Electrodynamics of the finite size particle with an arbitrary spin

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We construct the most general Hamiltonian for the electromagnetic interaction of the finite size particle like nucleus with an arbitrary spin, magnetic dipole, and electric quadrupole moments. It includes all the terms, which are important for obtaining atomic energy levels up to the order α^6. The result is verified against spin $s = 0, 1/2$, and 1 cases, where the Foldy-Wouthuysen transformation is performed of the corresponding relativistic equation.

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I. INTRODUCTION

In the calculation of atomic properties, the nucleus is usually treated as a static source of the electromagnetic field, and its kinetic energy is being neglected. In some cases, for example in the determination of the isotope shift in atomic transition frequencies, the finite nuclear mass effects are included, but only in the nonrelativistic (nucleus) approximation, Only in some special cases, relativistic nuclear recoil corrections become visible and important. It is when the orbiting electron is replaced by a heavier particle, like the muon or the antiproton. The orbiting particle and the nucleus may have a comparable mass and thus shall be treated on equal footing, for example in the muonic hydrogen [1] or in the antiprotonic helium ion [2]. It is not obvious however, how to describe the bound system involving finite size particles with an arbitrary spin. Here, we aim to construct a nonrelativistic expansion for the electromagnetic interaction of an arbitrary spin particle, which can be used for the accurate description of bound states. The Hamiltonian includes all possible electric and magnetic moments, the finite size and polarizability effects in a perturbative manner. Namely, we assume that the characteristic wavelength of the electromagnetic field is much larger than the particle size and the Compton wavelength. This make possible the expansion in powers of particle momenta and derivatives of the electromagnetic field. The main result of this paper is a construction of the universal Hamiltonian of an arbitrary spin particle, including all the couplings to the electromagnetic field, which contribute to atomic energy levels up to the order α^6. This Hamiltonian is presented below and is split for convenience into three parts, consisting of no-spin, single-spin and double-spin terms,

$$H = H_1 + H_2 + H_3\,, \tag{1}$$

These parts are the following:

$$\begin{aligned}
H_1 &= eA_0 + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} - \frac{e}{6}\,\langle r_E^2\rangle\,\vec\nabla\vec E - \frac{e}{120}\,\langle r_E^4\rangle\,\nabla^2\vec\nabla\vec E \\
&\quad + \frac{e\,(g-1/2)}{24\,m^4}\,s(s+1)\{\pi^2,\vec\nabla\vec E\} + \frac{ie}{32m^4}\Big(1+\frac{s(s+1)}{3}\Big)\big[\pi^2,\vec\pi\,\vec E+\vec E\,\vec\pi\big] \\
&\quad - \frac{e}{12\,m}\Big(\langle r_E^2\rangle - \frac{g}{2\,m^2}\,s\,(s+1)\Big)\big\{\vec\pi,\partial_t\vec E-\vec\nabla\times\vec B\big\}\,, \qquad (2)\\
H_2 &= -\frac{e\,g}{2\,m}\,\vec s\,\vec B + \frac{e}{8m^3}\big(\{\pi^i,\{\pi^i,\vec s\,\vec B\}\} + (g-2)\,\{\vec\pi\,\vec B,\vec\pi\,\vec s\}\big) - \frac{e}{6m}\,\langle r_M^2\rangle\,\vec s\,\nabla^2\vec B \\
&\quad - \frac{e\,(g-1)}{4m^2}\,\vec s\,(\vec E\times\vec\pi-\vec\pi\times\vec E) + \frac{e\,(g-1/2)}{16m^4}\,\vec s\,\{\pi^2,\vec E\times\vec\pi-\vec\pi\times\vec E\} \\
&\quad - \frac{e}{4m^2}\bigg(\frac{\langle r_M^2\rangle}{3} - \frac{\langle r_E^2\rangle}{6} + \frac{Q_E}{12} - \frac{1}{8m^2}\bigg)\vec s\,\big(\nabla^2\vec E\times\vec\pi-\vec\pi\times\nabla^2\vec E\big)\,, \qquad (3)\\
H_3 &= -\frac{e}{2}\,Q_E\,(s^is^j)^{(2)}\Big(\nabla^jE^i + \frac{1}{2m}\big\{\pi^k,\delta^{ik}\partial_tE^j+\epsilon^{ikl}\nabla^jB^l\big\}\Big) - \frac{e}{12}\,Q_E\,\langle r_Q^2\rangle\,(s^is^j)^{(2)}\nabla^2\nabla^jE^i \\
&\quad - \frac{e\,g}{8m^3}(s^is^j)^{(2)}\,\epsilon^{ikl}\,\big(\{\pi^k,\nabla^jB^l\}-\{\pi^j,\nabla^kB^l\}\big) - \frac{e}{30}\,Q_M(s^is^js^k)^{(3)}\nabla^j\nabla^kB^i \\
&\quad - \frac{e\,(g-1/2)}{8\,m^4}\,(s^is^j)^{(2)}\,\big(2\,m\,\{\pi^i,\partial_tE^j\} + 2\,\pi^i\,\vec\nabla\vec E\,\pi^j + \{\pi^2,\nabla^iE^j\}\big) \\
&\quad - \frac{e^2}{2}\,E^iE^j\big(\alpha_E\delta^{ij}+\gamma_E(s^is^j)^{(2)}\big) - \frac{e^2}{2}\,B^iB^j\big(\alpha_M\delta^{ij}+\gamma_M(s^is^j)^{(2)}\big)\,. \qquad (4)
\end{aligned}$$

