Spectroscopy of light hydrogenic atoms: a theoretical approach*

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Abstract Theoretical methods for the calculation of bound state energies in quantum electrodynamics are described. Review of results for hydrogen and positronium atoms is presented and further applications to the two electron systems are proposed.

1 Introduction

Quantum Electrodynamics (QED) aims to give an accurate description of all physical phenomena connected with the electromagnetic force from macroscopic scales down to the subatomic scale. QED has not yet been formulated in a mathematically strict manner. It exists as a compound of prescriptions of how to perturbatively calculate physical quantities such as cross sections or bound state energies. It is understood that all the ultraviolet divergences appearing in multiloop integrations can be absorbed into the redefinition of the mass and the charge of the electron. This renormalisation theory, which has been proved to work in all orders of perturbation theory, has been applied in practice through eighth order in e, see [1].

The first decisive confirmation of QED came from the measurement of the $2S_{1/2} - 2P_{1/2}$ splitting in the hydrogen atom [2]. According to Dirac equation both levels $2S_{1/2}$ and $2P_{1/2}$, while shifted by relativistic effects, remain degenerate. The electron self–energy and vacuum–polarization effects split these states by an amount in good agreement with the measured value. Currently, the Lamb

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shift in hydrogen has been measured very precisely $\delta E/E \approx 10^{-6}$ by several groups, which serves to rule out many exotic theories.

Many other experimental tests on hydrogenic atoms have been performed since the original Lamb shift measurement. The most precise are the $1S - 2S$ [3] and $2S - 4S$ [4] two–photon transition in hydrogen by the Hänsch group in Garching (T.W. Hänsch contribution to this proceedings), the $2S - 8S$ transition by Biraben *et al.* in Paris [5], $2S - 4P$ by Boshier at Yale [6], hydrogen [7] and muonium [8] hyperfine splitting (T. Kinoshita, contribution to this proceedings), the hyperfine splitting [9] and the $1S - 2S$ transition [10] in positronium. In this short review I present a treatment of the QED effects on energy levels in hydrogen and positronium, and propose a new method for studying the helium atom.

2 Hydrogen Lamb shift

Hydrogen is the most intensively studied bound state system, both experimentally and theoretically. While progress in high precision hydrogen spectroscopy is described by Hänsch in his contribution to this Workshop, I concentrate mostly on the theoretical description. Usually, as a first approximation one assumes that the proton is a static source of a Coulomb field, and later considers so called recoil corrections: for details see [11]. QED predicts electron self–interactions that leads to the shift of its energy. The derivation of this effect involves the perturbative expansion in the number of "loops", i.e. number of photons involved in the self–interaction. One derives the following complete relativistic expression for the one–loop self–energy in the field of a nucleus of charge $Z\,e$:

$$\begin{aligned}\Delta E &= e^2 \int \frac{d^4k}{(2\pi)^4 i}\, g_{\mu\nu} \left(\frac{1}{k^2} - \frac{1}{k^2+\Lambda^2} \right) \\ &\quad \times \langle \overline{\psi} | \gamma^\mu \frac{1}{\not{p} - \not{k} + \frac{Z\,\alpha}{r}\gamma^0 - m} \gamma^\nu | \psi \rangle - \langle \overline{\psi} | \delta m(\Lambda) | \psi \rangle \,, \end{aligned} \tag{1}$$

which is represented by the Feynman graph of Fig. (1). Here p_0 is the bound state

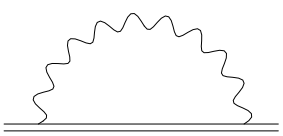

Figure 1: Electron self-energy diagram

energy, and δm is a mass counter term. It is conventional to keep the Z dependence in the above even for our case, as it distinguishes radiative from binding

