

Heavy-to-Light Form Factors from QCD Sum Rules on the Light-Cone

Alexander Khodjamirian^{a,1} and Reinhold Rückl^{a,b}

^aSektion Physik der Universität München, D-80333 München, Germany

^bMax-Planck-Institut für Physik, Werner-Heisenberg-Institut, D-80805 München, Germany

Abstract

We discuss the calculation of the $B \rightarrow \pi$ and $D \rightarrow \pi$ form factors based on an expansion in terms of pion wave functions on the light-cone with increasing twist and QCD sum rule methods. The results are compared with predictions of conventional QCD sum rules and other approaches.

1. Introduction

The method of QCD sum rules ¹ has proved to be particularly useful in heavy quark physics, where a small distance scale is provided by the inverse heavy quark mass. In order to calculate form factors of heavy hadrons one can consider suitable three-point vacuum correlation functions and apply the operator product expansion in terms of vacuum condensates which take into account nonperturbative quark-gluon dynamics. Some recent calculations along these lines can be found in refs. ^{2,3}. In this report we present an alternative, more economical method based on an expansion of vacuum-to-pion matrix elements near the light-cone. This method is used in ref. ⁴ to calculate the $B \rightarrow \pi$ and $D \rightarrow \pi$ form factors. After outlining the calculational procedure we show numerical results and discuss their sensitivity to various input parameters. We also compare our predictions with other estimates.

2. Derivation of the sum rules.

For definiteness, we focus on the $D \rightarrow \pi$ form factor f_D^+ entering the matrix element

$$\langle \pi | \bar{d} \gamma_\mu c | D \rangle = 2f_D^+(p^2)q_\mu + [f_D^+(p^2) + f_D^-(p^2)]p_\mu \quad (1)$$

with $p+q$, q and p being the D and π momenta and the momentum transfer, respectively. The corresponding $B \rightarrow \pi$ form factor f_B^+ can be treated in parallel by obvious formal replacements. These form factors are measurable in $B, D \rightarrow \pi l \nu_l$ semileptonic decays.

Let us consider the matrix element

$$F_\mu(p, q) = i \int d^4x e^{ipx} \langle \pi(q) | T \{ \bar{d}(x) \gamma_\mu c(x), \bar{c}(0) i \gamma_5 u(0) \} | 0 \rangle$$

¹Alexander von Humboldt Fellow, on leave from Yerevan Physics Institute, 375036 Yerevan, Armenia

$$= F((p+q)^2, p^2)q_\mu + \tilde{F}((p+q)^2, p^2)p_\mu \quad (2)$$

between the vacuum and an on-shell pion state. This object is represented diagrammatically in Fig. 1. The pion momentum squared, $q^2 = m_\pi^2$, vanishes in the chiral limit adopted throughout this discussion. Then, contracting the c -quark fields in (2) and keeping only the lowest order term, i.e. the free c -quark propagator, yields

$$F((p+q)^2, p^2) = i \int d^4x \int \frac{d^4P}{(2\pi)^4} e^{i(p-P)x} \sum_a \frac{\phi_a(x^2, q \cdot x)}{P^2 - m_c^2}, \quad (3)$$

where

$$\phi_a(x^2, q \cdot x) = \langle \pi(q) | \bar{d}(x)\Gamma_a u(0) | 0 \rangle, \quad (4)$$

Γ_a denoting certain combinations of Dirac matrices. This approximation corresponds to Fig. 1a.

If $(p+q)^2$ is taken sufficiently large and negative, and the time-like momentum transfer squared p^2 is far from the kinematical limit, $p^2 = m_D^2$, the c -quark propagating between the points x and 0 is far off-shell. In that case, it is justified to keep only the first few terms in the expansion of the matrix elements (4) around $x^2 = 0$, that is near the light-cone:

$$\phi_a(x^2, q \cdot x) = \sum_n \int_0^1 du \varphi_a^n(u) \exp(iuq \cdot x) (x^2)^n. \quad (5)$$

The form of the expansion is dictated by translational invariance. Logarithms in x^2 which may also appear in (5) are disregarded for simplicity. These terms can be consistently treated by means of QCD perturbation theory. They give rise to normalization scale dependence. Inserting (5) into (3) and integrating over x and P , one obtains, schematically,

$$F((p+q)^2, p^2) = \sum_a \sum_n \int_0^1 du \frac{\varphi_a^n(u)}{[m_c^2 - (p+qu)^2]^{2n}}. \quad (6)$$

It is thus possible to calculate the invariant function F with reasonable accuracy in the kinematical region of highly virtual c -quarks provided one knows the distribution functions $\varphi_a^n(u)$ at least for low values of n . The latter are nothing but the light-cone wave functions of the pion introduced in the context of hard exclusive processes⁵⁻⁷.