where: $[X\,,Y] \equiv X\,Y - Y\,X$, $\{X\,,Y\} \equiv X\,Y + Y\,X$, $\vec{\pi} = \vec{p} - e\,\vec{A}$, e is the charge, m is the mass, and s is the spin, g is the gyromagnetic factor of the considered particle, $e\,Q_E$, $e\,Q_M$ are the electric quadrupole, magnetic octupole moments respectively, $\langle r_E^2\rangle$ and $\langle r_M^2\rangle$ are the square of the charge and the magnetic radii respectively, $\langle r_Q^2\rangle$ is the square of the quadrupole radius, $\langle r_E^4\rangle$ is the averaged fourth power of the charge radius, $e^2\,\alpha_E$, $e^2\,\gamma_E$, $e^2\,\alpha_M$, $e^2\,\gamma_M$ are the electric, magnetic scalar and tensor polarizabilities. For an arbitrary spin, Q_E should be replaced by $Q_E/(s\,(2\,s-1))$, but for simplicity we use normalization as for $s=1$. The irreducible second $(s^i\,s^j)^{(2)}$ and third order $(s^i\,s^j\,s^k)^{(3)}$ tensors in Eq. (4) are, by definition, symmetric with respect to all indices and traceless, for example

$$(s^i\,s^j)^{(2)} = \frac{s^i\,s^j + s^j\,s^i}{2} - \frac{\delta^{ij}}{3}\,s^k\,s^k\,. \tag{5}$$

There are no further constants involved in Eqs. (2-4), as the requirement of the Lorentz symmetry fixes the form of couplings to the electromagnetic field. In particular, the terms which are linear in spin can be obtained from the Bargman-Michel-Telegdi (BMT) equation, and such derivation was presented in Ref. [3]. However, the terms which are quadratic in spin, can not be derived from the classical equation of motion, or at least we have not been able to do so. The results, which we obtained in the former work [4] on the basis of Ref. [3] is in disagreement with that obtained here from the Foldy-Wouthuysen (FW) transformation of the most general spin $s=1$ relativistic equation.

The form of the Hamiltonian in Eqs. (2-4) is not unique. It can be a subject of the unitary transformation

$$\tilde{H} = e^{-i\,S}\,(H - i\,\partial_t)\,e^{i\,S}\,, \tag{6}$$

for example, let S be of the form

$$S = \epsilon_{ij}\,\frac{e}{m^3}\,\{\pi^i\,,E^j\} \tag{7}$$

with some arbitrary constant tensor ϵ_{ij}, then the transformed Hamiltonian up to the neglected terms of higher orders in S is

$$\begin{aligned}\tilde{H} &= H + i\,[H - i\,\partial_t, S] \\ &= H + \epsilon_{ij}\,\frac{e}{m^3}\,\{\pi^i\,,\partial_t E^j\} + \epsilon_{ij}\,\frac{2\,e}{m^3}\,E^i\,E^j \\ &\quad + \epsilon_{ij}\,\frac{i\,e}{2\,m^4}\,\big[\vec{\pi}^2, \{\pi^i\,,E^j\}\big]\,. \end{aligned} \tag{8}$$

The Hamiltonian in Eqs. (2-4) can be regarded as the definition of the nuclear charge radius and nuclear polarizabilities. In fact, there are several different definitions of nuclear charge radii in the literature [5], depending on the nuclear spin and there is no a uniquely accepted one. Moreover, there is no established definition of the electric polarizability of nuclei (as any charged particle), because α_E depends on the form of the other interaction terms in the Hamiltonian, namely one can use the unitary transformation in Eq. (8) to eliminate $\vec{\pi}\,\partial_t\vec{E}$ at the expense of appearance of an additional $\vec{E}^2$ term, which contributes to the polarizability.

We will justify below the Hamiltonian in Eqs. (2-4) by considering three examples of the scalar $s=0$, spinor $s=1/2$ and vector $s=1$ particles with all possible electromagnetic moments. We will perform FW transformation of the corresponding relativistic equation and show that the obtained FW Hamiltonian is a particular case of that in Eqs. (2-4).

II. FOLDY-WOUTHUYSEN TRANSFORMATION FOR $S=0$

The scalar charged particle satisfies the Klein-Gordon equation

$$\big[(\partial_\mu + ieA_\mu)^2 + m^2\big]\phi = 0\,, \tag{9}$$

which in the three-dimensional notation takes the form

$$\big[(\partial_0 + ieA_0)^2 - (\nabla_j + ieA_j)^2 + m^2\big]\phi = 0. \tag{10}$$

In order to perform the FW transformation we follow Ref. [6] and introduce the ξ field

$$\xi = (\partial_0 + i\,eA_0)\phi \tag{11}$$

to rewrite the Klein-Gordon equation into the set of two first order differential equations

$$\begin{aligned}\partial_0\phi &= \xi - i\,eA_0\,\phi\,, \\ \partial_0\xi &= -(\pi^2 + m^2)\,\phi - i\,eA_0\xi\,. \end{aligned} \tag{12}$$

