}_{\rm se,evp1}$	$-0.000\,001\,70$	Eq. (89)
$\delta^{(3)}_{\rm se,evp2}$	$-0.000\,000\,70$	Eq. (93), Ref.[24]
$\delta^{(1)}_{\rm fns}$	$-0.008\,237(21)$	Eqs. (94,95), Ref.[25]
$\delta^{(1)}_{\rm rec}$	$0.001\,672(3)$	Eq. (96), Ref.[20]
$\delta^{(1)}_{\rm pol}$	$0.000\,200\,6(52\,4)$	Eq. (97), Ref.[26]
$\frac{\alpha}{\pi}\,c_1\,\delta^{(1)}_{\rm fns}$	$-0.000\,033\,12$	Eq. (98)
$\frac{\alpha}{\pi}\,c_1\,\delta^{(1)}_{\rm rec}$	$0.000\,006\,72$	Eq. (98)
$\delta^{(2)}_{\rm evp,fns}$	$-0.000\,149\,81$	Eq. (65)
$\delta^{(2)}_{\rm evp,rec}$	$0.000\,025\,24$	Eq. (66)
$\delta^{(2)}_{\mu\rm vp,fns}$	$-0.000\,025\,10$	Eq. (70)
$\delta^{(2)}_{\mu\rm vp,rec}$	$-0.000\,001\,18$	Eq. (71)
$\delta^{(2)}_{\rm se,fns}$	$0.000\,016\,18$	Eq. (81)
$\delta^{(2)}_{\rm se,rec}$	$0.000\,016\,48$	Eq. (84)
$\delta^{(2)}_{\rm rel,fns}$	$-0.000\,050\,85$	Eq. (103), Ref.[27]
$\delta^{(2)}_{\rm rel,rec}$	$0.000\,118\,86$	Eq. (104), Ref.[4]
$\delta^{(2)}_{\rm rel,rec2}$	$0.000\,000(12)$	$(Z\,\alpha)^2\,(m/M)^2$
$\delta^{(2)}_{\rm rel,rec,fns}$	$0.000\,000(12)$	$(Z\,\alpha)^2\,m^2/M\,r_Z$
$\delta^{(2)}_{\rm hvp}$	$0.000\,011\,80(8)$	Eq. (105), Ref.[28]
$\delta^{(3)}_{\rm fns}$	$0.000\,000(2)$	$\alpha^3\,m\,r_Z$
$\delta_{\rm weak}$	$0.000\,011\,99$	Eq. (106), Ref.[30]
δ	$0.000\,869\,(60)$	total value
$\frac{m_\mu}{m_e}\,[\delta_{\rm exp}({\rm H}) - \delta({\rm H})]$	$0.000\,133$	Sec. IX
$\delta_{\rm corr}$	$0.001\,002\,(30)$	corrected total value

$$
\begin{aligned}
& - \frac{72}{25}\,\Gamma\biggl(\frac{3}{4}\biggr)^2\,\Gamma\biggl(\frac{5}{4}\biggr)^{-2}\sqrt{\frac{m_e}{m_\mu}} + O\biggl(\frac{m_e}{m_\mu}\biggr)\biggr] \\
= & \; -0.70\ {\rm ppm}\,. \qquad (93)
\end{aligned}
$$

$\delta^{(1)}_{\rm fns}$ is the finite nuclear size correction related to the Zemach radius

$$
\begin{aligned}
\delta^{(1)}_{\rm fns} &= -\,2\,Z\,\alpha\,m\,r_Z, \qquad (94)\\
r_Z &= 1.054(3), \qquad (95)
\end{aligned}
$$

where the numerical value is taken from Ref. [25]. $\delta^{(1)}_{\rm rec}$ is the leading nuclear recoil correction

$$
\delta^{(1)}_{\rm rec} = 837.6^{+2.8}_{-1.0}\,{\rm ppm} + 2\,Z\,\alpha\,m\,r_Z\,\frac{m}{m+M}, \qquad (96)
$$

where the numerical value is from Ref. [20]. Because r_Z is scaled here by the muon mass rather than the reduced mass, our recoil correction includes an additional term. $\delta^{(1)}_{\rm pol}$ is the leading nuclear polarizability correction taken from Ref. [26],