corrections. The expression in Eq. (1) contains ultraviolet divergences which are characteristic of the perturbative formulation of QED. The integral is divergent in the limit $\Lambda \to \infty$. This divergence is canceled out by the δm term, using the requirement that for a free electron the energy shift, or in other words the correction to the electron mass, is exactly zero. The evaluation of the integral in Eq. (1) is a highly nontrivial problem. Two approaches have been introduced. The first, which has been applied with highest accuracy by P. Mohr [12], relies on a partial wave decomposition of the electron propagator and numerical integration. The other approach is based on an analytic expansion in $Z\,\alpha$ that is particularly suited for hydrogen [13], [14]. The original calculation of the leading term by Bethe [15] and later by Feynman and others explored the fact that $Z\,\alpha$ is small. To demonstrate specific features of the bound state calculation let me describe the derivation of the leading term. There are two characteristic energies in Eq. (1): the bound state energy $m\,\alpha^2/2$ and the electron mass m. One splits this integral into two parts and performs approximations specific to each part. In the low energy part the photon energy is limited by some arbitrarily chosen parameter ϵ that lies somewhere between 0 and the electron mass m. In the high energy part ϵ plays a role of an infrared cut-off. The key ingredient of this calculation is the fact that after expansion in $Z\,\alpha$ one goes to the limit of $\epsilon \to 0$. It is very helpful in the evaluation of the low energy part, where the binding field enters in a nonperturbative way. One can perform a non relativistic expansion of Eq. (1), use the Coulomb gauge, and find that the leading terms take the form

$$E_L = e^2 \int^{\epsilon} \frac{d^3k}{(2\,\pi)^3\,2\,k} \left(\delta^{ij} - \frac{k^i\,k^j}{k^2}\right) \langle\phi|p^i \frac{1}{E-\omega-H}\, p^j|\phi\rangle \qquad (2)$$

$$= m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\,\frac{4}{3}\ln\left(\frac{2\,\epsilon}{m\,(Z\,\alpha)^2\,k_0(n,l)}\right)\,. \qquad (3)$$

It is understood that in the expression (2) the limit $\epsilon \to 0$ is performed after the expansion in $Z/, \alpha$. For the high energy part, E_H, one makes an expansion of the electron propagator in Eq. (1) in powers of Coulomb potential $V = -Z\,\alpha/r$. It is sufficient, for the leading order, to consider only terms up to V^2. A peculiar feature of the QED bound state calculations is the internal cancellation between terms containing different powers of V. In our case it is sufficient to calculate only the single V term with the external legs on mass shell, i.e. we need only two electromagnetic formfactors, F_1 and F_2. Expression for these can be found in any textbook on quantum field theory. The effective interaction (radiatively corrected) of an electron with the electromagnetic field takes the form

$$\gamma_\rho\,A^\rho \to \left(\gamma_\rho\left\{1 + \frac{\alpha\,q^2}{3\,\pi\,m^2}\left[\ln\left(\frac{m}{2\,\epsilon}\right) + \frac{11}{24}\right]\right\} + \frac{i}{2\,m}\,\frac{\alpha}{2\,\pi}\,\sigma_{\rho\nu}\,q^\nu\right)\,A^\rho(q)\,. \quad (4)$$

The correction to the energy E_H is given by the expectation value of γA on the electron state,

$$E_H = e\,\langle\overline{\psi}|\gamma A|\psi\rangle = m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\,\left[\frac{4}{3}\,\ln\left(\frac{m}{2\,\epsilon}\right) + \frac{11}{18}\,\delta_{l0} - \frac{1}{2\,\kappa\,(2\,l+1)}\right)\,, \tag{5}$$

where

$$\kappa = \begin{cases} -(l+1) & \text{for } j = l + \frac{1}{2} \\ \\ \quad l & \text{for } j = l - \frac{1}{2} \end{cases} \tag{6}$$

The leading order one–loop self–energy contribution to the Lamb shift is a sum of E_L and E_H. However, it has been calculated in the non recoil limit, i.e. with the assumption that proton is a static source. Recoil corrections, in leading order, can be incorporated by a simple mass rescaling: for details see [11] and [16]. The complete formula which is valid for any mass ratio takes the form

$$\Delta E_{SE} = m\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\quad\left\{\quad\left[\left(\frac{4}{3}\,\ln\frac{m}{\mu\,(Z\,\alpha)^2} + \frac{10}{9}\right)\,\delta_{l0} - \frac{4}{3}\,\ln k_0(n,l)\right]\,\frac{\mu^3}{m^3}\right.$$
$$\left. -\frac{1}{2\,\kappa\,(2\,l+1)}\,(1-\delta_{l0})\,\frac{\mu^2}{m^2}\right\}. \tag{7}$$

This trick of introducing an auxiliary parameter ϵ without giving its any value could be extended to higher orders and in fact, most of the modern analytic QED calculations applies this.

