Black hole entropy from Quantum Geometry

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PLAN OF THE TALK

- The Ashtekar-Baez-Corichi-Krasnov QBH
- Combinatoric formulation of the QBH entropy calculation
- Results:
  - Constribution of higher spins is not negligible
  - A correct value of entropy from the ABCK model
  - The puncture spin statistics
- The status of the Quasinormal Modes relation
- What else could we count?

The issue of a quantum black hole from Quantum Geometry was raised by Krasnov (1998) and Rovelli (1996). The (classical and) quantum Isolated Horizon framework was introduced by Ashtekar, Baez, Corichi and Krasnov (1999, 2000).

The classical phase space: the set of space-times containing an isolated horizon, a location of the horizon and the area a are fixed.

In the 3+1 framework: horizon is represented by a 2-sphere S, the bulk is a 3-manifold Σ bounded by S.
The degrees of freedom:
- a $U(1)$ spin connection defined on the horizon S
- gravitational field data in Σ

such that a consistency condition is satisfied ensuring that the world-surface of S is an isolated horizon.

The quantum Hilbert space: $\mathcal{H}_{\rm kin} \subset \mathcal{H}_S \otimes \mathcal{H}_\Sigma$

$\mathcal{H}_S$: the horizon Hilbert space is understood as the space of the black-hole quantum states. Mathematically, this is the union of the quantum $U(1)$ Chern-Simons theories on a punctured sphere, where all possible sets of punctures are admitted.
$\mathcal{H}_\Sigma$: the bulk Hilbert space is described by Quantum Geometry, consists of excitations of the 3-geometry, which define the quantum area of the horizon.

The quantum consistency condition is equivalent to the quantum Gauss constraint in $\mathcal{H}_S \otimes \mathcal{H}_\Sigma$ at S.

The **quantum constraints** commute with the quantum horizon area operator. All the solutions whose quantum areas fall into any given finite interval $[a-\delta a, a+\delta a]$ can be labeled by a finite number of the quantum black hole states and bulk labels.

Consider a finite set of points,

$$\mathcal{P} \ = \{p_1, ..., p_n\} \subset S, \tag{1}$$

and a labeling by numbers $j = (j_1, ..., j_n)$, and $m = (m_1, ..., m_n)$ where

$$0 \not\models \ j_i \in \frac{1}{2}\mathbb{N}, \qquad m_i \in \{-j_1, ..., j_i\} \tag{2}$$

The space $\mathcal{H}_\Sigma$ is the orthogonal sum

$$\mathcal{H}_\Sigma \ = \ \bigoplus_{(\mathcal{P},j,m)} \mathcal{H}_\Sigma^{\mathcal{P},j,m}, \tag{3}$$

where $\mathcal{P}$ runs through all the finite subsets of S, (j, m) through all the finite labelings (2).

The meaning of the quantum numbers j_i, m_i: consider a piece $S' \subset S$

$$\text{area of } S' : \ a_{S'}^{\mathcal{P},j} \ = \ 8\pi\gamma\ell_{\text{Pl}}^2 \sum_{p_i \in \mathcal{P}\cap S'} \sqrt{j_i(j_i+1)}, \tag{4}$$

$$\text{flux across } S' : \ e_{S'}^{\mathcal{P},m} \ = \ 8\pi\gamma\ell_{\text{Pl}}^2 \sum_{p_i \in \mathcal{P}\cap S'} m_i \tag{5}$$

where only those points $p_i \in \mathcal{P}$ contribute which are contained also in S', and $\gamma > 0$ is a free parameter of Quantum Geometry known as Barbero-Immirzi parameter.

In particular, the area and flux of the horizon S are

$$a_S^j \ = \ 8\pi\gamma\ell_{\text{Pl}}^2 \sum_i \sqrt{j_i(j_i+1)}, \tag{6}$$

$$e_{S'}^{\mathcal{P},m} \ = \ 8\pi\gamma\ell_{\text{Pl}}^2 \sum_i m_i. \tag{7}$$

