Light Front Hamiltonian for Transverse Lattice QCD

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Abstract

A calculational framework for determining masses of low lying hadrons using light front quantization is discussed. The method is based upon four theoretical tools: discrete light cone quantization, which has been very successful in $1+1$ dimensional models, a projector Monte Carlo method to extract low lying state data with 2 additional transverse dimensions, the transverse spatial lattice Hamiltonian of QCD, and exact form factors of the $SU(3)_L \otimes SU(3)_R$ symmetric $1+1$ dimensional non-linear sigma model (NLSM). How these tools are to be put together to provide a description of hadrons is the main topic of this lecture. I also focus on the NLSM form factors, which are given new physical relevance via this picture of QCD.

1. Introduction

I would like to suggest an alternative framework for the calculation of hadron masses. This framework is built upon the method of discrete light cone (front) quantization DLCQ[1], which combines the natural truncation of physical states on the light-front with a numerical cutoff in the number of states. The light front quantization guarantees that physical states have positive light front momentum p^+, and DLCQ splits the total p^+ into discrete bits. The number of states in the Hilbert space is given by the number of possible ways of partitioning the bits of momenta of the particles in the theory. The light front Hamiltonian p^- commutes with p^+ and is inversely proportional to the bit size, $p^- \sim M^2/p^+$, so a mass spectrum calculation of the lowest states becomes tractable. With this method, the particle data booklet of $1+1$ dimensional QCD with dynamical fermions can filled out using only a modern personal computer[2].

Extending this analysis to $3+1$ dimensions in a straightforward fashion has been unsuccessful so far, primarily because of strong mixing of high and low momentum particles. I propose to use a spatial lattice method to regulate this problem, much like in ordinary lattice gauge theory. In this context however, the lattice is two dimensional, and the full light front Hamiltonian is decomposed into $(1+1)$-D longitudinal, 2-D transverse, and mixed parts of the theory. One does not need to diagonalize the entire Hamiltonian to study properties of the lowest states; diagonalization via DLCQ of a small subset of the Hamiltonian which includes only nearest or next to nearest neighbor transverse interactions is sufficient. This reduces both the computer time and memory requirements of a full diagonalization problem. The true ground states of the theory are obtained by projecting out the excited state components of a trial wave-function. This has been formulated and successfully applied to the ϕ^4 theory in $2+1$ dimensions by Burkardt[3].

The classical action for QCD on a transverse lattice was given by Bardeen and Pearson a long time ago[4]. Introduce link fields $U_{x_\perp}$, which are scalars with respect to the two continuous space-time coordinates perpendicular to the lattice. In light cone gauge[1] $A^- = 0$, and after eliminating A^+ via the Gauss constraint, the action $A = \int dx^+ dx^- \sum_{x_\perp} \mathcal{L}$, where $\mathcal{L} = \mathcal{L}_{\rm LL} + \mathcal{L}_{\rm LT} + \mathcal{L}_{\rm TT}$, where L denotes longitudinal and T denotes transverse,

$$\mathcal{L}_{\rm LL} = \frac{1}{g_1^2} \sum_{\alpha=1,2} {\rm tr}(\partial_+ U_{x_\perp,\alpha} \partial_- U^\dagger_{x_\perp,\alpha}) \ , \tag{1}$$

$$\mathcal{L}_{\rm LT} = \frac{1}{g_2^2 a^2} \int dy^- |x^- - y^-| J_-(x^-) \cdot J_-(y^-) \ , \tag{2}$$

$$\mathcal{L}_{\rm TT} = \frac{1}{g_3^2 a^2} \sum_{\alpha\beta} {\rm tr}(U_{x_\perp,\alpha} U_{x_\perp+\hat\alpha,\beta} U^\dagger_{x_\perp+\hat\beta,\alpha} U^\dagger_{x_\perp,\beta}) \ . \tag{3}$$

Eqn. (1) denotes the pure longitudinal part of the action, and is a set of decoupled two-dimensional non-linear sigma models (NLSM). The non-linear constraints are $\det U = 1$ and $U^\dagger = U^{-1}$ for $SU(N)$ theory. Eqn. (2) denotes longitudinal gluon exchange and contains both local and nearest neighbor type interactions. The pure glue non-Abelian current is given by $J_-^a = i \sum_\alpha {\rm tr}[T^a (U^\dagger_{x_\perp,\alpha} \partial_- U_{x_\perp,\alpha} + U^\dagger_{x_\perp-\hat\alpha,\alpha} \partial_- U_{x_\perp-\hat\alpha,\alpha})]$. Eqn. (3) is the nearest neighbor transverse plaquette interaction which becomes $F^2_{\alpha\beta}$ in the continuum limit. The parameter a is the transverse lattice spacing.