We introduce now another two fields which will correspond to the particle and the antiparticle

$$\begin{aligned}\theta &= \frac{1}{2}\big(\phi + \frac{i}{m}\,\xi\big)\,, \\ \chi &= \frac{1}{2}\big(\phi - \frac{i}{m}\,\xi\big)\,. \end{aligned} \tag{13}$$

They satisfy the following set of equations

$$\begin{aligned}i\partial_0\theta &= -\frac{\pi^2}{2m}\,(\theta + \chi) + eA_0\theta + m\theta\,, \\ i\partial_0\chi &= \frac{\pi^2}{2m}\,(\theta + \chi) + eA_0\theta - m\chi\,. \end{aligned} \tag{14}$$

Let us define the following three matrices

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{15}$$

and Φ

$$\Phi = \begin{pmatrix} \theta \\ \chi \end{pmatrix}, \tag{16}$$

then this set of equation in a matrix form is

$$i\,\eta\,\partial_t\Phi = \left[(1+\sigma)\,\frac{\vec{\pi}^2}{2m} + m + e\,\eta\,A_0\,\phi\right]\Phi \equiv H\,\Phi \quad (17)$$

Let us introduce a transformation U

$$\Phi = U\,\Phi' \quad (18)$$

such that

$$U^+\,\eta\,U = \eta \;\Rightarrow\; U^+\,\eta = \eta\,U^{-1} \quad (19)$$

then the transformed equation preserves its form with the new Hamiltonian H' being

$$H' = U^+\,(H - i\,\eta\,\partial_t)\,U\,. \quad (20)$$

The transformation $U = e^S$ with

$$S = \sigma\,\frac{\pi^2}{4\,m} \quad (21)$$

is chosen to eliminate off diagonal terms from H up to the specified order in $1/m$. It is Hermitian, $S^+ = S$, and satisfies condition $S\,\eta = -\eta S$, so the Eq. (19) holds, and the transformed Hamiltonian H' neglecting higher order odd and even terms is

$$H' = m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} + e\,\eta\,A_0 + \frac{ie\,\eta}{32m^4}[\pi^2, \{\vec{\pi}, \vec{E}\}]. \quad (22)$$

We consider now only particles (no antiparticles), so $\eta \to 1$ and the transformed Hamiltonian H, dropping the symbol $'$ and the rest mass, is

$$H = e\,A_0 + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} + \frac{ie}{32m^4}[\pi^2, \{\vec{\pi}, \vec{E}\}]\,. \quad (23)$$

It coincides with the Hamiltonian in Eq. (1) for $s = 0$, and $r_E^2 = r_E^4 = r_M^2 = \alpha_E = \alpha_M = 0$.

Inclusion of the particle finite size corresponds to the replacement $A^\mu \to \tilde{A}^\mu$, where

$$\begin{aligned} \tilde{A}^\mu(q) &= \varrho\left(\frac{q^2}{m^2}\right) A^\mu(q) &(24)\\ &= \left[1 + \varrho'(0)\,\frac{q^2}{m^2} + \varrho''(0)\,\frac{q^4}{2\,m^4} + \ldots\right] A^\mu(q). \end{aligned}$$

In the coordinate representation

$$\tilde{A}_\mu(x) = \left[1 - \varrho'(0)\frac{\Box}{m^2} + \varrho''(0)\frac{\Box^2}{2!\,m^4} + \ldots\right] A^\mu(x)\,, \quad (25)$$

where $\Box = \partial^\mu\partial_\mu$. The Hamiltonian $\tilde{H}$ including the finite size is therefore

$$\begin{aligned} \tilde{H} &= H + e\,(\tilde{A}^0 - A^0) - \frac{e}{2\,m}\left\{\pi^i\,,\,\tilde{A}^i - A^i\right\} &(26)\\ &= H - \frac{e\,\langle r_E^2\rangle}{6}\,\vec{\nabla}\cdot\vec{E} - \frac{e\,\langle r_E^4\rangle}{120}\,\nabla^2\,\vec{\nabla}\cdot\vec{E} - \frac{e\,\langle r_E^2\rangle}{6}\Bigg[\\ &\quad \vec{\nabla}\cdot\frac{\partial\vec{A}}{\partial t} + \frac{\partial^2 A^0}{\partial t^2} - \frac{1}{2\,m}\left\{\pi^i, \left(\vec{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right)A^i\right\}\Bigg] \end{aligned}$$

where we introduced the notation

$$\langle r_E^2\rangle = \frac{6}{m^2}\,\varrho'(0)\,, \quad (27)$$

$$\langle r_E^4\rangle = \frac{60}{m^2}\,\varrho''(0)\,, \quad (28)$$

which corresponds to the averaged quadratic and the fourth power of the charge radius. In order to simplify $\tilde{H}$, one performs the unitary transformation in Eq. (6) with

$$S = \frac{e\,\langle r_E^2\rangle}{6}\,(\vec{\nabla}\cdot\vec{A} + \partial_t\,A^0) \quad (29)$$

and obtains

$$\begin{aligned} \tilde{H} &= H - \frac{e}{6}\,\langle r_E^2\rangle\left(\vec{\nabla}\cdot\vec{E} + \frac{1}{2\,m}\,\{\vec{\pi}\,,\,\partial_t\vec{E} - \vec{\nabla}\times\vec{B}\}\right)\\ &\quad - \frac{e\,\langle r_E^4\rangle}{120}\,\nabla^2\,\vec{\nabla}\cdot\vec{E}\,. &(30) \end{aligned}$$