III. THE HORIZON GEOMETRY, $\mathcal{H}_S$, THE FULL $\mathcal{H}_{\rm KIN}$

In order to quantize consistently the horizon degrees of freedom, it is assumed that the fixed classical area a is,

$$a \;=\; 4\pi\gamma\ell_{\rm Pl}^2 k, \quad k\in\mathbb{N}. \tag{8}$$

where k is arbitrary.
Orthogonal decomposition:
labels:

$$\vec{\mathcal{P}} \;=\; (p_1, ..., p_n) \quad b \;=\; (b_1, ..., b_n)\,, \tag{9}$$

$$b_i \;\in\; \mathbb{Z}_k, \quad p_i \in S \quad \sum_{i=1}^{n} b_i = 0\,. \tag{10}$$

the decomposition:

$$\mathcal{H}_S = \bigoplus_{(\vec{\mathcal{P}},b)} \mathcal{H}_S^{\vec{\mathcal{P}},b}, \tag{11}$$

$$\dim\mathcal{H}_S^{\vec{\mathcal{P}},b} \;=\; 1. \tag{12}$$

The meaning of b_is:

$$\text{holonomy}(\delta S') \;=\; \prod_{i:p_i\in S'} e^{i\frac{2\pi b_i}{k}}. \tag{13}$$

Now, the quantum condition that S be a section of a spherically symmetric isolated horizon is

$$\vec{\mathcal{P}} = (p_1, ..., p_n), \quad \mathcal{P} \;=\; \{p_1, ..., p_n\} \tag{14}$$

$$b_i = -2m_i \mod k. \tag{15}$$

$$\mathcal{H}_{\rm kin} \;=\; \bigoplus_{\vec{\mathcal{P}},j,m} \mathcal{H}_S^{\vec{\mathcal{P}},b(m)} \otimes \mathcal{H}_\Sigma^{\mathcal{P},j,m}\,, \tag{16}$$

IV. $\mathcal{H}_{\rm PHYS}$ AND THE ENTROPY DEFINITION

Solving the vector constraints amounts to the averaging with respect to the S prerving diffeomorphisms. The scalar constraint and the Gauss are already solved on S.
The physical quantum horizon states:
an orthogonal basis $|b_1, ..., b_n$ is labelled by all the sequences,

$$b = (b_1, ...b_n), \quad b_i \in b_i \ \in \ \mathbb{Z}_k. \tag{17}$$

The full, horizon-bulk physical quantum states:

$$\mathcal{H}_{\rm phys} \ = \ \bigoplus_{j,m} \mathcal{H}^{b(m),j,m}, \tag{18}$$

where

$$b_i = -2m_i \mod k, \quad m_i \in \{-j_i, -j_i+1, ..., j_i\}, \ \sum_i b_i = 0. \tag{19}$$

The horizon area operator $\hat{A}_S$ commutes with all the constraints in this framework, and passes to the physical Hilbert space. *The sequences $j = (j_1, ...j_n)$ are responsible for the area assigned to the 2-surface S of the horizon by the bulk Quantum Geometry, whereas the sequences b represent the intrinsic quantum degrees of freedom of the horizon.*

The ABCK horizon entropy is defined by the number of the quantum horizon states $|b_1, ...b_n >\in \mathcal{H}_{S,\rm phys}$ which correspond to non-trivial subspaces $\mathcal{H}^{b(m),j,m}$ such that

$$a_S^j \ = \ 8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^n \sqrt{j_i(j_i+1)} \ \leq \ a. \tag{20}$$

The entropy S of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework is

$$S \;=\; \ln N(a), \tag{21}$$

where $N(a)$ is 1 plus the number of all the finite sequences $(m_1, ..., m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality and inequality are satisfied:

$$\sum_{i=1}^{n} m_i = 0, \tag{22}$$

$$\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{8\pi\gamma\ell_{\rm Pl}^2}, \tag{23}$$

where γ is the Barbero-Immirzi parameter of Quantum Geometry.

To find an upper bound for the number $N(a)$ introduced in the previous section (recall that $a = 4\pi\gamma\ell_{\rm Pl}^2 k$, and $k \in \mathbb{N}$), define the area-flux operator whose eigen subspaces are the spaces $\mathcal{H}_\Sigma^{\mathcal{P},j,m}$, but the eigen-values are given by taking the absolute value of each term in the sum defining the flux operator, namely

$$a_S^{+,m} \ := \ 8\pi\gamma\ell_{\rm Pl}^2 \sum_i |m_i|. \tag{24}$$

Incidently, it may be argued, that this is the way area could be quantized. Consider the set

$$M_k^+ \ := \ \{(m_1, ..., m_n) \ \Big| \ 0 \neq m_i \in \frac{1}{2}\mathbb{Z}, \ a_S^{+,m} \leq \ a = 4\pi\gamma\ell_{\rm Pl}^2 k\} \, . \tag{25}$$