$$
\delta^{(1)}_{\rm pol} = 200.6(52.4)\,{\rm ppm}\,. \qquad (97)
$$

It is convenient to separately consider EVP corrections to the square of the wave function at the origin $\phi^2(0)$

$$
\begin{aligned}
\phi^2(0)_{\rm evp} &= \phi^2(0)\,\biggl(1 + \frac{\alpha}{\pi}\,c_1 + \frac{\alpha^2}{\pi^2}\,c_2\biggr), \qquad (98)\\
c_1 &= 1.73115\,, \qquad (99)\\
c_2 &= 7.2558\,, \qquad (100)
\end{aligned}
$$

where c_1, c_2 are one- and two-loop EVP correction respectively, see Ref. [29]. Consequently, $\delta^{(1)}_{\rm fns}$ and $\delta^{(1)}_{\rm rec}$ receive corrections due to $\phi^2(0)_{\rm evp}$, along with additional EVP corrections to the hard two-photon exchange, denoted by $\delta^{(2)}_{\rm evp,fns}$ in Eq. (65) and $\delta^{(2)}_{\rm evp,rec}$ in Eq. (66), respectively. $\delta^{(2)}_{\mu\rm vp,fns}$ is the μVP combined with FNS, see Eq. (70). $\delta^{(2)}_{\mu\rm vp,rec}$ is the μVP combined with REC, see Eq. (71). $\delta^{(2)}_{\rm se,fns}$ is the μSE combined with FNS, see Eq. (81). $\delta^{(2)}_{\rm se,rec}$ is the μSE combined with REC, see Eq. (84). $\delta^{(2)}_{\rm rel,fns}$ is the nonrecoil relativistic correction with FNS, given by [27]

$$
\delta^{(2)}_{\rm rel,fns} = \frac{4}{3}\,(m r_p Z\alpha)^2\biggl[\gamma - 1 + \ln(2\,m r_{pp}\,Z\alpha) + \frac{1}{4}\,\frac{r_m^2}{r_p^2}\biggr]\,. \qquad (101)
$$

For the dipole parametrization of the nuclear form factors

$$
r_m = r_p\,, \quad r_{pp}/r_p = 5.274\,565\ldots\,, \qquad (102)
$$

where $r_p = 0.840\,60\,(39)$ fm is the proton charge radius; therefore,

$$
\delta^{(2)}_{\rm rel,fns}(\mu{\rm H}) = -\,50.85\ {\rm ppm} \qquad (103)
$$

$\delta^{(2)}_{\rm rel,rec}$ is the relativistic recoil correction [4]

$$
\begin{aligned}
\delta^{(2)}_{\rm rel,rec} = {}& \frac{m}{M}\,\frac{(Z\alpha)^2}{1+\kappa}\biggl[\frac{65}{18} + \frac{13}{18}\kappa + \frac{31}{36}\kappa^2 \qquad (104)\\
& - \biggl(8 + 2\kappa - \frac{1}{4}\kappa^2\biggr)\ln 2 - \biggl(2 + 2\kappa + \frac{7}{4}\kappa^2\biggr)\ln(Z\alpha)\biggr].
\end{aligned}
$$

$\delta^{(2)}_{\rm rel,rec2}$ is the relativistic second-order recoil correction $\sim (m/M)^2$ for which we provide only an estimate. $\delta^{(2)}_{\rm rel,rec,fns}$ is the relativistic recoil finite nuclear size correction and is also estimated only. $\delta_{\rm hvp}$ represents the hadronic VP, and we adopt the result from Ref. [28]

$$
\delta_{\rm hvp} = 11.80(8)\ {\rm ppm}\,. \qquad (105)
$$

$\delta^{(3)}_{\rm fns}$ is FNS correction at the order α^3, for which we provide estimation only. Finally, $\delta_{\rm weak}$ is the nonrecoil weak interaction correction for a point nucleus [30]

$$
\delta_{\rm weak}(\mu{\rm H}) = \frac{m_\mu}{m_e}\,58\cdot 10^{-9}\,. \qquad (106)
$$

IX. μH VS H

One can use HFS measurement in H to extract $\delta^{(1)}_{\rm nuc}({\rm H}) = \delta^{(1)}_{\rm fns}({\rm H}) + \delta^{(1)}_{\rm rec}({\rm H}) + \delta^{(1)}_{\rm pol}({\rm H})$ and use it to improve theoretical predictions for μH. This is possible, because all other contributions to the HFS in H are well known. Therefore, let us consider the proton structure corrections to the scaled difference between μH and H

$$\Delta = \delta(\mu{\rm H}) - \frac{m_\mu}{m_e}\,\delta({\rm H})\,. \tag{107}$$