The most general Hamiltonian for scalar particle including the electric and magnetic polarizabilities is thus

$$\begin{aligned} H &= \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} + eA_0 + \frac{ie}{32m^4}[\pi^2, \{\vec{\pi}\,,\,\vec{E}\}]\\ &\quad - \frac{e\,\langle r_E^2\rangle}{6}\left[\vec{\nabla}\cdot\vec{E} + \frac{1}{2m}\{\vec{\pi}, \partial_t\vec{E} - \vec{\nabla}\times\vec{B}\}\right]\\ &\quad - \frac{e\,\langle r_E^4\rangle}{120}\nabla^2\vec{\nabla}\vec{E} - \frac{\alpha_E}{2}\,e^2\vec{E}^2 - \frac{\alpha_M}{2}\,e^2\vec{B}^2\,. &(31) \end{aligned}$$

It coincides with the Hamiltonian in Eqs. (1-4) by setting $s = 0$.

III. FOLDY-WOUTHUYSEN TRANSFORMATION FOR $S = 1/2$

The next example begins with the minimally coupled spin 1/2 particle, for which Hamiltonian is

$$H = \vec{\alpha}\cdot\vec{\pi} + \beta\,m + e\,A^0\,. \quad (32)$$

The FW transformation of the Dirac equation [7] leads to a new Hamiltonian which does not couple upper and lower components of the Dirac wave function up to some specified order in powers of momenta. The transformed Hamiltonian $\tilde{H}$ is expanded in a power series in S

$$\begin{aligned} \tilde{H} &= H + [H - i\,\partial_t\,,\,i\,S] + \frac{1}{2!}\,[[H - i\,\partial_t\,,\,i\,S]\,,\,i\,S]\\ &\quad + \frac{1}{3!}\,[[[H - i\,\partial_t\,,\,i\,S]\,,\,i\,S]\,,\,i\,S] + \ldots &(33) \end{aligned}$$

As a result of these multiple commutations with adequately chosen S one obtains a nonrelativistic expansion of the Dirac Hamiltonian H [8]

$$\begin{aligned}\tilde{H} &= e\,A^0 + \frac{(\vec\sigma\cdot\vec\pi)^2}{2\,m} - \frac{(\vec\sigma\cdot\vec\pi)^4}{8\,m^3} + \frac{(\vec\sigma\cdot\vec\pi)^6}{16\,m^5} \\ &\quad -\frac{i\,e}{8\,m^2}\,[\vec\sigma\cdot\vec\pi,\vec\sigma\cdot\vec E] - \frac{e}{16\,m^3}\Big\{\vec\sigma\cdot\vec\pi\,,\,\vec\sigma\cdot\partial_t\vec E\Big\} \\ &\quad -\frac{i\,e}{128\,m^4}\,[\vec\sigma\cdot\vec\pi,[\vec\sigma\cdot\vec\pi,[\vec\sigma\cdot\vec\pi,\vec\sigma\cdot\vec E]]] \\ &\quad +\frac{i\,e}{16\,m^4}\Big\{(\vec\sigma\cdot\vec\pi)^2\,,\,[\vec\sigma\cdot\vec\pi,\vec\sigma\cdot\vec E]\Big\}\,. \qquad (34)\end{aligned}$$

One notes that $\tilde H$ is not unique. Different S may lead to a different Hamiltonian but equivalent up to some transformation, for example that in Eq. (7). Hamiltonian $\tilde H$ can be further simplified by reducing powers of $\vec\sigma$, for example

$$(\vec\sigma\cdot\vec\pi)^2 = \vec\pi^{\,2} - e\,\vec\sigma\cdot\vec B \qquad (35)$$

to obtain

$$\begin{aligned}\tilde H &= H_1 + H_2 + H_3\,, \\ H_1 &= eA_0 + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} - \frac{e}{8m^2}\,\vec\nabla\vec E \qquad (36) \\ &\quad + \frac{5\,i\,e}{128\,m^4}\left[\pi^2,\vec\pi\vec E + \vec E\vec\pi\right] + \frac{3}{64m^4}\{\pi^2, e\vec\nabla\vec E\}\,, \\ H_2 &= -\frac{e}{m}\,\vec s\,\vec B - \frac{e}{4m^2}\,\vec s\,(\vec E\times\vec\pi - \vec\pi\times\vec E) \\ &\quad + \frac{e}{8m^3}\{\pi^i,\{\pi^i,\vec s\vec B\}\} - \frac{e}{8m^3}\,\nabla^2(\vec s\vec B) \\ &\quad + \frac{3\,e}{32m^4}\,\vec s\,\{\pi^2,\vec E\times\vec\pi - \vec\pi\times\vec E\}\,, \qquad (37) \\ H_3 &= \frac{e^2}{8m^3}\,\vec E^2 - \frac{e^2}{8\,m^3}\,\vec B^2\,. \qquad (38)\end{aligned}$$

This Hamiltonian is equal to the one in Eqs. (1-4) for $g = 2$, $\langle r_E^2\rangle = \langle r_M^2\rangle = 3/(4\,m^2)$, $\langle r_E^4\rangle = 0$, $\alpha_M = -\alpha_E = 1/(4\,m^3)$, $\gamma_M = \gamma_E = Q_E = 0$. We observe that even the point charge particle has nonvanishing charge and magnetic radii according to definition in Eqs. (2) and (3).