Let N_k^+ be the number of elements of M_k^+ plus 1 (the empty sequence). Certainly,

$$N(a) \ \leq \ N_k^+ \tag{26}$$

Next, since k is arbitrarily fixed integer, let it become a variable of the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$. To establish a recurrence relation satisfied by the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$, notice that if $(m_1, ..., m_n) \in M_{k-1}^+$, then both $(m_1, ..., m_n, \frac{1}{2})$, $(m_1, ..., m_n, -\frac{1}{2}) \in M_k^+$. In the same way, for arbitrary natural $0 < l \leq k$,

$$(m_1, ..., m_n) \in M_{k-l}^+ \ \Rightarrow \ (m_1, ..., m_n, \pm\frac{1}{2}l) \in M_k^+. \tag{27}$$

Obviously, if we consider all $0 \ < \ l \ \leq \ k$, and all the sequences $(m_1, ..., m_n) \in M_{k-l}^+$, then the resulting $(m_1, ..., m_n, \pm\frac{1}{2}l)$ form the entire set M_k^+. Also, for two different $l \neq l'$,

$$(m_1, ..., m_n, \pm\frac{1}{2}l) \ \neq \ (m_1, ..., m_n, \pm\frac{1}{2}l'). \tag{28}$$

This proves the following recurrence relation,

$$N_k^+ \;=\; 2N_{k-1}^+ + ... + 2N_0^+ + 1. \qquad (29)$$

The (unique) solution is

$$N_k^+ \;=\; 3^k. \qquad (30)$$

In conclusion,

$$N(a) \leq 3^k\,. \qquad (31)$$

Incidently, if we defined the quantum area operator $\hat{A}^+(S)$ by taking the absolute value flux above, the entropy (the leading term) would be

$$S^+(a) \;=\; \ln N^+(a) \;=\; \frac{\ln 3}{\pi\gamma}\frac{a}{4\ell_{\rm Pl}^2} \qquad (32)$$

To find a lower bound for $N(a)$, we use the inequality

$$\sqrt{|m_i|(|m_i|+1)} \;\leq\; |m_i| + \frac{1}{2}, \qquad (33)$$

and consider the number N_k^- equal to 1 plus the number of elements in the set

$$M_k^- \;:=\; \{(m_1, ..., m_n) \,\Big|\, 0 \neq m_i \in \frac{1}{2}\mathbb{Z},\ \sum_{i=1}^{n}(|m_i| + \frac{1}{2}) \;\leq\; \frac{k}{2}\} \qquad (34)$$

Notice that this time, ignoring the constraint that the elements m_i of each sequence sum to zero makes an in-equivalence relation between $N(a)$ and N_k^- a priori not known. But let us postpone this problem for a moment. Using the same construction as above, we find the recurrence relation satisfied by N_k^-,

$$N_k^- \;=\; 2N_{k-2}^- + ... + 2N_0^- + 1. \qquad (35)$$

The unique solution is

$$N_k^- = \frac{2}{3}2^k + \frac{(-1)^k}{3}. \tag{36}$$

A lower bound for $N(a)$ is the number N'^-_k of the elements of M_k^- which additionally satisfy $m_1 + ... + m_n = 0$,

$$N'^-_k \leq N(a). \tag{37}$$

A statistical physics argument giving the value of the desired number N'^-_k is as follows (this argument is due to Meissner, who also provided an exact proof). We can think of each sequence $(m_1, ..., m_n)$ as of a sequence of random steps on a line. The total length of each path is bounded by k owing to the inequality in the definition of the set M_k^-. The number of sequences in M_k^- of a given, fixed value of the sum

$$m_1 + ... + m_n = \delta \tag{38}$$

depends on δ. The average value of δ is $\bar{\delta} = 0$. For large values of k, the number of the paths corresponding to the random walk distance δ should be given by the Gaussian function $\frac{C}{\sqrt{k}} e^{-\frac{\delta^2}{\beta k}} \ N_k^-$. In particular, the value

$$N'^-_k = \frac{C}{\sqrt{k}} N_k^- \tag{39}$$

corresponds to $\delta = 0$.