The next contribution. which is presented here without a derivation, is associated with vacuum polarization. The Coulomb field of the nucleus is modified at distances of order $1/m$, the electron Compton wavelength, by electron–positron pair creation, what leads to the so called Ühling potential. In leading order the energy shift is given by

$$\Delta E_{VP} = \frac{\mu^3}{m^2}\,\frac{\alpha}{\pi}\,\frac{(Z\,\alpha)^4}{n^3}\,\left(-\frac{4}{15}\right)\,. \tag{8}$$

These leading order terms are not sufficient to compare with the measurements. There are higher order terms $O(Z\,\alpha)$ and $o(Z\,\alpha)^2$ which have been calculated over a period of 5 decades, and for recent results I refer the reader to reviews in [11] and [17]. There are many other contributions to the Lamb shift which have been worked out. The most important are the two–loop contribution and recoil corrections. At the time this article is being written, the two–loop

contribution has been calculated only by means of expansion in $Z\,\alpha$ [18–20],

$$\Delta E = m\,\left(\frac{\alpha}{\pi}\right)^2\,\frac{(Z\,\alpha)^4}{n^3}\quad\left[\;0.867\,420\,\delta_{l0} + 0.328\,478\,\frac{1}{\kappa(2\,l+1)}\right.$$
$$\left. -(Z\,\alpha)\,21.558(3)\,\delta_{l0} + \cdots\right]. \qquad (9)$$

The unexpectedly large value of the $(Z\,\alpha)$ coefficient for S-states causes the theoretical prediction for the Lamb shift in hydrogen and singly ionized helium to shift beyond previous uncertainty estimates. Surprisingly, Mallampali and Sapirstein [21] have obtained partial numerical unperturbative results for the two–loop contribution. Their result indicates that higher order terms are significant. The perturbed orbital term, according to the authors gives an extra correction beyond the known perturbative contribution of the amount of -8.9 kHz. This correction changes the overall sign for this particular term, what makes it the most unperturbative contribution known in QED. It is not in an agreement with the estimate for $\ln^3(Z\,\alpha)^2$ term found by Karshenboim [22]. This difficult numerical calculations deserve a independent confirmation.

Further corrections, due to the finite nuclear mass are considered separately. Pure recoil corrections may be understood as a correction to the Dirac equation due to motion of the nucleus excluding any self–interaction. They require a much more complicated analysis involving the Bethe–Salpeter equation. This equation is frequently made more tractable by, for example, elimination of the relative time from the generalized wave function. Let me mention here that there exist alternative formalisms, for example: NRQED by P. Lepage [23] or a similar effective Hamiltonian method which was developed by me in [24]. They take advantage of the natural cancellation of many terms before performing the actual calculation. These formalisms are particularly suited for systems consisting of particles with comparable masses. Coming back to hydrogen, the Bethe Salpeter equation was used by Salpeter [25] for the derivation of $(Z\,\alpha)^5$ contributions. Later on, Shabayev [26] found beautiful formulas for the recoil correction in first order of the mass ratio that are valid to all orders in $(Z\,\alpha)$. These formulas have been rederived in a much simpler way in [27]. They were the basis of the calculation of recoil effects in order $(Z\,\alpha)^6$ which I have performed with H. Grotch [27]. Our result is

$$\Delta E(nS) = \frac{m^2}{M}\,\frac{(Z\,\alpha)^6}{n^3}\,\left(4\,\ln(2) - \frac{7}{2}\right)\,. \qquad (10)$$

In the similar treatment one accounts for the additional radiative recoil corrections that have not been accounted for by the reduced mass rescaling of the wave function at the origin [16]. One writes a scattering amplitude for two photon exchange between the electron and nucleus, subtracts lower order terms, and

obtains the correction

$$\Delta E = \frac{m^2}{M}\,\alpha\,\frac{(Z\,\alpha)^5}{n^3}\,(-1.364\,49)\,. \tag{11}$$