Bardeen, Pearson and Rabinovici performed a DLCQ calculation of the glueball spectrum in the large N limit[4] by approximating the NLSM part of the Lagrangian eqn. (1) with a complex linear matrix field and adding potential terms to mimic the non-linear constraints. However for finite N, this scheme is problematic because the number of induced potential terms (suppressed by powers of $1/N$ in the large N calculation) is infinite. In addition, DLCQ is not able to describe behavior of systems about phase transition critical points very well without addressing the 'zero mode' problems of the vacuum, i.e. the coherent state of the vacuum in the new phase is made out of infinitely many Fock state particles, and is difficult to simulate numerically.

I propose to solve the NLSM part of the theory exactly, by calculating matrix elements of the asymptotic states $P_i(\theta_i)$ with operators in the interacting quantum Hamiltonian, where $\theta = \ln(P^+/m)$ is the rapidity of state P with mass m. The relevant form factors of the NLSM are

$$\langle J_-(0)|P_1(\theta_1)P_2(\theta_2)\rangle \ , \ \langle J_-(0)|P_1(\theta_1)P_2(\theta_2)P_3(\theta_3)P_4(\theta_4)\rangle \ , \ \cdots \tag{4}$$

$$\langle U(0)|P_1(\theta_1)\rangle \ , \ \langle U(0)|P_1(\theta_1)P_2(\theta_2)P_3(\theta_3)\rangle \ , \ \cdots \tag{5}$$

[1]The 'zero mode' issue associated with this gauge is neglected in this lecture.

Symmetry arguments imply that odd (even) state form factors vanish in the series of eqn. (4)((5)), the matrix elements between incoming and outgoing states are related to these by crossing symmetry and do not require additional calculations, and relativistic invariance lets us trivially relate these to $x^- \neq 0$ form factors. Form factors of series (4) are required for LT interactions of eqn. (2). We must insert a complete set of states between $J(x)$ and $J(y)$. Form factors of series (5) are required for TT interactions in eqn. (3). For the $SU(3)_L \otimes SU(3)_R$ NLSM, they have not been explicitly calculated to my knowledge, so the rest of the lecture will discuss some of the theoretical tools that help calculate form factors . To simplify the presentation, I focus on the $SU(2)$ case.

2. The Integrable $SU(2)$ NLSM

The quantum principle chiral $SU(2)\otimes SU(2)$ NLSM is a completely integrable system. This means that it has an infinite number of conserved currents – for two dimensional systems with pure elastic scattering, not only is momentum conserved, but the entire momentum distribution is conserved.[2] This implies that full S matrix factorizes into a product of two particle S matrices [5]. Symmetry arguments – unitarity, crossing symmetry, and the Yang-Baxter equation (the time ordering of the scattering process is irrelevant to the final S matrix), are almost sufficient to determine unique two particle S-matrices. This is elegantly discussed in the 'Zamolochikov^2' paper[6]. The remaining ambiguities are the initial group multiplets of the particles, and a minimality assumption on the number bound states. For the $SU(2)$ NLSM, we assume that particles are in the fundamental representations of the left and right global $SU(2)$ symmetries of the action, and (self consistently) that there are no additional states. Each particle is labeled by four states of the $2_L \otimes 2_R$ representation, which I will parameterize in a matrix notation, $P = P^0 + i\sigma^a P^a$, $P^\dagger = P^0 - i\sigma^a P^a$, where $a = 1, 2, 3$, and σ^a are the Pauli sigma matrices. The P^A states, where $A = 0, 1, 2, 3$, transform under global left symmetry as $[Q_L, P] = i\sigma^a\epsilon^a P$, and $[Q_L, P^\dagger] = -iP^\dagger\sigma^a\epsilon^a$, or in terms of components, $\delta^a_L P^0 = -P^a$, $\delta^a_L P^b = (\delta^{ab}P^0 + \epsilon^{abc}P^c)$. The transformation rules under the $SU(2)_R$ group are obtained by exchanging P and $P^\dagger$. The two particle S matrix for the $SU(2)$ NLSM was given by Wiegmann[7], based on the work of Berg, Karowski, Weisz, and Kurak[8]. In the basis of states given above, the S matrix $\mathcal{S} = S\Sigma$ is

