SUPERSYMMETRY
Problems: set 1

November 6, 2008

1. Find the spectrum of the fermionic oscillator:

$$\{a, a^\dagger\} = 1,\ H = \tfrac{1}{2}(aa^\dagger - a^\dagger a)\omega.$$

2. Let the superpotential W be of the form $W = ax^3,\ a > 0$. Write down the supersymmetric partner potentials and plot them.

3. Let the ground state of a supersymmetric Hamiltonian be $\psi_0 = Ax^5 e^{-\beta x}$. Write down W, V_1, V_2, make the plots.

4. Consider $W = ax^2 + bx + c,\ \ a, b, c > 0$. Is supersymmetry broken or not? Assume $a = 1/5, b = 1, c = 0$. Plot the partner potentials, compare the spectra (no need to solve the potentials explicitly).

5. Prove the following identities:

$$(\sigma^\mu)_{\alpha\dot\beta}(\bar\sigma_\mu)^{\dot\gamma\delta} = 2\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}$$
$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$$
$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\beta^\alpha = 2\eta^{\mu\nu}\delta_\beta^\alpha$$
$$tr(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu}$$

6. Demonstrate the decomposition $(1/2, 0) \otimes (0, 1/2) = (1/2, 1/2)$:

$$\psi_\alpha\bar\chi_{\dot\alpha} = \tfrac{1}{2}(\sigma^\mu)_{\alpha\dot\alpha}(\psi\sigma_\mu\bar\chi).$$

7. Find explicitly the form of the conserved supercurrent in the case of interacting supersymmetric quantum mechanics.

8. Consider 2×2 complex matrices M belonging to the $SL(2,\mathcal{C})$ group: $\det(M) = 1$. They act on left spinors as follows: $\psi \rightarrow \psi'_\alpha = M_\alpha^\beta \psi_\beta$. Show that $\epsilon^{\alpha\beta} = \epsilon^{\gamma\delta} M_\gamma^\alpha M_\delta^\beta$. Furthermore, demonstrate the mapping from $SL(2,\mathcal{C})$ to $SO(1,3)$:

$$\Lambda_\nu^\mu = \tfrac{1}{2} tr(\bar{\sigma}^\mu M \sigma_\nu M^\dagger).$$

9. Show that $e^{\frac{1}{2} i\vec{\theta}\vec{\sigma} \pm \frac{1}{2}\vec{\eta}\vec{\sigma}}$, with real parameters θ^i, η^i, belongs to $SL(2,\mathcal{C})$.

10. Define $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} V_\mu$, where V_μ transforms as a four-vector. Express V_μ through $V_{\alpha\dot{\alpha}}$.

11. Demonstrate that:

$$\eta\sigma^{\mu\nu}\psi = -\psi\sigma^{\mu\nu}\eta$$
$$\bar{\chi}\bar{\sigma}^\mu\psi = -\psi\sigma^\mu\bar{\chi}$$
$$(\sigma^{\mu\nu})_\beta^\alpha(\sigma^{\mu\nu})_\gamma^\delta = \epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta_\alpha^\delta\delta_\gamma^\beta$$

12. Using the results of the previous problem show the decomposition $(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0)$:

$$\psi_\alpha\chi_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\psi\chi + \tfrac{1}{2}(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi).$$

13. Show that

$$(\theta\phi)(\bar{\chi}\bar{\xi}) = -\tfrac{1}{2}(\theta\sigma^\mu\bar{\xi})(\bar{\chi}\bar{\sigma}_\mu\phi)$$
$$\theta^\alpha\theta_\beta = \tfrac{1}{2}(\theta\theta)\delta_\beta^\alpha$$
$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \tfrac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta})$$

14. Write down the N-extended supersymmetry algebra with central charges in four-component spinor notation. Show explicitly that it reduces down to the two-component form. Use Majorana spinors.

15. Write explicitly general $N = 1$, $N = 2$ supersymmetry multiplets which contain (a) massive particles of maximum spins 1/2, 1, 3/2 and (b) massless particles with maximum helicities 1, 3/2, 2. Take note of the CPT invariance.