One can expect that the nuclear structure contributions, as well as their associated uncertainties, cancel out to a high degree in this difference. The nuclear part is

$$\begin{aligned}\Delta_{\rm nuc} &= \delta^{(1)}_{\rm nuc}(\mu{\rm H}) - \frac{m_\mu}{m_e}\,\delta^{(1)}_{\rm nuc}({\rm H}) \\ &= \Delta_{\rm fns} + \Delta_{\rm rec} + \Delta_{\rm pol}\,. \end{aligned}\tag{108}$$

The FNS contribution $\Delta_{\rm fns}$ vanishes exactly; so does the related uncertainty. The recoil contribution $\Delta_{\rm rec}$

$$\begin{aligned}\Delta_{\rm rec} &= \delta^{(1)}_{\rm rec}(\mu{\rm H}) - \frac{m_\mu}{m_e}\,\delta^{(1)}_{\rm rec}({\rm H}) \\ &= 0.000\,578(1), \end{aligned}\tag{109}$$

is decreased by a factor of 3 compared to $\delta^{(1)}_{\rm rec}(\mu{\rm H})$. Thus, we assume that its uncertainty is also decreased by this factor, rendering it negligible. The proton polarizability contribution

$$\begin{aligned}\Delta_{\rm pol} &= \delta^{(1)}_{\rm pol}(\mu{\rm H}) - \frac{m_\mu}{m_e}\,\delta^{(1)}_{\rm pol}({\rm H}) \\ &= \,-0.000\,025(25), \end{aligned}\tag{110}$$

is decreased by a factor of 8 compared to $\delta^{(1)}_{\rm pol}(\mu{\rm H})$, and we estimate its uncertainty to be as large as the entire value of $\Delta_{\rm pol}$. It is therefore less than half the size of the uncertainty in $\delta_{\rm pol}(\mu{\rm H})$. We believe that this can be further improved with a more detailed analysis.

Let us now consider the theoretical result for $\delta(\mu{\rm H})$ and replace it with

$$\begin{aligned}\delta_{\rm corr}(\mu{\rm H}) &= \delta(\mu{\rm H}) + \frac{m_\mu}{m_e}\,[\delta_{\rm exp}({\rm H}) - \delta({\rm H})] \\ &= \Delta + \frac{m_\mu}{m_e}\,\delta_{\rm exp}({\rm H}) \\ &= 0.001\,002(30)\,, \end{aligned}\tag{111}$$

where the correction without uncertainty is [4]

$$\frac{m_\mu}{m_e}\,[\delta_{\rm exp}({\rm H}) - \delta_{\rm theo}({\rm H})] = 0.000\,133\,. \tag{112}$$

The uncertainty of $\delta_{\rm corr}(\mu{\rm H})$ is obtained from the uncertainty of Δ, which originates from the 25 ppm of $\Delta_{\rm pol}$, the 1 ppm of $\Delta_{\rm rec}$, and all other uncertainties listed in Table I. The corrected theoretical predictions for μH is therefore

$$E_{\rm hfs}(\mu{\rm H}) = E_F\,(1+\delta_{\rm corr}) = 0.182\,626(5)\ {\rm eV}. \tag{113}$$

It is interesting to note that this value is close to the one obtained from the nonrelativistic formula in Eq. (4) with the muon magnetic moment anomaly

$$E_F\,(1+a_\mu) = 0.182\,656\ {\rm eV}, \tag{114}$$

indicating a tendency for higher-order corrections to cancel out.

In comparison to the previous work on this topic by Faustov and Martynenko in Ref. [31], we observe an agreement only for the one-loop EVP correction. Moreover, it was difficult to perform further comparison term by term, due to inconsistent classification of their corrections, but whenever we were able to compare, we observed a disagreement. In contrast, our result is in good agreement with the value $E_{\rm hfs}(\mu{\rm H}) = 0.182\,636(8)$ eV, presented by Antognini *et al.* in Ref. [32], but we were also not able to perform comparison term by term, due to lack of detailed results in that work.