$$P^A(\theta_1)P^B(\theta_2) = S(\theta)\Sigma^{ABCD}(\theta)\, P^C(\theta_2)P^D(\theta_1)\ ,\ \theta \equiv \theta_1 - \theta_2\ , \tag{6}$$

$S(\theta) = [\Gamma(-\theta/2\pi i)\Gamma(1/2 + \theta/2\pi i)]^2/[\Gamma(\theta/2\pi i)\Gamma(1/2 - \theta/2\pi i)(\theta - i\pi)]^2$, and the factor $\Sigma^{ABCD}(\theta) = ((i\pi)^2 - i\pi\theta)\delta^{AC}\delta^{BD} + (\theta^2 - i\pi\theta)\delta^{AD}\delta^{BC} + (i\pi\theta)\delta^{AB}\delta^{CD}$. As $\theta \to \infty$ states commute.

[2] This can be demonstrated on a pool table by lining up a straight column of touching balls and shooting the cue ball straight onto the leading ball. Only one ball comes out the other end. This is in contrast to the $2 + 1$ dimensional case which occurs when one begins the pool game with a 'break' of a two dimensional array of balls.

The Zamolochikov – Faddeev (zf) algebra is the full realization of the S matrix and normalization in terms of creation and annihilation operators acting on a vacuum state $|0\rangle$. Asymptotic 'in' states are a representation of the S matrix, and their algebra is given by eqn. (6). Normalization for two particle 'in' states is ${}^{\rm out}\langle P^D(\theta_2')P^C(\theta_1')|P^A(\theta_1)P^B(\theta_2)\rangle^{\rm in}$ $= \delta^{AC}\delta^{BD}\delta(\theta_1-\theta_1')\delta(\theta_2-\theta_2')$, for $\theta_1 > \theta_2$ and $\theta_1' > \theta_2'$. We define $\overline{P}^A$ to be annihilation operators satisfying $\overline{P}^A|0\rangle = 0$. I find the additional self-consistant algebraic relations

$$\overline{P}^A(\theta_1)\overline{P}^B(\theta_2) = S(\theta)\Sigma^{ABCD}(\theta)\,\overline{P}^C(\theta_2)\overline{P}^D(\theta_1)\ , \tag{7}$$

$$\overline{P}^A(\theta_1)P^B(\theta_2) = S(-\theta)\Sigma^{ACBD}(-\theta)\,P^C(\theta_2)\overline{P}^D(\theta_1) + \delta^{AB}\delta(\theta_1-\theta_2)\ . \tag{8}$$

Eqns. (6, 7, 8) denote the full zf algebra for the case at hand. The Fock space of quantum states is given by an ordered (with respect to rapidity, lesser rapidity particles to the left is the convention) set of creation operators acting on the vacuum. The operator $\rho(\theta) = P^A(\theta)\overline{P}^A(\theta)$ has simple commutation relations with the particles,

$$[\rho(\theta_1),\, P^A(\theta_2)] = P^A(\theta_1)\delta(\theta_1-\theta_2)\ ,\ [\rho(\theta_1),\, \overline{P}^A(\theta_2)] = -\overline{P}^A(\theta_1)\delta(\theta_1-\theta_2)\ . \tag{9}$$

With these commutators, it is trivial to construct the light-front Hamiltonian and momentum: $P^{\pm} = m\int d\theta e^{\pm\theta}\rho(\theta)$, where m is the mass of the particles. Clearly, all operators of the form $\int d\theta e^{n\theta}\rho(\theta)$ are diagonal in this basis of states. These operators for integer n correspond to moments of the light-front momentum of a multi-particle system, which we expected to exist by assumption. It is remarkable that they have such a simple form in this basis of states!