The leading twist 2 wave function is defined by

$$\langle \pi(q) | \bar{d}(x)\gamma_\mu\gamma_5 P \exp\{i \int_0^1 d\alpha x_\mu A^\mu(\alpha x)\} u(0) | 0 \rangle = -if_\pi q_\mu \int_0^1 du e^{iuq \cdot x} \varphi_\pi(u), \quad (7)$$

where the exponential factor involving the gluon field is necessary for gauge invariance. The asymptotic form of φ_π is well known: $\varphi_\pi(u) = 6u(1-u)$. In our calculation of f_D^+ and f_B^+ we have included quark-antiquark wave functions up to twist four. In addition, we have also calculated the first-order correction to the free b -quark propagation shown in

Fig. 1b which involves quark-antiquark-gluon wave functions of twist 3 and four. On the other hand, the perturbative $O(\alpha_s)$ corrections corresponding to inserting gluon exchanges between quark lines in Fig. 1a have not been evaluated directly but have only been taken into account in a rough indirect way, as explained below.

In order to extract the desired form factor f_D^+ from the result on the invariant function $F((p+q)^2, p^2)$ sketched in (6) we employ a QCD sum rule with respect to the D -meson channel. Writing a dispersion relation in $(p+q)^2$, we approximate the hadronic spectral function in the D -channel by the pole contribution of the D meson and a continuum contribution. In accordance with quark-hadron duality, the latter is identified with the spectral function derived from the QCD representation (6) above the threshold $(p+q)^2 = s_c$. Formally, subtraction of the continuum then amounts to simply changing the lower integration boundary in (7) from 0 to $\Delta = (m_c^2 - p^2)/(s_c - p^2)$. After Borel transformation one arrives at a sum rule for the product $f_D f_D^+$, where f_D is the D meson decay constant:

$$f_D f_D^+(p^2) = \frac{f_\pi m_c^2}{2m_D^2} \left\{ \int_\Delta^1 \frac{du}{u} \exp \left[\frac{m_D^2}{M^2} - \frac{m_c^2 - p^2(1-u)}{uM^2} \right] \Phi_2(u, M^2, p^2) \right. \\ \left. - \int_0^1 u du \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{\Theta(\alpha_1 + u\alpha_2 - \Delta)}{(\alpha_1 + u\alpha_2)^2} \right. \\ \left. \times \exp \left[\frac{m_D^2}{M^2} - \frac{m_c^2 - p^2(1 - \alpha_1 - u\alpha_2)}{(\alpha_1 + u\alpha_2)M^2} \right] \Phi_3(u, M^2, p^2) \right\}, \quad (8)$$

where

$$\Phi_2 = \varphi_\pi(u) + \frac{\mu_\pi}{m_c} \left[u\varphi_p(u) + \frac{1}{6}\varphi_\sigma(u) \left(2 + \frac{m_c^2 + p^2}{uM^2} \right) \right] + \dots, \quad (9)$$

$$\Phi_3 = \frac{2f_{3\pi}}{f_\pi m_c} \varphi_{3\pi}(\alpha_1, 1 - \alpha_1 - \alpha_2, \alpha_2) \left[1 - \frac{m_c^2 - p^2}{(\alpha_1 + u\alpha_2)M^2} \right] + \dots \quad (10)$$

Here, φ_p , φ_σ , and $\varphi_{3\pi}$ represent twist 3 pion wave functions, while the ellipses denote contributions of higher twist. The contributions of twist 4 are given explicitly in refs. ^{4,8}. The analogous sum rule for the $B \rightarrow \pi$ form factor follows from the above by formally changing $c \rightarrow b$ and $D \rightarrow \bar{B}$.

3. Numerical evaluation

The numerical values to be substituted for m_c , f_D and the threshold s_c are interrelated by the QCD sum rule for the two-point correlation function $\langle 0 | T\{j_5(x), j_5^+(0)\} | 0 \rangle$, $j_5 = \bar{c}i\gamma_5 u$. This sum rule should be used without $O(\alpha_s)$ corrections in order to be consistent with the neglect of these corrections in the sum rule for $f_D f_D^+$ given above. A similar interrelation exists for m_b , f_B and s_b from the analogous correlation function of b -flavoured currents. For the wave functions we use the parametrization suggested in ref. ⁹. Other details on the choice of parameters are given in refs. ^{4,8}.