X. CONCLUSIONS

We have accounted for all QED and recoil corrections larger than 1 ppm to the ground state HFS of μH, and obtained numerical values or estimates, as summarized in Table I. Our result for $E_{\rm hfs}(\mu{\rm H})$ in Eq. (113) corresponding to $\lambda = 6.788\,97(19)$ μm might serve as a starting point to search for this hyperfine transition, which so far has not been observed. The most important outcome, however, was the identification of corrections that still need to be calculated or improved in order to reach 1 ppm accuracy. The first of these is the proton structure correction, and more precisely the weighted difference $\Delta_{\rm nuc}$ in Eq. (108), which can be obtained much more accurately than $\delta^{(1)}_{\rm nuc}$ itself. The second important contribution left for future work is the $(Z\,\alpha)^2$ correction, evaluated without expansion in the mass ratio and including the finite nuclear size. At present, we know the point nucleus values in the nonrecoil limit Eq. (91), the leading recoil correction Eq. (104), and the FNS in the nonrecoil limit Eq. (103). The omitted terms constitute the largest uncertainty besides $\Delta_{\rm nuc}$. Their calculation is certainly feasible, but most importantly, the contents of Table I with various corrections should be independently verified.

Our current theoretical predictions with incorporation of H HFS are presented in Eq. (113). It is clear that the HFS measurement in μH with 1 ppm accuracy will stand as a highly significant test of fundamental interactions when combined with H HFS, or alternatively will serve for the accurate determination of the proton Zemach radius.

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Appendix A: Lepton self-energy integrals

The lepton factors $t_{1\mathrm{se}}$ and $t_{2\mathrm{se}}$ can be expressed in terms of scalar integrals in the form

$$f(n,m,l)=\int\frac{\mathrm{d}^Dk}{i\pi^{D/2}}\,\frac{1}{[(k-t)^2]^n[k^2-1]^m[(k+q)^2-1]^l}\,,\tag{A1}$$

where $t^2=1$ and $D=4-2\,\epsilon$. Using integration by parts identities [33, 34], one can algebraically reduce all powers $n, m,$ and l to 0 and 1. These integrals with lowest powers can all be performed using the Feynman parameters approach. Neglecting $\mathcal{O}(\epsilon)$, they are

$$f(n,0,0)=0\,,\tag{A2}$$

$$f(0,1,0)=\frac{1}{\epsilon}+1-\gamma_E,\tag{A3}$$

$$f(0,2,0)=\frac{1}{\epsilon}-\gamma_E,\tag{A4}$$

$$f(0,0,1)=\frac{1}{\epsilon}+1-\gamma_E,\tag{A5}$$

$$f(0,0,2)=\frac{1}{\epsilon}-\gamma_E,\tag{A6}$$

$$f(1,1,0)=\frac{1}{\epsilon}+2-\gamma_E,\tag{A7}$$

$$f(1,2,0)=\frac{1}{2\,\epsilon}-\frac{\gamma_E}{2},\tag{A8}$$

$$f(0,1,1)=\frac{1}{\epsilon}-\gamma_E+2-2\sqrt{\frac{4}{q^2}-1}\;\arcsin\left(\frac{q}{2}\right),\tag{A9}$$

$$f(1,0,1)=\frac{1}{\epsilon}-\gamma_E+2+\frac{1-p^2}{p^2}\,\ln(1-p^2),\tag{A10}$$

$$f(0,2,1)=-\,\frac{2}{q^2}\left(\frac{4}{q^2}-1\right)^{-1/2}\arcsin\left(\frac{q}{2}\right),\tag{A11}$$

$$\begin{aligned}f(1,2,1)={}&\frac{1}{p^2-1}\left[\frac{1}{2\,\epsilon}-\frac{\gamma_E}{2}-\ln(1-p^2)\right.\\&\left.+2\left(\frac{2}{q^2}-1\right)\left(\frac{4}{q^2}-1\right)^{-1/2}\arcsin\left(\frac{q}{2}\right)\right],\end{aligned}\tag{A12}$$

$$f(1,1,1)=-\,J(-q^2,q^0),\tag{A13}$$

where γ_E is the Euler constant, $p=t+q$, and $q^0=q\cdot t$. The master integral J is

$$\begin{aligned}J(-q^2,q^0)={}&-\int\frac{d^4k}{\pi^2\,i}\,\frac{1}{k^2}\,\frac{1}{(t-k)^2-1}\,\frac{1}{(p-k)^2-1}\\={}&\int_0^1du\,\frac{1}{1-u(1-u)\,q^2-u(1-p^2)}\\&\times\ln\left(\frac{1-u(1-u)q^2}{u(1-p^2)}\right).\end{aligned}\tag{A14}$$