3. Form Factors

The relationship between the asymptotic particles and the operators J (and possibly even U) which occur in the interacting transverse lattice Hamiltonian is very complicated. In the remaining part of this lecture, I will discuss the first form factor of eqn. (5). Unfortunately, we do not know the exact relation between the asymptotic states and the current J[3]. However we are optimistic that these form factors can be evaluated, because there exist similar result for the $SU(2)$ Thirring model[9]. I will now construct the exact two point $SU(2)\otimes SU(2)$ form factor below, using rules which were applied in the Thirring model case. The two point form factor for the left current is defined as

$$e^{+ix^-[p^+(\theta_1)+p^+(\theta_2)]}\,\langle J_L^a(x^-)P^B(\theta_1)P^C(\theta_2)\rangle = \langle J_L^a(0)P^B(\theta_1)P^C(\theta_2)\rangle = F^{BC}_{J_L^a}(\theta_1,\theta_2) \tag{10}$$

where the amplitude is taken to be between asymptotic 'in' and 'out' vacua. Crossing symmetry implies that if $\theta_1 \to \theta_1 + i\pi$ in the amplitude, the P^B 'in' field becomes an 'out'

[3]This problem has an analogy in the Ising model about the critical point, which has a Lagrangian of free massive fermions. The order and disorder operators in the Ising model are very complicated functions of the fermion, and their form factors are very non-trivial, even though the S matrix elements of the fermions are just ± 1.

field: $F^{BC}_{J^a_L}(\theta_1+i\pi,\theta_2)=\langle\overline{P}^B(\theta_1)J^a_L(0)P^C(\theta_2)\rangle$. The current is assumed to be conserved, with normalization

$$\int dx^-\langle\overline{P}^B(\theta_1)J^a_L(x^-)P^C(\theta_2)\rangle=\langle\overline{P}^B(\theta_1)Q^a_LP^C(\theta_2)\rangle=\mathrm{tr}[\sigma^aT^{BC}_L]\delta(\theta_1-\theta_2)\ . \qquad (11)$$

By comparing to the left transformations on the particles, we find the T matrix to be antisymmetric with matrix elements $T^{00}_L=0$, $T^{0b}_L=\sigma^b/2$, $T^{bc}_L=-\epsilon^{bcd}\sigma^d/2$. Note that $\mathrm{tr}[\sigma^aT^{BC}_L]$ is simply the Clebsh-Gordon coefficient that corresponds to the left triplet channel in the product of two left-handed doublets.

The form factor must satisfy an additional set of reasonable constraints, the **Smirnov Axioms**[10]. In the context of the two point form factor, they can be stated as: Axiom 1: The function $F^{BC}_{J^a_L}(\theta_1,\theta_2)$ is analytic in θ in the strip $0\le\Im\theta\le2\pi$, except for single poles. It becomes the physical form factor for real θ_i and $\theta_1<\theta_2$. The poles are on the imaginary axis and of of two types, either the fusion of physical states onto intermediate physical bound states, or particle - anti-particle fusion onto the vacuum. Axiom 2: Relativistic invariance requires that the form factor satisfies $F^{BC}_{J^a_L}(\theta_1+\beta,\theta_2+\beta)$ $=\exp[s(J)\beta]F^{BC}_{J^a_L}(\theta_1,\theta_2)$, where $s(J)=-1$ is the spin of J_- in our case. Axiom 3: The form factor should satisfy the symmetry property (Watson's Theorem) $F^{BC}_{J^a_L}(\theta_1,\theta_2)=\Sigma^{BCDE}(\theta_1-\theta_2)S(\theta_1-\theta_2)F^{DE}_{J^a_L}(\theta_2,\theta_1)$. This seen by applying eqn.(6) to the form factor, or equivalently by inserting a complete set of states between P^B and P^C. Axiom 4: The form factor satisfies $F^{BC}_{J^a_L}(\theta_1,\theta_2+2\pi i)=e^{2\pi iw(J,U)}F^{CB}_{J^a_-}(\theta_2,\theta_1)$, where $w(J,U)$ is the relative monodromy index between the current and the fundamental field corresponding to U. This corresponds to a Euclidean continuation of one operator about the other, and $w=0$ in our bosonic case. The ansatz for the form of $F^{BC}_{J^a_L}$ which satisfies these axioms is

$$F^{BC}_{J^a_L}(\theta_1,\theta_2)=-\frac{1}{4\pi}[p^-(\theta_1)+p^-(\theta_2)]\,\mathrm{Tr}[\sigma^aT^{BC}_L]\tanh(\theta/2)F(\theta)\ . \qquad (12)$$

