

Multi-color QCD at High Energies and Exactly Solvable Lattice Theories

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Abstract

We examine the generalized leading-logarithmic approximation (LLA) equations for compound states of n -reggeized gluons. It is shown that in multi-color QCD, when $N_c \rightarrow \infty$, these equations have a sufficient number of conservation laws to be exactly solvable. Holomorphic factorization of the wave functions is used to reduce the corresponding quantum mechanical problem to the solution of the one-dimensional Heisenberg model with the spins being the generators of the Möbius group of conformal transformations.

1. Introduction

This talk is centered around obtaining the exact solution to a perturbative QCD evolution equation known as the Bartels-Kwiecinski-Praszalowicz (BKP)-equation¹ in the limiting case where the number of colors of gluons, N_c , is infinite. One may wonder what relevance any equations of perturbative QCD may have in understanding the low energy confining properties of the theory. The answer to this question is not well defined. However, what is clear is that only in the perturbative regime of QCD, we are able to exactly treat gauge and Lorentz invariance. Even then, within this regime one discovers that such a task is nontrivial. Thus the first lesson one gains from examination of perturbative QCD is experience with nonabelian gauge calculations that can be tested for their correctness.

That may be a useful reason for those working in low energy QCD to nevertheless study the high energy regime as a warm-up exercise. However that is not the primary reason for this talk. The general class of equations that we are considering here are the only known evolution equations in QCD that exactly respect gauge invariance and have a kinematic regime in which

they are exactly valid. Although their practical uses are for calculating high energy scattering amplitudes, it is natural to also examine what properties of these equations and their solutions correspond to what we believe to be true at low energy.

To start with, let us first introduce the names of the equations to which we are referring and give some history on their development. The first evolution equation in this class was the Gribov-Lipatov-Altarelli-Parisi (GLAP) equation² which was derived in the early 70's. This equation is a predecessor to the main equations we want to discuss here. From this group, the first was the Fadin-Kuraev-Lipatov (FKL)-equation³, which was derived in 1975. This equation was the initial form of a more contemporary version known as the Balitsky-Fadin-Kuraev-Lipatov (BFKL)-equation³. The FKL-equation was derived for massive Yang Mills theory with a massive Higgs particle and for arbitrary SU(N) gauge group. This is an equation for the two-to-two scattering amplitude in the Regge limit, $m^2 e^{1/g^2} \sim s \gg m^2 \sim t$, where m is the mass of the vector boson, \sqrt{s} is the center of mass energy and $\sqrt{-t}$ is the momentum transfer. We note that the Regge limit also implies the leading-log-approximation, where $g^2 \ln(s/t) \sim 1$ and $g^2 \ll 1$. To obtain the equation, working in momentum space using s-channel unitarity along with analyticity, the amplitude was computed to eighth-order. From this the general form could be deduced into what became the FKL-equation. Only the $t=0$ solution was obtained in [3]. The solution showed that the amplitude violated the Froissart bound. However it should be realized that the region where this violation occurs is also beyond the region where the FKL-approximations are valid.

In 1978 it was verified that there are no infrared divergences in QCD for scattering of colorless particles at arbitrary t in the BFKL-equation. In particular this held at $t=0$, where the problem is typically most pronounced. The solution for arbitrary t was found in 1986⁴. A relevant point for the present discussion is that the calculation was done in transverse coordinate representation (or impact parameter space). In this representation it was recognized that the BFKL-equation had two-dimensional Möbius invariance.

The shortcoming of the BFKL-equation is that it violates the unitarity bounds. To correct this, the suggestive approach is to consider diagrams with an arbitrary number of reggeized gluons. The BFKL-equation only accounts for two reggeized gluons. The equation with N reggeized gluons was obtained by Bartels and by Kwiecinski and Praszalowicz¹. The purpose of this talk is to examine the solutions of this equation for $N_c \rightarrow \infty$. What will be achieved here is a relation of this equation in the above limit, to exactly solvable models. The end result is a reduction of the problem to a one-dimensional lattice model.