There is a further correction to the Lamb shift due to the finite size of the nucleus. It presents the most difficult problem in the calculation of the QED predictions, because the proton charge radius is not precisely known. Moreover, if we use a standard definition of the charge radius,

$$\frac{\langle r^2\rangle}{6} = \left.\frac{\partial G_E(q^2)}{\partial(q^2)}\right|_{q^2=0}\,, \tag{12}$$

we encounter a conceptual problem with the nuclear self–energy correction. Part of it is included already in the charge radius. Moreover, radiative corrections to G_E are infrared divergent. Therefore, the definition in Eq. (12) is no longer proper at the current level of precision. One can use the enhanced definition proposed in [16], which is based on the forward scattering amplitude. With this definition the correction to the energy for S–states takes the form

$$\begin{aligned}\Delta E &= \frac{2}{3\,n^3}\,(Z\,\alpha)^4\,\mu^3\langle r^2\rangle \\ &\quad + \frac{4\,(Z^2\,\alpha)\,(Z\,\alpha)^4}{3\,\pi\,n^3}\,\frac{\mu^3}{M^2}\,\left[\ln\left(\frac{M}{\mu\,(Z\,\alpha)^2}\right) - \ln k_0(n)\right]\,,\end{aligned} \tag{13}$$

where $\ln k_0(n)$ is the Bethe logarithm and Z denotes the nucleus charge. The first term is due to the finite nuclear size, while the second term comes from the proton self–energy and is significant in the case of muonic hydrogen. The $(Z\,\alpha)$ corrections to this are roughly proportional to $\langle r_p^3\rangle$ and thus completely negligible for the case of the hydrogen Lamb shift. There is a serious problem with the measurement of the proton charge radius. There are two measurements that disagree [28], [29] and each one has a large uncertainty. Moreover, even a new reanalysis [30] of the former scattering data is in disagreement with both previous values. We therefore stress the need of a new independent determination of the proton charge radius. In fact, there are plans at the Paul-Scherrer Institute in Villingen to measure the Lamb shift in muonic hydrogen from which one can extract the proton charge radius, (Kottman, contribution to this proceedings).

For the comparison of theoretical predictions with experimental results for the Lamb shift in hydrogen we assume the proton charge radius [29] $r_p = 0.862(12)$ fm. The theoretical predictions are

$$L(2S-2P) = 1057.839(6)\,\mathrm{MHz}\,, \tag{14}$$

$$L(1S) = 8172.802(40)\,\mathrm{MHz}\,, \tag{15}$$

in moderate agreement with several independent measurements

$$L(2S-2P) = 1057.845(9)\,\mathrm{MHz}\,[31,32]\,, \quad (16)$$
$$L(1S) = 8172.876(29)\,\mathrm{MHz}\,[3,5,6]\,, \quad (17)$$

where only the most precise results are shown. Surprisingly, there is a significant discrepancy for singly ionized helium He^+, which is a hydrogen–like system:

$$L_{th}(2S-2P) = 14041.33(18)\,\mathrm{MHz}\,[17]\,, \quad (18)$$
$$L_{exp}(2S-2P) = 14042.52(16)\,\mathrm{MHz}\,[33]\,, \quad (19)$$
$$\Delta = L_{th} - L_{exp} = 1.19(18)(16)\,\mathrm{MHz}\,. \quad (20)$$

All the corrections obtained for hydrogen require only a simple rescaling due to the different properties of the nucleus, specifically the charge, mass, and charge radius. The fact that the helium nucleus has spin 0 rather than $1/2$ results in shifts of only a few kHz, which is not presently significant. If the measurement is confirmed, this will indicate a serious disagreement with quantum electrodynamic theory.