The pole at $\theta=i\pi$ in the tanh function satisfies axiom 1 for the scalar part of the form factor and corresponds to the fusion of the two doublets onto the vacuum state. For the full vector form factor, this pole is cancelled by the zero in $p^-(\theta_1)+p^-(\theta_2)$, giving a finite result at the normalization point $\theta=i\pi$, and the normalization condition eqn. (11) is satisfied when $F(i\pi)=1$. The zero in the tanh function at $\theta=0$ exists by axiom 3 since the S-matrix in this channel is -1 at $\theta=0$. Eqn. (12) satisfies relativistic invariance (axiom 2) as long as $F=F(\theta_1-\theta_2)$. F has no poles or zeros in the region $0\le\Im\theta\le2\pi$, and it satisfies

$$8\pi i\frac{d}{d\theta}F(\theta)=\int_C\frac{dz}{\sinh^2\frac{1}{2}(z-\theta)}\ln F(z)=\int_\infty^\infty\frac{dz}{\sinh^2\frac{1}{2}(z-\theta)}\ln\frac{F(z)}{F(z+2\pi i)}\ . \qquad (13)$$

where the contour C goes from $[-\infty,\infty]$ up to the $2\pi i$ line, from $[\infty+2\pi i,-\infty+2\pi i]$, and is closed by going back to the real line. Assuming that the form factor falls off faster than $\ln F \to e^z$ (which we will verify below), we can ignore the end caps to get the second equality above. From axiom 3, $F(\theta) = \bar{S}(\theta)F(-\theta)$ and from axiom 4, $F(\theta+2\pi i) = F(-\theta)$, so

$$\frac{F(\theta)}{F(\theta+2\pi i)} = S(\theta)(\theta - i\pi)(\theta + i\pi) \equiv \bar{S}(\theta) = \frac{\Gamma^2(1-\frac{\theta}{2\pi i})\Gamma(\frac{1}{2}+\frac{\theta}{2\pi i})\ \Gamma(\frac{3}{2}+\frac{\theta}{2\pi i})}{\Gamma^2(1+\frac{\theta}{2\pi i})\Gamma(\frac{1}{2}-\frac{\theta}{2\pi i})\ \Gamma(\frac{3}{2}-\frac{\theta}{2\pi i})}\ . \quad (14)$$

If $\bar{S}(\theta) = \exp\int_0^\infty \frac{dx}{x} f(x) \sinh\frac{x\theta}{i\pi}$, then $F(\theta) = \exp\int_0^\infty \frac{dx}{x} f(x)\sin^2[x(i\pi - \theta)/2\pi]/\sinh x$ [11]. For the case at hand, we find $f(x) = -(1-e^{-x})^2/\sinh x$. For large $Q^2 = me^\theta$, the result simplifies and one finds $F(\theta) = [\ln\frac{Q^2}{m^2}]^{-1/2}$. This is the expected asymptotic freedom of the $SU(2)$ NLSM[4]. Higher point form factors are more complicated because of the number of possible group theoretic ways the states can factor onto the current. Recently, a bosonization prescription has been given for the construction of higher point form factors in the $SU(2)$ Thirring model case[12].

We clearly need to compute more NLSM form factors before full computer simulations can begin. With the connection between the $SU(3)\otimes SU(3)$ NLSM and QCD made explicit by the transverse lattice, these calculations take on new phenomenological relevance. This work was partly supported by SSC fellowship FCFY9318, and DOE grant DE-FG05-86ER-40272.

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[4] In the transverse lattice picture, this form factor for large Q^2 is a dressed three point transverse gluon vertex. The above result means that the sum over a certain class of transverse gluons renormalizes the charge such that $\alpha(Q^2) \sim 1/[\ln\frac{Q^2}{m^2}]$.