We consider now a point spin 1/2 particle with the magnetic moment anomaly $g = 2\,(1+\kappa)$. Its Hamiltonian is

$$H = \vec\alpha\cdot\vec\pi + \beta\,m + e\,A^0 + \frac{\kappa\,e}{4\,m}\,\beta\,\sigma^{\mu\nu}\,F_{\mu\nu}\,, \qquad (39)$$

and last term can be rewritten to the three dimensional form

$$\sigma^{\mu\nu}\,F_{\mu\nu} = \frac{e\,\kappa}{2\,m}\left(i\,\beta\,\vec\alpha\,\vec E - \vec\Sigma\,\vec B\right). \qquad (40)$$

The FW transformation is similar (but not the same) to the case of the minimally coupled particle and the transformed Hamiltonian takes the form [9]

$$\begin{aligned}\tilde H &= H_1 + H_2 + H_3\,, \\ H_1 &= eA_0 + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} + \frac{\pi^6}{16m^5} - \frac{e(1+2\kappa)}{8m^2}\,\vec\nabla\vec E \\ &\quad + \frac{5\,i\,e}{128m^4}\,[\pi^2,\{\vec\pi\,,\,\vec E\}] - \frac{e\,\kappa}{16m^3}\,\{\vec\pi,\partial_t\vec E - \vec\nabla\times\vec B\} \\ &\quad + \left(1+\frac{4\kappa}{3}\right)\frac{3}{64m^4}\{\pi^2, e\vec\nabla\vec E\}\,, \qquad (41)\end{aligned}$$

$$\begin{aligned}H_2 &= -\frac{e}{m}\,(1+\kappa)\vec s\vec B - \frac{(1+2\kappa)}{4m^2}\,e\,\vec s\,(\vec E\times\vec\pi - \vec\pi\times\vec E) \\ &\quad + \frac{e}{8m^3}\big(\{\pi^i,\{\pi^i,\vec s\vec B\}\} + 2\kappa\{\vec\pi\vec B,\vec\pi\vec s\}\big) \\ &\quad - \frac{e(1+\kappa)}{8m^3}\times\nabla^2(\vec s\vec B) \\ &\quad + \left(1+\frac{4\kappa}{3}\right)\frac{3e}{32m^4}\,\vec s\,\{\pi^2,\vec E\times\vec\pi - \vec\pi\times\vec E\}\,, \qquad (42)\end{aligned}$$

$$H_3 = \frac{e^2}{8m^3}\,(\kappa^2+\kappa+1)\,\vec E^2 - \frac{e^2}{8\,m^3}\,\vec B^2\,. \qquad (43)$$

This Hamiltonian is equal to the one in Eqs. (1-4) for $s = 1/2$, $g = 2(1+\kappa)$, $\langle r_E^2\rangle = 3\,(1+2\,\kappa)/(4\,m^2)$, $\langle r_M^2\rangle = 3\,(1+\kappa)/(4\,m^2)$, $\langle r_E^4\rangle = 0$, $\alpha_M = 1/(4\,m^3)$, $\alpha_E = -(1+\kappa+\kappa^2)/(4\,m^3)$, $\gamma_M = \gamma_E = Q_E = 0$.

The only further generalization includes full electromagnetic formfactors F_1 and F_2.

$$\gamma^\mu A_\mu(q) \rightarrow \left[\gamma_\mu F_1\Big(\frac{q^2}{m^2}\Big) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}\,\kappa F_2\Big(\frac{q^2}{m^2}\Big)\right]A^\mu(q)\,. \qquad (44)$$

The transformed Hamiltonian with electromagnetic formfactors is again equivalent to that in Eqs. (2-4) with constants [9]:

$$\begin{aligned}\langle r_E^2\rangle &= \frac{3}{4m^2}\big(1+2\kappa+8F_1'(0)\big)\,, \qquad (45) \\ \langle r_M^2\rangle &= \frac{3}{4m^2}\big(1+\kappa+8\,F_1'(0)+8\,\kappa F_2'(0)\big)\,, \qquad (46) \\ \langle r_E^4\rangle &= \frac{15}{m^4}\,\big(4\,F_1''(0)+F_1'(0)+2\,\kappa\,F_2'(0)\big)\,, \qquad (47)\end{aligned}$$

and polarizabilities are the same as in the previous case. There are no any further generalization of spin 1/2 interactions which will conserve discrete symmetries like the parity and the time reversal, so we have shown that the Hamiltonian of Eqs. (1-4) is the most general for spin 1/2 particle.