The form factor f_D^+ derived from (8) is plotted in Fig. 2 as a function of Borel mass squared M^2 . Numerically, the twist 3 contributions turn out to be more important than the nonasymptotic corrections to the leading twist 2 wave function. In the range $3 < M^2 < 5 \text{ GeV}^2$ the corrections due to twist 4 in (8) remain subdominant and, simultaneously, the contribution from excited and continuum states does not exceed 30%. Restricting oneself to this interval, one obtains the value $f_D^+(0) = 0.66 \pm 0.03$ for the $D \rightarrow \pi$ form factor at zero momentum transfer. The analogous fiducial interval for the $B \rightarrow \pi$ form factor is $8 < M^2 < 12 \text{ GeV}^2$ yielding $f_B^+(0) = 0.29 \pm 0.01$.

The maximum momentum transfer p^2 to which these sum rules are applicable is estimated to be about 1 GeV^2 for f_D^+ and about 15 GeV^2 for f_B^+ . The p^2 -dependence of both form factors is plotted in Fig. 3. It is important to investigate the theoretical uncertainties in these results. Two main sources are the nonasymptotic corrections to the leading twist wave function φ_π and to the twist 3 three-particle wave function $\varphi_{3\pi}$. In order to estimate the sensitivity of our results to these corrections we drop them and recalculate the form factors. As can be seen from Fig. 3, the result changes by less than 10 %.

4. Conclusion

Summarizing our investigations, in Fig. 4 we compare our predictions on $f_D^+(p^2)$ and $f_B^+(p^2)$ with the results of other calculations. Within the uncertainties there is satisfactory agreement. In particular, we would like to emphasize the coincidence with the result $f_B^+(0) = 0.24 \pm 0.025$ derived from conventional QCD sum rule ² in which the large-distance effects are parametrized in terms of vacuum condensates rather than by pion wave functions on the light-cone. On the other hand, the value $f_D^+(0) = 0.5 \pm 0.1$ obtained in ref. ³ is smaller than ours. Also the p^2 -dependence of the form factors is rather similar in the different approaches. Note, however, that in the quark model ¹⁰ the momentum dependence of the form factors is not predicted but simply assumed to be given by a single pole:

$$f^+(p^2) = \frac{f^+(0)}{1 - p^2/m_*^2} \quad (11)$$

with $m_* = 2.01 \text{ GeV}$ in the case of f_D^+ and $m_* = 5.3 \text{ GeV}$ for f_B^+ as expected in the spirit of vector dominance. The authors of refs. ^{2,3} have fitted their calculated shape for f^+ to the form (11) and obtained $m_* = 1.95 \pm 0.10 \text{ GeV}$ for f_D^+ and $m_* = 5.2 \pm 0.05 \text{ GeV}$ for f_B^+ . In comparison to that we find a somewhat steeper p^2 -dependence.

In conclusion, we emphasize that light-cone sum rules such as the ones exemplified in this report represent a well defined alternative to the conventional QCD sum rule method. In this variant, the nonperturbative aspects are described by a set of wave functions on the light-cone with varying twist and quark-gluon multiplicity. These universal functions can be studied in a variety of processes involving the π and K meson, or other light mesons. The main problem to be solved if one wants to fully exploit the light-cone approach is the reliable determination of the nonasymptotic effects in the wave functions. In this respect, measurements of hadronic form factors, couplings etc. can provide important information. A second, mainly technical problem, concerns the higher order perturbative corrections which are still unknown.

The most important advantage of the light-cone sum rules is the possibility to take light hadrons on mass-shell from the very beginning. One thus avoids the notorious model-dependence of extrapolations from Euclidean to physical momenta in light channels. Furthermore, in many cases the light-cone approach is technically much easier than a conventional QCD sum rule calculation. Finally, the light-cone method is rather versatile. It can also be profitably employed to calculate heavy-to-light form factors such as $B \rightarrow \rho$ and $B \rightarrow K^*$, amplitudes of rare decays¹¹ and hadronic couplings such as $D^*D\pi$ and $B^*B\pi$ ⁸.

Acknowledgements

A.K. is grateful to the Institute of Theoretical Physics of the Warsaw University and to the A. von Humboldt Foundation for supporting his participation in the workshop. Useful discussions with D. Wyler are acknowledged.