3 Positronium energy levels

Positronium is the simplest atomic system that we can deal with. The small mass of it's constituents, the electron and positron, ensures that strong and weak interaction effects contribute at a negligible level. The energy spectrum may therefore be predicted with an accuracy limited only by the complexity in higher order QED calculations. The equal mass of the electron and the positron requires special theoretical treatment to incorporate relativistic and recoil effects on the same footing. Several approaches have been developed for the calculation of the positronium spectrum. The traditional one is based on the Bethe–Salpeter equation. This equation is a generalization of the Dirac equation for the case where the second particle is not static. However, the global wave function has an extra dependence on relative time, and therefore loses a probabilistic interpretation. It serves as a tool to calculate bound state energies. The cost one pays using this very general approach is the complicated implementation of the perturbation scheme. Nevertheless this method has been successfully used by Adkins *et. al* for the calculation of various corrections the hyperfine structure [34–36]. An alternative approach is based on the idea of introducing some effective interaction, which takes advantage of the cancellation mechanism between various corrections. Such an approach was first introduced by Caswell and Lepage [23]. It is called NRQED: nonrelativistic quantum electrodynamics. It contains many higher order couplings designed to match the low energy limit of the complete theory. This approach, after being applied to the calculation of the three–photon

exchange contribution to *hfs* in positronium, has recently been used for the calculation of muonium *hfs* [37] (Kinoshita and Nio contributions to this proceedings) and the one–photon annihilation channel contributions to the positronium *hfs* [38]. A similar approach has been introduced by me in [24,39], which is an extension of the Breit Hamiltonian to higher orders. One constructs an effective two–body Hamiltonian H_{EF} from the Hamiltonian of complete QED H_{QED} by comparing corresponding resolvents to the specified order n in α:

$$\left\langle \mathbf{p}', -\mathbf{p}' \left| \frac{1}{E - H_{QED}} \right| \mathbf{p}, -\mathbf{p} \right\rangle^{(n)} = \lim_{\lambda \to \infty} \left\langle \mathbf{p}' \left| \frac{1}{E - H_{EF}(\lambda, E)} \right| \mathbf{p} \right\rangle^{(n)} . \quad (21)$$

In this comparison one assumes that momenta and energy are of the order $m\,\alpha$ and $m\,\alpha^2$ respectively. This effective Hamiltonian H_{EF} is obtained in parts by the application of Breit expansion to the higher order, and by adding the forward scattering amplitude terms. Since higher order terms would be divergent, one regularizes the photon propagator by introducing a factor:

$$\frac{1}{k^2} \to \frac{1}{k^2} \, \frac{(\lambda\,\mu\,\alpha)^2}{\mathbf{k}^2 + (\lambda\,\mu\,\alpha)^2} \,, \quad (22)$$

where μ is a reduced mass of the system. This construction allows for an extension of the Khriplovich calculation [40] to the states with a zero angular momentum. In the case of for example P states the wave function vanishes at origin and all the Breit type terms are finite. In the case of S-states we have to include also delta-like terms to compensate for the regularization introduced in (22). These terms are derived from the forward scattering amplitude. The complete effective Hamiltonian reads

$$H_{EF} = \frac{p^2}{2\,\mu} - \frac{\alpha}{r} \, \left(1 - e^{-\lambda\,\mu\,\alpha\,r}\right) + H^{(4)} + H^{(6)} + \alpha^2 M_5 \, \delta^3(r) + \alpha^3 \, M_6 \, \delta^3(r) \,. \quad (23)$$

With this Hamiltonian the correction to energy in the order $m\,\alpha^6$ is

$$\begin{aligned} \Delta E &= \left\langle \phi | H^{(6)} | \phi \right\rangle + \left\langle \phi | H^{(4)} \, \frac{1}{(E - H)'} \, H^{(4)} | \phi \right\rangle \\ &\quad + \left\langle \phi | \alpha^2 \, M_5 \, \delta^3(r) | \phi \right\rangle^{(6)} + \left\langle \phi | \alpha^3 \, M_6 \, \delta^3(r) | \phi \right\rangle \,. \end{aligned} \quad (24)$$