IV. FOLDY-WOUTHUYSEN TRANSFORMATION FOR $S = 1$

The most general Lagrangian for particle with spin $s = 1$, which includes terms linear in electromagnetic field strength F and first derivatives of F is [10]

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\,u^{*\mu\nu}\,u_{\mu\nu} + m^2\,u^{*\mu}\,u_\mu \\ &+\frac{i\,e\,\gamma}{2}\,(u^*_\nu\,u_\nu - u_\mu\,u^*_\nu)\,F^{\mu\nu} \\ &+\frac{i\,e\,q}{4\,m^2}\,(u^*_{\mu\nu}\,u_\lambda - u_{\mu\nu}\,u^*_\lambda)\,\partial^\lambda F^{\mu\nu} \qquad (48)\end{aligned}$$

where:

$$\begin{aligned} u_{\mu\nu} &= \nabla_\mu u_\nu - \nabla_\nu u_\mu\,, &(49)\\ \nabla_\mu &= \partial_\mu + i\,e\,A_\mu\,, &(50)\\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu\,. &(51)\end{aligned}$$

The constants γ and q are related to magnetic dipole and the electric quadrupole moments by:

$$\begin{aligned}\gamma &= g-1\,, &(52)\\ q &= Q_E + g - 1\,, &(53)\end{aligned}$$

and this will be shown later by performing FW transformation. The equation of motion obtained from $\mathcal{L}$ is

$$\begin{aligned}&\nabla_\nu u^{\nu\rho} + m^2\,u^\rho + i\,e\,\gamma\,F^{\rho\nu}u_\nu \\ &-\frac{i\,e\,q}{2\,m^2}\left[\nabla_\mu u_\nu(\partial^\nu F^{\mu\rho} + \partial^\rho F^{\mu\nu}) + u_\nu\partial^\nu\partial_\mu F^{\mu\rho}\right] = 0\,. \qquad (54)\end{aligned}$$

In order to perform FW transformation, we follow the Ref. [10]. The coupling of the vector field u to A^0 is of the form

$$\begin{aligned}\mathcal{L} \approx i\,e\,A^0\,\Big[u^{*i}\Big(u_{0i} - \frac{ieq}{2\,m^2}\,u^\lambda\partial_\lambda F_{0i}\Big) \\ -u^i\Big(u^*_{0i} + \frac{ieq}{2\,m^2}\,u^{*\lambda}\partial_\lambda F_{0i}\Big)\Big]\,. \qquad (55)\end{aligned}$$

we thus introduce the following fields ϕ and ξ

$$\begin{aligned}\phi^i &= \frac{1}{m}\,u^{0i} - \frac{i\,e\,q}{2\,m^3}\,u^\lambda\partial_\lambda F^{0i}\,, &(56)\\ \xi^i &= u^i\,, &(57)\end{aligned}$$

and rewrite equation of motion in terms of them. For this, however, we neglect terms of higher order, compared to that in Eqs. (2-4), namely $\nabla_j(\nabla_k\phi^k\partial^0F^{ji})$, $\nabla^i(\nabla_j(\nabla_k\phi^k\partial^0F^{j0}))$, and $\nabla^i((\nabla_j\xi_k - \nabla_k\xi_j)\partial^0F^{jk})$, so

$$\begin{aligned}\nabla_0\phi^i &= -m\,\xi^i - \frac{1}{m}\,\nabla_j(\nabla^j\xi^i - \nabla^i\,\xi^j) - \frac{ie\gamma}{m}\Big(F^{ij}\,\xi_j \\ &+\frac{1}{m}\,F^{i0}\nabla_j\phi^j\Big) + \frac{ieq}{2m^3}\Big(\nabla_j(\xi^k\nabla_kF^{ji}) \\ &+\frac{1}{2}(\nabla_j\xi_k - \nabla_k\xi_j)\nabla^iF^{jk} + m\phi_j\nabla^iF^{0j}\Big) \qquad (58)\\ \nabla_0\xi^i &= m\,\phi^i + \frac{1}{m^2}\,\nabla^i\,(m\,\nabla^j\phi_j - ie\gamma F^{0j}\,\xi_j) + \frac{ieq}{2m^3}\Big(\\ &\nabla_k\phi^k\partial_0F^{0i} + m\,\xi^k\partial_kF^{0i} + \nabla^i(\phi_j\partial^0F^{0j})\Big) \qquad (59)\end{aligned}$$

and in the following we will implicitly neglect all the higher order terms. Using the field Ψ

$$\Psi^j = \begin{pmatrix}\phi^j + i\,\xi^j \\ \phi^j - i\,\xi^j\end{pmatrix} \qquad (60)$$

the equation of motion is

$$\begin{aligned} i\,\eta\,\nabla_0\Psi^i &= H^{ij}\,\Psi^j\,, \qquad (61)\\ H^{ij} &= -\Big(m - \frac{\nabla^2}{2\,m}\Big)\,\delta^{ij} - \frac{1}{2\,m}\,(\nabla^2\,\delta^{ij} - 2\,\nabla^i\,\nabla^j)\,\sigma \\ &+\frac{i\,e(\gamma+1)}{2\,m}\,F^{ij}\,(1-\sigma) + \frac{e\gamma}{2\,m^2}(F^{0i}\nabla^j \\ &-\nabla^iF^{0j})\,\eta + \frac{e\,\gamma}{2\,m^2}\,(F^{0i}\nabla^j + \nabla^iF^{0j})\,\rho \\ &+\frac{eq}{4\,m^2}\,(\partial^iF^{0j} + \partial^jF^{0i})\,\eta + \frac{ieq}{4\,m^3}\,\big(\nabla^k\partial^jF^{ki} \\ &+\partial^iF^{kj}\nabla^k + \partial^0F^{0i}\nabla^j + \nabla^i\partial^0F^{0j}\big)\,. \qquad (62)\end{aligned}$$