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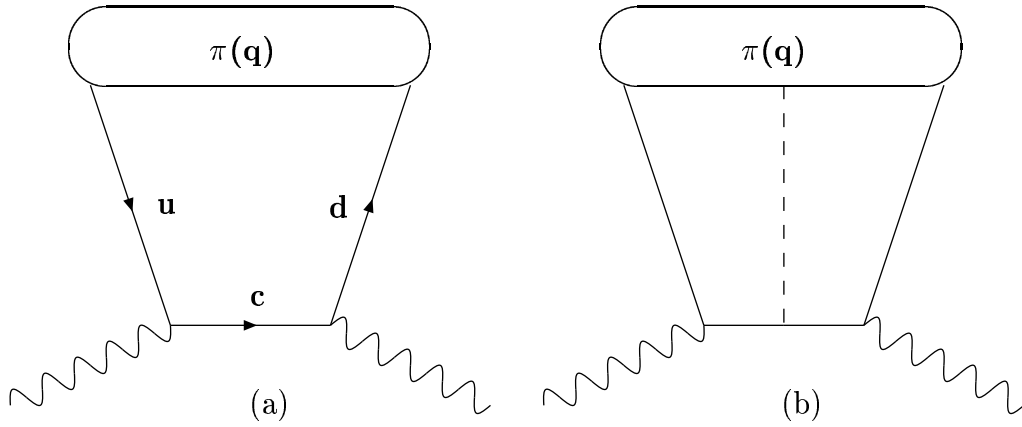


Figure 1: QCD diagrams contributing to the matrix element (2) involving (a) quark-antiquark light-cone wave functions; (b) three-particle quark-antiquark-gluon wave functions. Solid lines represent quarks, dashed lines gluons, wavy lines are external currents, and the ovals denote π meson wave functions on the light-cone.

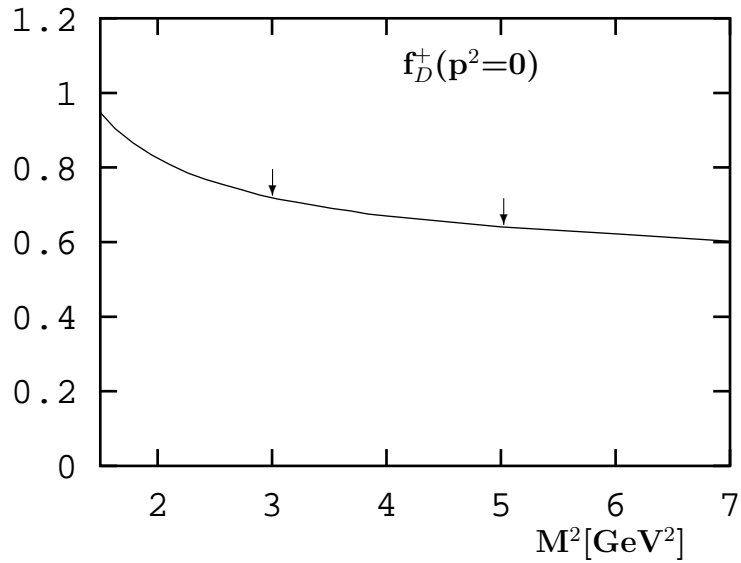


Figure 2: Form factor f_D^+ at zero momentum transfer as a function of the Borel mass squared M^2 . The arrows indicate the fiducial interval in M^2 as described in the text.