Before turning to the quantitative discussion, let us place into perspective what contact this development makes with the problem of confinement. We have believed since the early seventies that Yang-Mills theory is plausibly the low-energy limit of an appropriate string theory. In the high energy limit, one may then ask if any aspect of QCD's string-like nature manifests. There is no known reason from general principles to expect this. Nevertheless, in light of the results we discuss here, we do find a string-like remnant of QCD in this limit.

Examining the issue a little further, we next recall that high energy processes in fact have

an intrinsic dependence on the low energy properties of QCD. We have known since the 60's, that the dominant exchange in a high energy collision at fixed t is the pomeron. From what little we know about the dynamical make-up of the pomeron, we suspect it is some sort of collective excitation made of several gluons plus perhaps quarks. Thus to study the low energy regime of QCD, one way is to focus on particular bound states and try to derive their properties from QCD. However another option is to study the Regge families of hadrons, such as the pomeron, and try to calculate their parameters from QCD. We can not offer any reason why the latter option is better than the former. However, the one evident fact is that we have much better experimental data about Reggeons than about individual hadrons. Also from the point of view of light-cone kinematics, the description of Reggeons is more natural than of individual particles. If one accepts this line of reasoning to its fullest extent, one could imagine calculating masses of hadrons using Reggeon concepts. At present we do not have sufficient control on the approximations involved in our evolution equations to justify such calculations. However one could assume the radius of convergence for our equations is sufficiently large to make some sort of estimates. We will not discuss this point further in this talk.

2. Evolution Equations

The GLAP equation²,

$$\frac{dn_i(x)}{d\xi} = -\omega_i n_i(x) + \sum_j \int_x^1 \frac{dx'}{x} \omega_{j \rightarrow i}\left(\frac{x}{x'}\right) n_j(x') \quad (1)$$

where,

$$\xi = \frac{1}{c} \ln\left(1 + \frac{\alpha}{\pi} c \ln \frac{Q^2}{\mu^2}\right) \quad (2)$$

and

$$\omega_i = \sum_k \int_0^1 dx \omega_{i \rightarrow k}(x), \quad (3)$$

determines the Q^2 -evolution of the parton distributions $n_i(x)$, where $x = \frac{k_+}{p_+}$ is the ratio of the parton to hadron longitudinal momentum in the light-cone frame. The splitting kernels, $\omega_{i \rightarrow k}\left(\frac{x}{x'}\right)$, describe the inclusive probabilities of the parton decay into the opening phase space $d\xi$. Mellin transforming in $\ln \frac{x}{x'}$ gives the anomalous dimension matrix $\gamma(j)$ for the twist two operators in QCD. For example in the case of pure gluodynamics,

$$n_g(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j e^{\xi\gamma(j)} \quad (4)$$

where

$$\gamma(j) = \frac{2}{j(j-1)(j+1)(j+2)} - \frac{1}{12} - \psi(j-1) + \psi(1) \quad (5)$$

and $\psi(j) = \frac{\Gamma'(j)}{\Gamma(j)}$. Note that $\gamma(j) = \gamma(1-j) - \pi \text{ctg}(\pi j)$ and $\gamma(2) = 0$ due to the conservation of the stress tensor $T_{\mu\nu}$. From eq. (4) we obtain that for $x \rightarrow 0$, $n_g(x) \sim \frac{1}{x} \exp(c\sqrt{\xi \ln \frac{1}{x}})$. This implies that total cross-section $\sigma_t(x, Q^2)$, for γ^*p scattering at large energies $\sqrt{s} = \frac{Q}{\sqrt{x}}$ grows more rapidly than any power of lns. This is a violation of the Froissart bound $\sigma_t < c \ln^2 s$.