The regularized Breit Hamiltonian $H^{(4)}$ is obtained from only the single photon exchange between two particles with masses m_1 and m_2. It reads

$$\begin{aligned} H^{(4)} &= -\frac{p^4}{8\,m_1^3} - \frac{p^4}{8\,m_2^3} + \frac{\pi\,\alpha}{2}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\,\delta^3(r)_\lambda \\ &\quad -\frac{\alpha}{2\,m_1\,m_2\,r}\left(p^2 + \frac{r^i\,r^j}{r^2}\,p^i\,p^j\right)_\lambda \\ &\quad -\frac{\alpha}{4\,m_1\,m_2}\left(\frac{\sigma_1\,\sigma_2}{r^3} - \frac{3\,\sigma_1\mathbf{r}\;\sigma_2\mathbf{r}}{r^5} - \frac{8}{3}\,\pi\,\sigma_1\sigma_2\,\delta^3(r)\right)_\lambda , \end{aligned} \tag{25}$$

where subscript λ denotes regularization according to prescription in (22).

$H^{(6)}$, the low energy part, is given by the low momenta expansion of time ordered perturbation theory diagrams, as shown in Fig. 2. The details of these

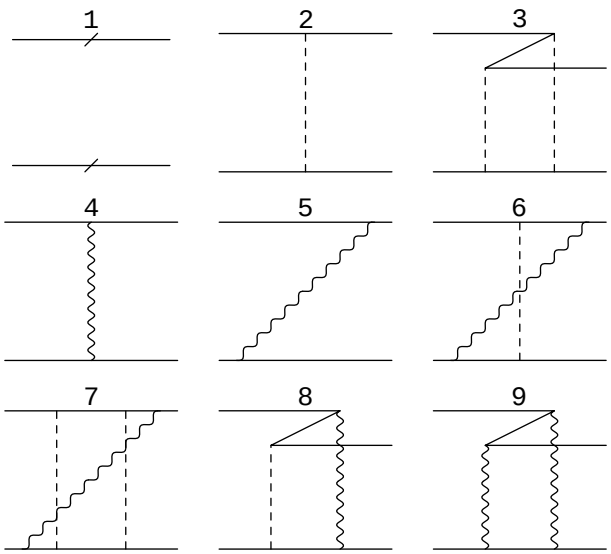

Figure 2: Time ordered diagrams contributing to positronium energy levels in order $m\,\alpha^6$

calculations may be found in Ref. [24]. The high energy part is given by the forward scattering amplitude involving exchange of three photons, see Fig. 3. When both parts are added, the contribution to energy does not depend on extra parameter λ. The sum of all terms for the hyperfine structure reads:

$$\begin{aligned} \Delta E^{(6)} &= \frac{8}{3\,n^3}\,\alpha^6\,\frac{\mu^3}{m_1\,m_2}\left[\frac{11}{6} - \frac{11}{6\,n^2} + \frac{3}{2\,n}\right] + \frac{8}{3\,n^3}\,\alpha^6\,\frac{\mu^5}{m_1^2\,m_2^2} \\ &\quad \left[-2\,\ln(\alpha) + F + \left(\frac{4}{3\,n^2} + \frac{1}{n} + 2\,\ln(n) - 2\,(\Psi(n) + C) - \frac{7}{3}\right)\right], \end{aligned} \tag{26}$$

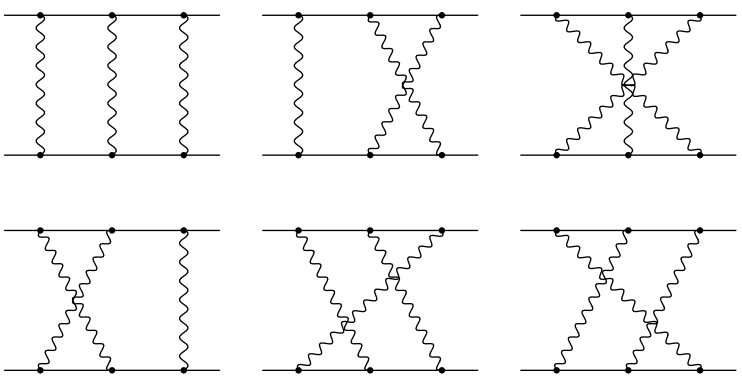

Figure 3: Three–photon exchange forward scattering amplitude

where F is a function depending on mass ratio only, and is known numerically, Ψ is a logarithmic derivative of Euler Gamma function and C is the Euler constant. The first term in the above is a known Breit term and the second agrees with the result of Bodwin *et al.* [41] for the recoil correction to the hydrogen hyperfine structure. For the ground state positronium hfs I obtained

$$\Delta E^{(6)} = m\,\alpha^6 \left(0.3767(17) - \frac{1}{6}\,\ln(\alpha) \right) . \tag{27}$$