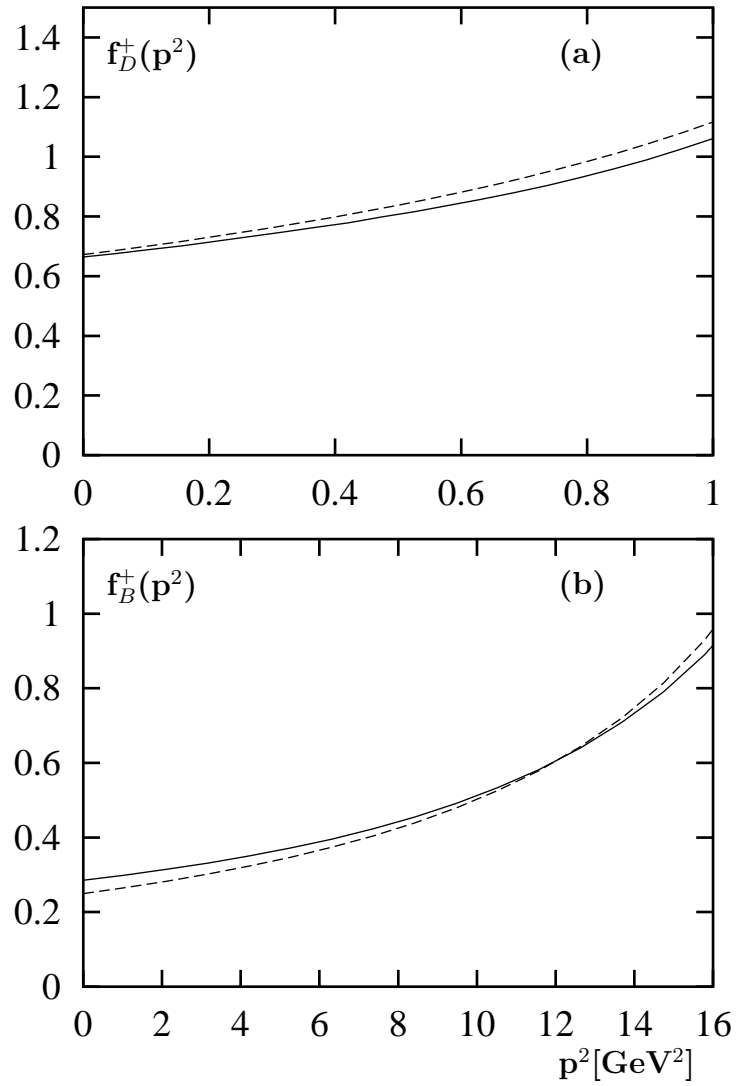


Figure 3: Sensitivity of the heavy-to-light form factors to the nonasymptotic effects in the light-cone wave functions. The dashed curves show the results for purely asymptotic wave functions, while the solid curves include nonasymptotic corrections.

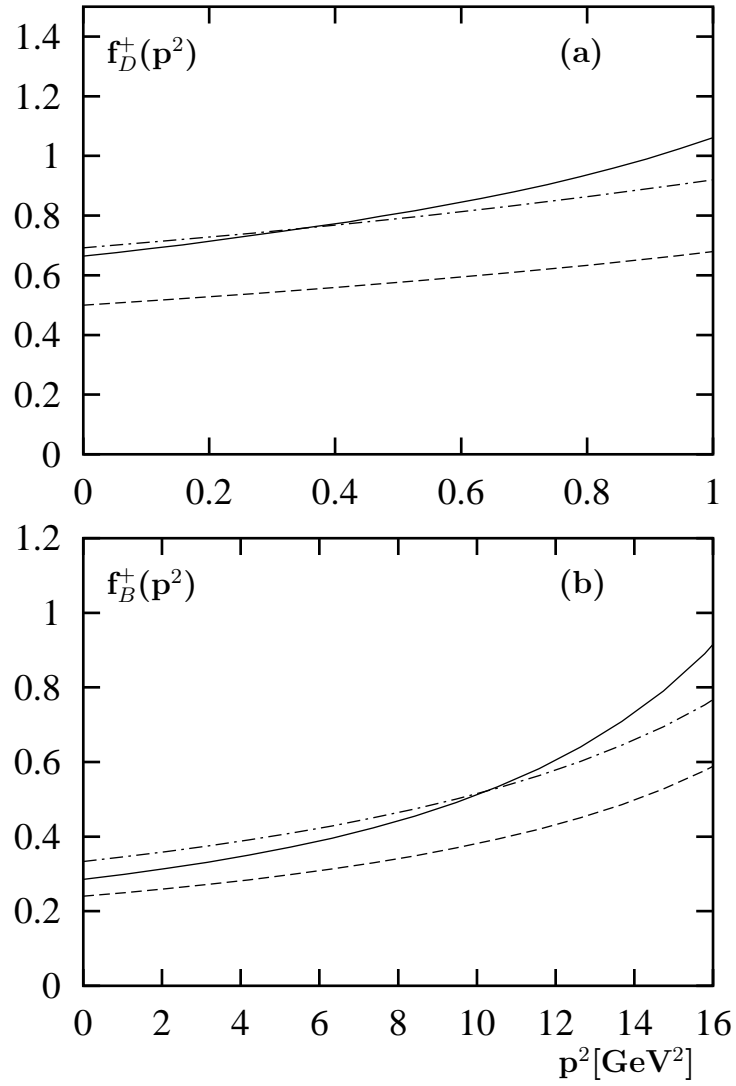


Figure 4: Comparison of our predictions (solid lines) with the form factors calculated from conventional QCD sum rules ^{2,3} (dashed curves) and from a quark model ¹⁰ (dash-dotted curves).