At small x, for parton distributions $n_g(x, k_\perp)$ depending on transverse parton momentum k_\perp , one should use the BFKL equation³,

$$\frac{dn(x, k_\perp)}{d \ln \frac{1}{x}} = 2\omega(-k_\perp^2)n(x, k_\perp) + \int d^2 k'_\perp K(k_\perp, k'_\perp)n(x, k'_\perp) \quad (6)$$

where $n(x) = \int d^2 k_\perp n(x, k_\perp)$ and

$$\omega(-k_\perp^2) = -\frac{g^2}{16\pi^3} N_c \int d^2 k'_\perp \frac{k_\perp^2}{(k - k')_\perp^2 k'_\perp^2}. \quad (7)$$

where $k_\perp^2 > 0$. Note that the gluon Regge trajectory $j(-k_\perp^2)$ is related to ω by $j(-k_\perp^2) = 1 + \omega(-k_\perp^2)$. The kernel K for $SU(N_c)$ gauge theory is,

$$K(k_\perp, k'_\perp) = \frac{g^2}{4\pi} N_c \frac{1}{(k - k')_\perp^2}. \quad (8)$$

Observe that the infrared divergences cancel in the right hand side of eq.(6).

The solution of eq. (6) can be written in the form³,

$$n(x, k_\perp) = \frac{1}{x} \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{\infty} d\nu \left(\frac{1}{x}\right)^{\omega(\nu, m)} k_\perp^{2i\nu} c_{m, \nu}, \quad (9)$$

where $c_{m, \nu}$ is determined by the initial conditions for $n(x, k_\perp)$ at fixed x, $\phi = \arctg(\frac{k_\perp^x}{k_\perp^y})$, and the eigenvalue $\omega(\nu, m)$ of the corresponding stationary equation is,

$$\omega(\nu, m) = \frac{g^2}{2\pi^2} N_c \int_0^1 \frac{dy}{1-y} [y^{\frac{-1+|m|}{2}} \cos(\nu \ln y) - 1] = \frac{g^2}{2\pi^2} N_c (\psi(1) - \text{Re}\psi(\frac{1}{2} + i\nu + \frac{|m|}{2})). \quad (10)$$

The biggest value of ω is $\omega(0, 0) = \frac{g^2}{\pi^2} N_c \ln 2$, and therefore from eq. (9) we obtain that $n(x, k_\perp) \sim \frac{1}{x} (\frac{1}{x})^{\omega(0,0)}$. This means that the solution of the BFKL-equation also does not agree with the Froissart bound. For this equation as well as for the GLAP-equation, the reason for this violation is that the evolution equations were obtained in the leading logarithmic approximation, where the S-matrix does not satisfy unitarity³.

Thus we find in both cases, the GLAP and BFKL equations, the result is incomplete. As such we will construct a modified leading logarithmic approximation (LLA) that is compatible with the Froissart bound.

3. Partonic Wave Functions

The partonic distributions $n_i(x, k_\perp)$ are proportional to the imaginary part of the scattering amplitude at the momentum transfer $q=0$. It is natural to generalize the evolution equations for arbitrary momentum transfer. In this case the resulting equations could be considered as equations for the hadronic wave function. The usual Schrödinger equation $E\psi = H\psi$ determines the mass of the hadron as a function of its spin, $m^2 = m^2(j)$. To determine $j=j(m^2)$, one can replace this equation by the BFKL equation

$$2H_{12}\psi = E\psi, \quad (11)$$

where

$$E = -16 \frac{\omega\pi^2}{g^2 N_c} \quad (12)$$

and $j = 1 + \omega$ is the position of the j -plane singularity of the t -channel partial wave. The high energy asymptotics of scattering amplitudes are determined by the eigenvalues of equation (11) as $A(s, t) \sim s^{1+\omega(t)}$. The eigenvalues ω could in general also depend on $t=-q^2$, but due to the conformal invariance of the BFKL equation³, in LLA this dependence is absent. The operator H_{12} on the left hand side of eq. (11) is⁴,

$$H_{12} = \frac{1}{|P_1|^2 |P_2|^2} P_1^* P_2 \ln |\rho_{12}|^2 P_1 P_2^* + h.c. + \ln(|P_1|^2 |P_2|^2) - 4\psi(1), \quad (13)$$

where $\rho_{12} = \rho_1 - \rho_2$, $\rho_r = x_r + iy_r$, the momenta $P_r = i \frac{\partial}{\partial \rho_r}$, and h.c means the complex conjugated expression.