However, the result for the ground state hyperfine structure 0.3767(17) is in serious disagreement with the former calculation of Caswell and Lepage [23] 0.167(33)), and with the most recent calculation of Adkins and Sapirstein [42] 0.0705(35). This problem remains to be resolved. In the similar way, the spin independent term has been obtained in [39]. Many other contributions have been calculated over the years. The two– and three–photon annihilation channel has been calculated by Adkins in Ref's [34,35]. One photon annihilation has been recently obtained by Adkins *et al* [36] and independently by Hoang *et al* [38]. Together with S. Karshenboim, we have completed the theoretical predictions for positronium energy levels through order $m\,\alpha^6$ by the calculation of the radiative recoil correction [43]. The final results for the 1S and 2S states, using Ref. [24] are:

$$E^{(6)}(1S) = m\,\alpha^6 \left[0.16107(63) - 0.3925(17)\,\mathbf{s}_-\,\mathbf{s}_+ \right] , \tag{28}$$

$$E^{(6)}(2S) = \frac{m\,\alpha^6}{8} \left[0.34557(63) - 0.1048(17)\,\mathbf{s}_-\,\mathbf{s}_+ \right] , \tag{29}$$

where s_- and s_+ are spin operators for electron and positron respectively. Taking this α^6 contributions the theoretical predictions for the two most precisely

measured transitions read

$$
\begin{aligned}
E_{HFS}(1S) &= 203\,388.09(80)\,\mathrm{MHz}\ \text{Caswell and Lepage}\ [23]\,, && (30)\\
&\quad 203\,392.02(50)\,\mathrm{MHz}\ \text{Pachucki}\ [24]\,, && (31)\\
&\quad 203\,386.41(50)\,\mathrm{MHz}\ \text{Adkins and Sapirstein}\ [42]\,, && (32)\\
E(1S-2S) &= 1\,233\,607\,211.0(1.0)\,\mathrm{MHz}\,. && (33)
\end{aligned}
$$

Comparison to experimental values

$$
\begin{aligned}
E_{HFS}(1S) &= 203\,389.10(74)\,\mathrm{MHz}\ [9]\,, && (34)\\
E(1S-2S) &= 1\,233\,607\,216.4(3.2)\,\mathrm{MHz}\ [10]\,, && (35)
\end{aligned}
$$

gives a fair agreement for the 1S-2S transition. Because of the discrepancy between the three calculations of three photon exchange contribution, we cannot draw final conclusions regarding *hfs*. However, the Caswell, Lepage result is in agreement, and my result in $3.4\,\sigma$ disagreement with the measurement of *hfs*. Other transitions are not significantly affected by this discrepancy. For example the 2^3S_1 state might be shifted by 0.01 MHz,

4 Prospects of QED tests on the helium atom

The helium atom is one of the simplest systems, where quantum electrodynamics (QED) effects could be studied with the high precision [44]. Several transitions involving S-states in helium have been precisely measured: $2^3S_1 - 2^3P_J$ by Shiner and coworkers in [45], $2^3S_1 - 3^3P_J$ by Inguscio and coworkers in [46], $2^3S_1 - 3^3D_1$ by the Paris group in [47], and $1^1S_0 - 2^2S_0$ by Bergeson *et al.* in [48]. From the last but one measurement authors derived 2^3S_1 Lamb shift which is two orders more precise than the current theoretical predictions.