Note that H in the above is Hermitian, and we use the notation where ∇^i is an operator equal to $-\partial/\partial x^i + ieA^i$ and we raised all indices up. In order to obtain the nonrelativistic expansion, we perform the transformation $U = e^S$ in Eqs. (19,20) with S of the form

$$S^{ij} = -\frac{\sigma}{8\,m^2}\left\{\nabla^2\delta^{ij} - 2\,\nabla^i\nabla^j\,,\,1 + \frac{1}{2}\frac{\nabla^2}{m^2} + \frac{1}{3}\frac{\nabla^4}{m^4}\right\}. \qquad (63)$$

The new Hamiltonian $\tilde{H} = U^+(H - i\,\eta\partial_t)U$ is

$$\begin{aligned}\tilde{H}^{ij} &= m\Big(-1 + \frac{1}{2}\,\frac{\nabla^2}{m^2} + \frac{1}{8}\frac{\nabla^4}{m^4} + \frac{1}{16}\,\frac{\nabla^6}{m^6}\Big)\,\delta^{ij} \\ &+\frac{ie(\gamma+1)}{2\,m}\,F^{ij} + \frac{e\,\gamma}{2\,m^2}\,\eta\,(F^{0i}\,\nabla^j - \nabla^i\,F^{0j}) \\ &+\frac{eq}{4\,m^2}\,\eta\,\big(\partial^iF^{0j} + \partial^jF^{0i}\big) + \frac{ie}{4\,m^3}\,\big(\nabla^kF^{ik}\nabla^j \\ &+F^{ik}\nabla^k\nabla^j - \nabla^iF^{jk}\nabla^k - \nabla^i\nabla^kF^{jk}\big) \\ &+\frac{ie(\gamma+1)}{8\,m^3}\big(\tilde{\nabla}^2\,F + F\,\tilde{\nabla}^2\big)^{ij} + \frac{ieq}{4\,m^3}\,\big(\nabla^k\partial^jF^{ki} \\ &+\partial^iF^{kj}\nabla^k\big) + \frac{e\gamma}{8\,m^4}\,\eta\,\big[\tilde{\nabla}^2\,,\,F\nabla\big]^{ij} \\ &+\frac{e}{32\,m^4}\,\eta\,\big[\nabla^kF^{0k} + F^{0k}\nabla^k - 2\,F\nabla\,,\,\tilde{\nabla}^2\big] \\ &+\frac{ieq}{4\,m^3}\big(\partial^0F^{0j}\,\nabla^j + \nabla^i\,\partial^0F^{0j}\big) \qquad (64)\end{aligned}$$

where:

$$\begin{aligned}(\tilde{\nabla}^2)^{ij} &= \nabla^2\,\delta^{ij} - 2\,\nabla^i\,\nabla^j\,, &(65)\\ (F\nabla)^{ij} &= F^{0i}\nabla^j + \nabla^iF^{0j}\,. &(66)\end{aligned}$$

The Hamiltonian for particles is the one with $\eta \to -1$, and because of Eq. (61), $H \to -H$. Following [10], we replace in Eq. (64) Cartesian indices by spin 1 operators. Namely, the antisymmetric operator F is represented as

$$F \to \hat{F} = \frac{i}{2}\,\epsilon^{ijk}\,F^{ij}\,s^k, \qquad (67)$$

the symmetric traceless operator Q as

$$Q \to \hat{Q} = -Q^{ij}\,(s^i\,s^j)^{(2)} \tag{68}$$

and unity operator is replaced by 1. Further on, the electromagnetic tensor F in Eq. (64) in terms of B and E fields is $F^{ij} = -\epsilon^{ijk}\,B^k$, and $F^{0i} = -E^i$, constants γ, q are expressible in terms of g, Q_E, see Eqs. (52,53), and the Hamiltonian for spin 1 particle becomes