To unitarize the results of the LLA, one must generalize eq.(11) for compound states with an arbitrary number of gluons. Such a generalization was done in [1]. Here we discuss the BKP-equation for the large N_c case. Thus we consider the equation,

$$H\psi = E\psi \quad (14)$$

with E as given in eq. (12) and where the Hamiltonian H contains only interactions of neighboring particles,

$$H = \sum_{r=1}^m H_{r,r+1}. \quad (15)$$

The pair Hamiltonian $H_{r,r+1}$ acts on the coordinates r and $r+1$ of the gluons as given by eq. (13).

Note that there is also a generalization of the GLAP-equation (1) for matrix elements of quasipartonic operators of high twist⁵. This B'F'KL-equation is also simplified in the region of

large N_c . It takes the form of eqs. (14) and (15) with the pair kernel describing the evolution of twist two operators. The eigenvalues of this equation are proportional to the anomalous dimensions of the quasi-partonic operators whose contributions are important in the small- x region. We will not discuss this equation any further below.

4. The BKP-Equation in the Large N_c Limit

From eqs. (13) and (14) one can derive the holomorphic separability of the Hamiltonian, which is a central property for our present discussion. Thus we can write

$$H = H + H^*, \quad (16)$$

where H and H^* act on the holomorphic (ρ_j) and antiholomorphic (ρ_j^*) coordinates respectively with

$$H = \sum_{j=1}^n H_{j,j+1}, \quad (17)$$

and similarly for H^* . The pair holomorphic Hamiltonian is,

$$H_{j,j+1} = \frac{1}{P_j} \ln(\rho_{j,j+1}) P_j + \frac{1}{P_{j+1}} \ln(\rho_{j,j+1}) P_{j+1} + \ln(P_j, P_{j+1}) - 2\psi(1). \quad (18)$$

An important outcome of holomorphic separability is that the solution of eq. (14) separates as⁶,

$$\psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum \psi(\rho_1, \rho_2, \dots, \rho_n) \tilde{\psi}(\rho_1^*, \rho_2^*, \dots, \rho_n^*) \quad (19)$$

where the sum is over all degenerate solutions of the Schrödinger equation in the holomorphic and antiholomorphic subspaces,

$$E = \epsilon + \tilde{\epsilon}, H\psi = \epsilon\psi, H^*\psi = \tilde{\epsilon}\psi \quad (20)$$

The pair Hamiltonian $H_{j,j+1}$ in eq. (18) can also be written in the forms⁶,

$$H_{j,j+1} = \ln(\rho_{j,j+1}^2 P_j) + \ln(\rho_{j,j+1}^2 P_{j+1}) - 2 \ln \rho_{j,j+1} - 2\psi(1) \quad (21)$$

$$= \rho_{j,j+1} \ln(P_j P_{j+1}) \rho_{j,j+1}^{-1} + 2 \ln \rho_{j,j+1} - 2\psi(1) \quad (22)$$

From the above representations, it is obvious that H is invariant under the Möbius transformations⁴,

$$\rho_j \rightarrow \frac{a\rho_j + b}{c\rho_j + d}, \quad (23)$$

where a,b,c, and d are arbitrary complex parameters. The generators of these transformations are

$$\vec{M} = \sum_{i=1}^n \vec{M}_i, M_i^z = \rho_i \partial_i, M_i^- = \partial_i, M_i^+ = -\rho_i^2 \partial_i \quad (24)$$

One can also obtain the transposed Hamiltonian H^T from H by two different similarity transformations,

$$H^T = P_1 P_2 \cdots P_n H P_1^{-1} P_2^{-1} \cdots P_n^{-1} \quad (25)$$

$$= \rho_{12}^{-1} \rho_{23}^{-1} \cdots \rho_{n1}^{-1} H \rho_{12} \rho_{23} \cdots \rho_{n1}. \quad (26)$$