The particular form at $q^0=0$ of the master integral $J(q^2)\equiv J(q^2,0)$ is

$$\begin{aligned}J(q^2)={}&\int_0^1du\,\frac{1}{1-u^2\,q^2}\,\ln\left(\frac{1+u(1-u)q^2}{u\,q^2}\right)\\={}&1+\frac{5\,q^2}{18}+\frac{11\,q^4}{150}-\left(1+\frac{q^2}{3}+\frac{q^4}{5}\right)\ln(q^2)+\ldots\\={}&\frac{2}{q^2}+\frac{2}{9\,q^4}+\left(\frac{1}{q^2}-\frac{5}{3\,q^4}\right)\ln(q^2)+\ldots\end{aligned}\tag{A15}$$

The derivative of $J(q^2)$ satisfies

$$\frac{\partial}{\partial q}\left[qJ(q^2)\right]=-\,\frac{4}{q^2}\,\frac{\mathrm{arcsinh}\frac{q}{2}}{\sqrt{1+\frac{4}{q^2}}}-\frac{\ln q^2}{1-q^2}\,.\tag{A16}$$

Other properties of J can be found in the Appendix B of Ref. [35]

[1] G.T. Bodwin, D.R. Yennie, Some recoil corrections to the hydrogen hyperfine splitting, Phys. Rev. D **37**, 498 (1988).

[2] A.V. Volotka, V.M. Shabaev, G. Plunien, and G. Soff, Zemach and magnetic radius of the proton from the hyperfine splitting in hydrogen, Eur. Phys. J. D **33**, 23 (2005).

[3] Savely G. Karshenboim, Precision physics of simple atoms: QED tests, nuclear structure and fundamental constants, Phys. Rep. **422**, 1 (2005).

[4] J. Hevler, A. Maroń, and K. Pachucki, Relativistic nuclear recoil effects in hyperfine splitting of hydrogenic systems, arXiv:2602.15616 [physics.atom-ph] (2026).

[5] S. G. Karshenboim, Some possibilities for laboratory searches for variations of fundamental constants, Can. J. Phys. **78**, 639 (2000).

[6] L. Essen, R.W. Donaldson, E.G. Hope, M.J. Bangham, Frequency of the Hydrogen Maser, Nature **229**, 110 (1971).

[7] L. Essen, R.W. Donaldson, E.G. Hope, M.J. Bangham, Hydrogen Maser Work at the National Physical Laboratory, Metrologia **9**, 128 (1973) .

[8] P. Amaro, A. Adamczak, M. Abdou Ahmed, L. Affolter, F. D. Amaro, P. Carvalho, T. -L. Chen, L. M. P. Fernandes, M. Ferro, D. Goeldi *et al.*, Laser excitation of the 1s-hyperfine transition in muonic hydrogen, SciPost Phys. **13**, 020 (2022).

[9] A. Vacchi, E. Mocchiutti, A. Adamczak, D. Bakalov, G. Baldazzi, M. Baruzzo, R. Benocci, R. Bertoni, M. Bonesini, H. Cabrera *et al.*, Investigating the Proton Structure: The FAMU Experiment, Nuclear Physics News, **33** 9 (2023).

[10] C. Itzykson and J. B. Zuber, Quantum Field Theory, International Series In Pure and Applied Physics (McGraw-Hill, New York) (1980).

[11] K. Pachucki, V. Lensky, F. Hagelstein, S.S. Li Muli, S. Bacca, and R. Pohl, Comprehensive theory of the Lamb shift in light muonic atoms, Rev. Mod. Phys. 96, 015001 (2024).

[12] V. Patkos and K. Pachucki, Antiprotonic atoms with nonperturbative inclusion of vacuum polarization and finite nuclear mass, Phys. Rev. A **112**, 052808 (2025).

[13] V. M. Shabaev, Generalizations of the virial relations for the Dirac equation in a central field and their applications to the

Coulomb field, J. Phys. B **24**, 4479 (1991).