$$\begin{aligned}
H \;=\;& e\,A^0 + \frac{\pi^2}{2\,m} - \frac{\pi^4}{8\,m^3} + \frac{\pi^6}{16\,m^5} - \frac{e\,g}{2\,m}\,\vec{B}\,\vec{s} - \frac{e\,Q_E}{2\,m^2}\,\partial_i E^j\,(s^i\,s^j)^{(2)} \\
&+\frac{e\,Q_E}{6\,m^2}\,\vec{\nabla}\vec{E} - \frac{e\,(g-1)}{4\,m^2}\,(\vec{E}\times\vec{\pi} - \vec{\pi}\times\vec{E})\,\vec{s} - \frac{e\,(Q_E+g)}{12\,m^3}\,\{\vec{\pi}\,,\,\vec{\nabla}\times\vec{B}\} \\
&+\frac{e\,s^l}{8\,m^3}\,\left(\{\vec{\pi}\,,\,\{\vec{\pi}\,,\,B^l\}\} + (g-2)\,\{\pi^l,\vec{\pi}\,\vec{B}\} + (Q+g-1)\,\nabla^2(B^l)\right) \\
&-\frac{e}{4\,m^3}\,(s^i\,s^k)^{(2)}\,\epsilon^{ijl}\,\Big(Q_E\,\{\pi^j\,,\,\partial_k B^l\} - \frac{g}{2}\big(\{\pi^k\,,\,\partial_j B^l\} - \{\pi^j\,,\,\partial_k B^l\}\big)\Big) \\
&-\frac{i\,e}{24\,m^4}\,\Big(Q_E + g - \frac{5}{4}\Big)\,\big[\pi^2\,,\vec{\pi}\vec{E} + \vec{E}\vec{\pi}\big] + \frac{e}{12\,m^4}\,\Big(g - \frac{1}{2}\Big)\,\Big(\frac{1}{2}\nabla^2(\vec{\nabla}\vec{E}) + \{\pi^2\,,\vec{\nabla}\vec{E}\}\Big) \\
&+\frac{e}{8\,m^4}\,\Big(g-\frac{1}{2}\Big)\,s^l\,\Big(i\,\epsilon^{ikl}\,\pi^i\,\vec{\nabla}\vec{E}\,\pi^k - \frac{1}{2}\,\{\pi^2\,,\vec{\pi}\times\vec{E} - \vec{E}\times\vec{\pi}\}^l\Big) - \frac{e}{8\,m^4}\,(s^i\,s^k)^{(2)} \\
&\Big(i\,\Big(Q_E + g - \frac{1}{2}\Big)\,\big[E^i\,\pi^k + \pi^i\,E^k\,,\pi^2\big] + 2\,\Big(g-\frac{1}{2}\Big)\,\pi^i\,\vec{\nabla}\vec{E}\,\pi^k + \Big(g-\frac{1}{2}\Big)\,\{\pi^2\,,\partial_i E^k\}\Big).
\end{aligned} \tag{69}$$

Here we returned to the standard notation, namely $\vec{\nabla}$ denotes the differentiation $\nabla^i \equiv \partial/\partial x^i$ of the term which follows. The Hamiltonian in Eq. (69) is equivalent to that in Eq. (1) up to the transformation (8) for $s = 1$,

$$\langle r_E^2\rangle \;=\; -\frac{Q_E}{m^2}\,, \tag{70}$$

$$\langle r_E^4\rangle \;=\; -\frac{5}{m^4}\,\Big(g - \frac{1}{2}\Big)\,, \tag{71}$$

$$\langle r_M^2\rangle \;=\; -\frac{3}{4\,m^2}\,(Q_E + g - 1)\,. \tag{72}$$

and $Q_M = r_Q = 0$. Polarizabilities are not determined, as we have been neglecting (for simplicity) during the course of calculations all the terms quadratic in the field strength. We have shown thus, that the Hamiltonian in Eqs. (1-4) is the most general one for a particle with the spin $s = 1$.

V. SUMMARY

We have constructed, what we think, the most general Hamiltonian for the finite size particle with an arbitrary spin and electromagnetic moments. It consists of all the terms which contribute to atomic energy levels up to the order α^6. The corresponding derivation of the effective interaction potential up to this order is presented in Ref. [8]. The obtained Hamiltonian involves several constants characterizing the particle: mass, charge, g-factor, and also charge and magnetic radii, polarizability. We point out nevertheless, that there is no consistent definition of charge radii valid for all the nuclei. For spin $s = 1/2$, the Darwin term $3/(4\,m^2)$ is usually subtracted from r_E^2, so the point particle has the charge radius $r_E = 0$. However it is not obvious what is the point $s = 1$ particle, or in other words what is its quadrupole moment. If we assume that $g = 2$ and $q = 0$, then $Q_E = -1$ and for point particle $r_E^2 = 1/m^2$, but this has not yet been widely accepted. For higher spins Khriplovich and coworkers in [11] assumed some definition and has found the general formula for the Darwin term. We stress, however that for higher spins a problem exists with construction of the relativistic and the causal differential equation.

A similar Hamiltonian to that in Eqs. (1-4) with terms of lower order only, have already been presented in the literature, also recently in Refs. [4, 12] where the problem of nuclear recoil correction to the bound electron g-factor has been considered. Our goal however, is the accurate description of exotic atoms, where electron is replaced by a heavier particle, such as the muon or the antiproton. To some extend one can use the Dirac equation, but then finite nuclear mass corrections are not correctly included. The only way, in our opinion, is the NRQED approach (see implementation in Ref. [8] and references therein), which allows one for consistent inclusion of all effects order by order in powers of α. Such exotic systems, like muonic hydrogen, muonic helium or antiprotonic helium ions are currently under investigation. In muonic hydrogen, however, we observe significant, yet unexplained discrepancies with theoretical predictions [1], so it is certainly worth to investigate two-body relativistic systems to a high accuracy.

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