Summarizing,

$$\frac{C}{\sqrt{k}} N_k^- \ \leq \ N(a) \leq N_k^+, \tag{40}$$

where the numbers N_k^- and N_k^+ were calculated in (31, 36). Therefore the entropy is bounded in the following way

$$\frac{\ln 2}{4\pi\gamma\ell_{\rm Pl}^2} a + o(a) \leq S(a) \leq \frac{\ln 3}{4\pi\gamma\ell_{\rm Pl}^2} a. \tag{41}$$

A necessary condition for the agreement of the entropy $S(a)$ with the Bekenstein-Hawking entropy:

$$\frac{\ln 2}{\pi} \ \leq \ \gamma \ \leq \frac{\ln 3}{\pi}. \tag{42}$$

VII. EXACT CALCULATION (MEISSNER)

Generalization of the recurrence relation used above to a relation satisfied by the desired number $N(a)$ itself:

$$N(\tilde{a}) = \theta(\tilde{a} - \sqrt{3}/2)\left(2N(\tilde{a} - \sqrt{3}/2) + 2N(\tilde{a} - \sqrt{2}) + \ldots \right. \\ \left. +2N(\tilde{a} - \sqrt{|m_i|(|m_i|+1)}) + \ldots + 2\left[\sqrt{4\tilde{a}^2+1} - 1\right]\right) \tag{43}$$

where [.] stands for the integer part, and $a = \tilde{a}8\pi\gamma\ell_{\mathrm{Pl}}^2$. The Laplace transform of $N(\tilde{a})$:

$$P(s) \ = \ \int_0^\infty \mathrm{d}\tilde{a}\, N(\tilde{a})\, e^{-s\tilde{a}} \tag{44}$$

$$= \ \frac{2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/4}}}{s\left(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/4}}\right)}. \tag{45}$$

The simple real pole is $s_o = 2\pi\gamma_M$, where

$$1 - \sum_{0\neq m\in\frac{1}{2}\mathbb{Z}} e^{-2\pi\gamma_{\mathrm{M}}\sqrt{|m|(|m|+1)}} = 0. \tag{46}$$

And

$$S(a) \ = \ \frac{\gamma_{\mathrm{M}}}{\gamma}\frac{a}{4\ell_{\mathrm{Pl}}^2} \ + \ O(a). \tag{47}$$

Taking into account the condition $\sum_i m_i \ = \ 0$ produces the sub-leading term

$$S(a) \ = \ \frac{\gamma_{\mathrm{M}}}{\gamma}\frac{a}{4\ell_{\mathrm{Pl}}^2} - \frac{1}{2}\ln a + O(1). \tag{48}$$

The numerically calculated value:

$$\gamma_{\rm M} = 0.23753295796592\ldots \tag{49}$$

VIII. THE SPIN PROBABILITY DISTRIBUTION

Given any value a of the classical horizon area, and the $N(a)$ quantum states of the horizon labeled by all the finite sequences $(m_1, ..., m_n)$ which contrubute to the entropy, one can fix any arbitrary value

$$0 \neq m \in \frac{1}{2}\mathbb{Z} \tag{50}$$

and consider the subset of states corresponding to the sequences such that

$$m_1 = m. \tag{51}$$

Denote the number of the elements of this subset by $N_{(a)}(m)$. The ratio

$$P_{(a)}(m) \ := \ \frac{N_{(a)}(m)}{N(a)} \tag{52}$$

can be considered as the probability that the first puncture is labeled by $m_1 = m$. The question is, what $P_{(a)}(m)$ is when a is large compared to the minimal area a_m created at the puncture[1],

$$a_m \ = \ 8\pi\gamma\ell_{\rm Pl}^2\sqrt{|m|(|m|+1)}. \tag{53}$$

(Notice, that the answer could make no probabilistic sense, if the $\lim_{a\to\infty} P_{(a)}(m)$ were 0 for all the values of m, for example.) The

[1] This issue has been raised recently by John Baez.

number $N_{(a)}(m)$ defined above can be also thought of as a number of all the finite sequences $(m_2, ..., m_n)$ such that

$$\sum_{i=2}^{n} m_i \ = \ -m, \quad \sum_{i=2}^{n} \sqrt{|m_i|(|m_i|+1)} \ \leq \ \frac{a - a_m}{8\pi\gamma\ell_{\rm Pl}^2}. \tag{54}$$

By the same randon walk argument as the one used above in the treatment of the number ${N'}_k^-$ we have

$$N_{(a)}(m) \ = \ e^{-\frac{m^2}{a-a_m}} N(a - a_m) + ... \tag{55}$$

where the other terms can be neglected when we go to the limit $a \to \infty$. Since the number $N(a)$ is given by exponentiating (47), we can see that in the limit $a \to \infty$

$$P_{(a)}(m) \ \to \ e^{-2\pi\gamma_{\rm M}\sqrt{|m|(|m|+1)}} \ =: P(m). \tag{56}$$

Now, the equality defining $\gamma_{\rm M}$ means that the limits $P(m)$ of the probabilities still sum to 1,

$$\sum_{0 \neq m \in \frac{1}{2}\mathbb{Z}} P(m) \ = \ 1. \tag{57}$$

That (limit) probability distribution $P(m)$ can also be written in terms of the entropy $S(a_m)$ corresponding to the area a_m, namely

$$P(m) \ = \ e^{-S(a_m)}, \tag{58}$$

regardless of whether we fix the value of γ by the agreement with the Bekenstein-Hawking entropy or not.