The presence of the electron–electron interaction is an essential ingredient in the comparison of helium to hydrogenic systems and brings QED calculations to the next level of difficulty. A long standing problem is the treatment of the electron–electron interaction on the same footing as the electron–nucleus interaction. Commonly accepted Bethe-Salpeter formalism [15] is a well starting point for the derivation of various QED corrections in the helium atom. This treatment has been developed by Araki [49] and independly by Sucher [50] for the complete calculation of $m\,\alpha^5$ correction to energy levels. Although, this correction could be presented in a compact form, the calculation of the one term: the Bethe–logarithm, has not yet been performed with the satisfactory precision. The extension to the next order of $m\,\alpha^6$ has been achieved by Douglas and Kroll [51] but only for the fine splitting between triplet P-levels. Using the Bethe-Salpeter equation they have derived corresponding nonrelativistic operators, which can be treated in a standard perturbation calculus using Schrödinger wave functions.

These operators are the extension of Breit Hamiltonian [52] to the order of $m\,\alpha^6$. Usually, such an nonrelativistic expansion gives a very singular operators which could not for example be evaluated in the second order of perturbation calculus. However, triplet P-state wave functions are much more regular at small distances, making matrix elements of these operators finite at order $m\,\alpha^6$ for the difference between P-levels.

It is proposed the calculation of helium energy of S–levels through the effective Hamiltonian approach with the regularized photon exchange propagator, like it has been done for the positronium case. I have already derived in [53] a complete set of operators for the calculation of energies of helium triplet n^3S_1-states at order $m\,\alpha^6$ and demonstrated the cancelation of all divergences. The similar set of operators for singlet S-states is presented in [54]. However, the treatment for singlet states is much more complex. One has to regularize the electron–electron interaction to keep matrix elements finite. The divergence in some chosen parameter λ is canceled by the terms derived from the high energy scattering amplitude, which is the same as in positronium. For helium the effective Hamiltonian is a sum of electron-nucleus, electron–electron and three–body terms. The complete expression is quite long, however the electron–nucleus part is quite simple. It is derived from the Dirac equation only and reads

$$
\begin{aligned}
H_{EN} &= H_{EN}^{(4)} + H_{EN}^{(6)}\,, && (36)\\
H_{EN}^{(4)} &= -\frac{p^4}{8\,m^3} + \frac{Z\,\alpha\,\pi}{2\,m^2}\,\delta^3(r)_\lambda\,,\\
H_{EN}^{(6)} &= \frac{p^6}{16\,m^5} - \frac{1}{8\,m^3}\left[\mathbf{p};\left(\frac{Z\,\alpha}{r}\right)_\lambda\right]^2 - \frac{5}{128\,m^4}\left[p^2;\left[p^2;\left(\frac{Z\,\alpha}{r}\right)_\lambda\right]\right]\\
&\quad -\frac{3}{64\,m^4}\left(p^2\left[\mathbf{p};\left[\mathbf{p};\left(\frac{Z\,\alpha}{r}\right)_\lambda\right]\right] + \left[\mathbf{p};\left[\mathbf{p};\left(\frac{Z\,\alpha}{r}\right)_\lambda\right]\right]p^2\right). && (37)
\end{aligned}
$$

The subscript λ denotes regularization according to prescription in (22). Using this Hamiltonian one could derive the hydrogenic energy levels up to the order $m\,\alpha^6$. For the helium atom, although we have the complete effective Hamiltonian the problem is with the calculation of its matrix elements. The wave function is found only numerically using the variational approach. However, relativistic corrections are sensitive to the behaviour of this wave function at the origin, what requires a large basic set for their calculation. Author is currently pursuing this project.

Recently particular attention has been focussed on the fine structure of n^3P_J-levels. Several experimental groups are attempting to measure it with a precision allowing for the determination of the fine structure constant α with a precision better than that available from solid state experiments. This will require the calculation of $m\,\alpha^7$ corrections, which is a highly nontrivial task.

5 Summary

Quantum electrodynamics gives a self–consistent description of atomic systems. It has been successfully developed for hydrogenic atoms, such us hydrogen, muonium and positronium. Its predictions in most cases are in agreement with the experimental results. However a few disagreements, specifically the He^+ Lamb shift and *hfs* and decay rates in positronium, remain to be resolved. The QED of many electron atoms has not yet been as well developed. Particularly interesting is the helium atom. Theoretical predictions for helium energy levels are an order of magnitude less precise, because of difficulties in calculating multi photon exchange contributions. I believe the effective Hamiltonian approach, which was briefly presented here, is the most suitable method for the calculation of these contributions.

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