This implies that there are two different normalization conditions for the solutions of eq. (14) which are compatible with eqs. (25) and (26). These are,

$$\|\psi_1\|^2 = \int \psi^* \prod_{r=1}^n d\rho_r P_r \psi \quad (27)$$

$$\|\psi_2\|^2 = \int \psi^* \prod_{r=1}^n \frac{d\rho_i}{\rho_{r,r+1}} \psi. \quad (28)$$

From eqs. (25) and (26) we conclude that there is a nontrivial differential operator⁶,

$$A = \rho_{12} \rho_{23} \cdots \rho_{n1} P_1 P_2 \cdots P_n, \quad (29)$$

which commutes with H ,

$$[A, H] = 0 \quad (30)$$

Below we will show that there are an infinite number of operators that commute with H .

5. Equivalence Between Multi-color QCD at High Energies and an Exactly Solvable Spin Model

We can write down the operator A in eq.(29) as follows⁷,

$$A = i^n \text{tr}(M_1 M_2 \cdots M_n), \quad (31)$$

where M_i is the 2*2 matrix constructed from the Möbius generators M_i in eq. (24),

$$M_i = \begin{pmatrix} \rho_i \partial_i & \partial_i \\ -\rho_i^2 \partial_i & -\rho_i \partial_i \end{pmatrix} \quad (32)$$

In representation (31) the operator A can be interpreted as a transfer matrix for a lattice theory. On the links in the "space" direction (the auxiliary subspace) there are discrete variables ξ taking values $\xi = \pm 1$ and on the links in the "time" direction (the quantum subspace), there are continuous variables ρ .

To solve eq (20) exactly, one should find the one parameter family of integrals of motion, including the operator A of eq.(31). It turns out⁷ that such a family is the following,

$$t(\theta) = \text{tr}(L_1(\theta) L_2(\theta) \cdots L_n(\theta)), \quad (33)$$

where,

$$L_i(\theta) = \begin{pmatrix} \theta + \rho_i \partial_i & \partial_i \\ -\rho_i^2 \partial_i & \theta - \rho_i \partial_i \end{pmatrix} \quad (34)$$

is the so called L-operator. Let us also introduce the monodromy matrix,

$$T(\theta) = L_1(\theta)L_2(\theta) \cdots L_n(\theta). \quad (35)$$

One can verify⁸ that it satisfies the following Yang-Baxter equation:

$$T^{i_1 i_1'}(u)T^{i_2 i_2'}(v)(u - v - P_{12}) = (u - v + P_{12})T^{i_2 i_2'}(v)T^{i_1 i_1'}(u), \quad (36)$$

where P_{12} is the operator that interchanges the matrix spin indices (the right and left ones correspondingly). By taking the traces over indices i_r and $i_{r'}$, we obtain:

$$t(u)t(v) = t(v)t(u), \quad (37)$$

so that the operators defined in eq. (33) commute with each other.

Now we want to prove that the operator $t(\theta)$ of eq. (33) also commute with the holomorphic Hamiltonian in eq. (20). For this purpose the idea we use is⁹ that the spin model with the transfer matrix (33) can be considered as a modification of the Heisenberg model. However, instead of the fundamental representation of the group $SU(2)$ with spin $S=\frac{1}{2}$, here we have the infinite-dimensional representation of the Möebius group $SU(1,1)$ with spin $S=0$. For this new spin model, there is a unique Hamiltonian describing the interaction of nearest spins for which the model is exactly solvable. The general method of obtaining this Hamiltonian was developed many years ago⁸. Briefly, to do this one should construct the operator $L^{1,2}(\theta)$, which satisfies the trilinear Yang-Baxter equation for the case when both the quantum and auxiliary subspaces are one-dimensional (ρ_1 and ρ_2). Then for this new spin model, the Hamiltonian is given by eq. (17), where $H_{1,2}$ can be calculated from the small- θ expansion of $L^{1,2}(\theta)$:

$$L^{1,2}(\theta) = P^{1,2}(1 + \theta H_{1,2} + \cdots). \quad (38)$$

Here $P^{1,2}$ is the operator which interchanges the coordinates ρ_1 and ρ_2 . According to the general theory⁸, $L^{1,2}(\theta)$ should also satisfy the linear Yang-Baxter equation:

$$L_1(u)L_2(v)L^{1,2}(v - u) = L^{1,2}(v - u)L_2(v)L_1(u) \quad (39)$$

In this equation, the L_i operators are 2*2 matrices (34). From eq. (39) we find³ that $H_{1,2}$ depends only on the Casimir operator of the conformal group. This can be written in the form:

$$(\vec{M}_1 + \vec{M}_2)^2 = \hat{m}(\hat{m} - 1) \quad (40)$$

We also find that $H_{1,2}$ satisfies the equation

$$[H_{12}(\hat{m}) - H_{12}(\hat{m} - 1)](\hat{m} - 1) = 2, \quad (41)$$

for which the solution is

$$H_{1,2} = \psi(\hat{m}) + \psi(1 - \hat{m}) - 2\psi(1) \quad (42)$$

up to an additional term $\Delta(\hat{m})$, which is a periodic function (ie. $\Delta(\hat{m}) = \Delta(\hat{m} + 1)$). Using eq. (10), we can verify that the expression for $H_{1,2}$ determined by eqs. (18) and (42) coincide. Thus, according to the general theory in [8], the Hamiltonian (17) commutes with all operators of the type $t(\theta)$ in eq.(33).

6. Conclusion

We have shown above that in the generalized leading logarithmic approximation, the equation for the compound states of n -reggeized gluons is significantly simplified in the large N_c -limit. In particular, it is conformally invariant and the Hamiltonian has the remarkable property of holomorphic separability. In addition, the equations for holomorphic and antiholomorphic wave functions have a sufficient number of conservation laws to be exactly solvable. This is related with the fact that the Hamiltonians in the corresponding subspaces coincide with the local Hamiltonians of the exactly solvable Heisenberg model for spin $S=0$. As such, the quantum mechanical problem posed in eq. (20) is reduced to the pure algebraic one of constructing the representations of the Yang-Baxter algebra in eq. (36). The simple method of finding these representations was developed in [10]. It is based on the solution of the Baxter equation

$$\Lambda(\lambda)Q(\lambda) = (\lambda + i)^n Q(\lambda + i) + (\lambda - i)^n Q(\lambda - i), \quad (43)$$

where n is the number of reggeized gluons, $\Lambda(\lambda)$ are the eigenvalues of the operator $t(i\lambda)$ in eq. (33) and the function Q is the integer function in the complex λ - plane. The eigenvalue $\Lambda(\lambda)$ has the polynomial expansion in λ ,

$$\Lambda(\lambda) = 2\lambda^n - j(j+1)\lambda^{n-2} + \dots + A, \quad (44)$$

where n is the number of reggeized gluons, $m=j-1$ is the conformal weight of the corresponding composite operator, $j(j+1)$ is the eigenvalue of the Casimir operator $(\sum_i \vec{M}_i)^2$, and A is the eigenvalue of the integral of motion A . The eigenvalues and eigenfunctions of eqs. (20) can be expressed through $Q(\lambda)$. For $n=2$ eq. (43) is solved in terms of hypergeometric functions. For $n=3$ the solution of eq. (43) for integer j can be expanded as a linear combination of its solutions $Q_j^{(2)}$ for $n=2$ as,

$$Q(\lambda) = \sum_{k=1}^j d_k(A) Q_k^{(2)}(\lambda), \quad (45)$$

where the parameter A is determined in eq.(44) and $d_k(A)$ are orthogonal polynomials satisfying the recurrence relations,

$$Ad_k(A) = \frac{k(k+1)}{2k+1}(k-j)(k+j+1)(d_{k+1}(A) + d_{k-1}(A)). \quad (46)$$

The quantization condition for the eigenvalues A is,

$$d_j(A) = 0. \quad (47)$$

It is possible to express the energies ϵ in eq. (20) directly in terms of $d_k(A)$, when eq.(46) is analytically continued to complex j . The solution of eq. (46) would give a possibility to find the Odderon intercept¹¹.

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