IX. REANIMATION OF THE QUASI NORMAL MODES LINK

Dreyer's calculations connecting the ABCK entropy from Quantum Geometry, the BH entropy, the quasinormal-modes and the quantum mechanical formula $\Delta E = \omega\hbar$ rely on the 'entropy' derived from the number of states of the type $(\pm j_{\rm min}, ..., \pm j_{\rm min}$ only. Our results show, this is not the full entropy.

Remarkably however (Baez, private communication), the link can be found if we use instead of our area operator the flux-area operator defined above by the eigen values

$$a_S^{+,m} \;:=\; 8\pi\gamma\sum_i |m_i|, \tag{59}$$

$(G = \hbar = 1)$ which also arguably corresponds to the classical area observable. Recall, that the corresponding entropy was

$$S^+(a) \;=\; \ln N^+(a) \;=\; \frac{\ln 3}{\pi\gamma}\frac{a}{4\ell_{\rm Pl}^2}. \tag{60}$$

The consistency with BH implies

$$\gamma \;=\; \frac{\ln 3}{\pi}. \tag{61}$$

The area spectrum gap is a multiple of

$$\Delta a_S^+ \;=\; 8\pi\gamma\frac{1}{2} \;=\; 4\pi\gamma \;=\; 4\ln 3 \tag{62}$$

Defining after Dreyer the frequency $\omega_{\rm SU(2)}$ by

$$\Delta a_S^+ \;=\; 32\pi M\Delta M = 32\pi M\omega_{\rm SU(2)} \tag{63}$$

we find

$$M\omega_{\rm SU(2)} \;=\; \frac{\ln 3}{8\pi}, \tag{64}$$

and notice, that $\omega_{\mathrm{SU}(2)}$ coincides with the limit of the quasinormal-mode frequences in which the dumping is maximal.

Remarks:

- the calculation uses the SU(2) gauge group of Quantum Geometry

- if one assumes that Quantum Geometry is an SO(3) theory and repeats the calculation of the area, entropy, γ and finally $\omega_{\mathrm{SO}(3)}$, the resulting value of $\omega_{\mathrm{SO}(3)}$ is

$$\omega_{\mathrm{SO}(3)} = \omega_{\mathrm{SU}(2)}. \qquad (65)$$

- The status of the flux-area operator: there are several remarks related to this point. The flux-area operator used above is given by the flux of the vector field normal to the horizon. The ACBK framework distinguishes this operator in a very special way, speciphic for the horizon only. The normal vector field is defined by the extra structure provided at the horizon, namely the given internal, su(2) valued vector field r. Contracted with the Ashtekar frame field E^i_a, $r^i E^a_i$ defines the vector normal to the horizon. Itself r is NOT a dynamical field. The flux-area operator Baez proposes to use is obtained from the flux integral corresponding to r with the extra absolute value inserted under the integral defining the flux. Classically this an equivalent to the usual definition of the area of the horizon. The quantum operators, on the other hand, are different. If we want to take the QN mode relation seriously, we choose the flux-area definition. Consider now instead of the horizon, a regular 2-surface contained in the bulk. Do we have again two area operatotrs? In this case the internal vector field r normal (via the soldering form) to the surface is not apriori given. We need to construct it from the surface and from the frame. Therefore it involves the dynamical fields which get quantized,

$$r = r[E]. \tag{66}$$

I have calculated the resulting flux operator. It COINCIDES with our legal AREA OPERATOR.

Conclusion: it is not radicules to assume that the classical area of the BH should be replaced by the flux area in the quantum theory; thus the quanta are just $|m|$s. However, this new operator does NOT generalize to other surfaces contained in the bulk.

- the Quantum Geometry quantum flux-area becomes compatible with the CS quantum consistency condition

X. WHAT ELSE SHOULD WE COUNT? (DISCUSSION)