[14] V. Shabaev, Virial relations for the Dirac equation and their applications to calculations of hydrogen-like atoms, in Precision Physics of Simple Atomic Systems, edited by S. G. Karshenboim and V. B. Smirnov, Lecture Notes in Physics, pages 97 – 113, Berlin, 2003, Springer.

[15] K. Pachucki, Nuclear recoil correction to the hyperfine splitting in atomic systems, Phys. Rev. A **106**, 022802 (2022).

[16] C.K. Iddings, Structure of the Proton and the Hyperfine Shift in Hydrogen, Phys. Rev. **138**, B446 (1965).

[17] K. Pachucki, Theory of the Lamb shift in muonic hydrogen, Phys. Rev. A **53**, 2092 (1996).

[18] M. I. Eides, H. Grotch, V. A. Shelyuto, Theory of light hydrogenic ions, Phys. Rep. **342**, 63 (2001).

[19] A. C. Zemach, Proton Structure and the Hyperfine Shift in Hydrogen, Phys. Rev. **104**, 1771 (1956).

[20] Aldo Antognini, Yong-Hui Lin, Ulf-G. Meißner, Precision calculation of the recoil–finite-size correction for the hyperfine splitting in muonic and electronic hydrogen, Phys. Lett. B **835**, 137575 (2022).

[21] S.G. Karshenboim, Nuclear structure-dependent radiative corrections to the hydrogen hyperfine splitting, Phys. Lett. A **225**, 97 (1997).

[22] P. J. Mohr, D. B. Newell, B. N. Taylor, and E. Tiesinga, CODATA recommended values of the fundamental physical constants: 2022, Rev. Mod. Phys. **97**, 025002 (2025).

[23] V. Patkos, V. A. Yerokhin, and K. Pachucki, Nuclear polarizability effects in $^3\mathrm{He}^+$ hyperfine splitting, Phys. Rev. A **107**, 052802 (2023).

[24] Petr A. Krachkov and Roman N. Lee, Two-loop corrections to Lamb shift and hyperfine splitting in hydrogen via multi-loop methods, J. High Energ. Phys. 2023, 211 (2023).

[25] Yong-Hui Lin, Hans-Werner Hammer, and Ulf-G. Meißner, New Insights into the Nucleon's Electromagnetic Structure, Phys. Rev. Lett. **128**, 052002 (2022).

[26] David Ruth, Karl Slifer, Jian-Ping Chen, Carl E. Carlson, Franziska Hagelstein, Vladimir Pascalutsa, Alexandre Deur, Sebastian Kuhn, Marco Ripani, Xiaochao Zheng, Ryan Zielinski, Chao Gu, New spin structure constraints on hyperfine splitting and proton Zemach radius, Phys. Lett. B **859**, 139116 (2024).

[27] M. Kalinowski, K. Pachucki, and V.A. Yerokhin, Nuclear-structure corrections to the hyperfine splitting in muonic deuterium, Phys. Rev. A **98**, 062513 (2018).

[28] F. Hagelstein, V. Lensky, B. Malaescu, and V. Pascalutsa, in preparation (2026).

[29] Vladimir G. Ivanov, Evgeny Yu. Korzinin, Savely G. Karshenboim, Second-order corrections to the wave function at origin in muonic hydrogen and pionium, Phys. Rev. D **80**, 027702 (2009).

[30] Michael I. Eides, Weak-interaction contributions to hyperfine splitting and Lamb shift, Phys. Rev. A **53**, 2953 (1996).

[31] R. N. Faustov, A. P. Martynenko, Muonic hydrogen ground state hyperfine splitting, J. Exp. Theor. Phys. **98**, 39 (2004).

[32] Aldo Antognini, Franziska Hagelstein, and Vladimir Pascalutsa, The Proton Structure in and out of Muonic Hydrogen, Annu. Rev. Nucl. Part. Sci. **72**, 389 (2022).

[33] F. V. Tkachov, A theorem on analytical calculability of 4-loop renormalization group functions, Phys. Lett. **100B**, 65 (1981).

[34] K. G. Chetyrkin and F. V. Tkachov, Integration by parts: The algorithm to calculate β-functions in 4 loops, Nucl. Phys. B **192**, 159 (1981).

[35] K. Pachucki, Radiative corrections to the nuclear size and polarizability effects in atomic systems, Phys. Rev. A **112**, 012816 (2025).