The Limiting Transition To The Light-Front Hamiltonian

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Abstract

We consider Hamiltonian of field theory at $x^0 = 0$ between fastly moving states and investigate the limit of infinite momentum $(p^3 \to \infty)$ of the states. In this way we get the information about the Hamiltonian on the light-front. We fix ultraviolet cutoff in transversal momenta and introduce partial renormalization of the remaining ultraviolet divergences. For the "$\lambda\phi^4$" and Yukawa models we assume the approximate decomposition of the fields in "hard" and "soft" parts with respect to momentum argument of the Fourier modes (in x^3) of these fields ("hard" momenta increasing proportionally to the momentum p^3 of the states). The resulting light-front Hamiltonian in such approximation contains new terms comparatively to the canonical form. They depend on vacuum averages of "soft" field operators which are unknown but can play the role of vacuum parameters in the light-front Hamiltonian. For the Yukawa model such parameters form entire functions in transversal space of (noncoinciding) arguments of field operators entering this vacuum averages.

1. Introduction

It exists an open question whether the light-front field theory is equivalent to usual equal-time one. A definite answer was found only for simple 2 - dimensional models (like $\lambda\phi^4$ or Schwinger model[1,2]). Here we analyse this problem using, as in[1,2], the limiting transition to the light-front.

2. Light-Front Hamiltonian

For explanation we start with (1+1) - dimensional $\lambda\phi^4$- model. Lorentz coordinates are x^0 and x^3.

Consider matrix elements of equal-time Hamiltonian $(P^0)_{x^0=0}$ and of momentum operator $(P^3)_{x^0=0}$ between "fastly moving states" (with total momentum $p^3 \to \infty$). Then the matrix elements of the light-front Hamiltonian $(P^-)_{x^+=0}$ can be obtained by following limiting transition (equivalent to Lorentz boost to infinite momentum frame):

$$\langle q'^+|(P^-)_{x^+=0}|q^+\rangle = \lim_{\eta\to 0}\ (\frac{1}{\sqrt{2}\,\eta}\langle p'^3 = \frac{q'^+}{\sqrt{2}\,\eta}|(P^0-P^3)_{x^0=0}|\ p^3 = \frac{q^+}{\sqrt{2}\,\eta}\rangle), \tag{1}$$

where q^+,q'^+ are finite light-front momenta.

Let $H \equiv (P^0 - P^3)_{x^0=0}$, $x \equiv x^3$, $p \equiv p^3$, $\partial \equiv \partial_3$. In terms of canonical variables $\phi(x), \Pi(x) = \partial_0\phi$ we have at $x^0 = 0$

$$H \;=\; \int dx\,\{\frac{1}{2}(\Pi+\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4\}, \qquad P^3 = -\int dx\Pi\partial\phi. \tag{2}$$

Next we estimate the η - dependence of matrix elements of operators between fastly moving states (for which $|p| \geq \eta^{-1}\delta$, δ is a fixed parameter, and $\eta \to \infty$). We suppose that this dependence is determined by Lorentz boost properties of the operators. Accordingly we rescale the momentum integration variables in the "Hamiltonian" H:

$$p \equiv q\eta^{-1},\ a(p) \equiv \sqrt{\eta}\,\check{a}(q),$$

$$[a(p), a^\dagger(p')] = \delta(p-p'),\ [\check{a}(q), \check{a}^\dagger(q')] = \delta(q-q'), \tag{3}$$

where $a^\dagger(p), a(p)$ are "bare" creation and annihilation operators (see eq.(8) below). Then some power of η is extracted from each term of the Hamiltonian.

If we keep only the modes with $|p| \geq \eta^{-1}\delta$, we get, after limiting transition, the naive canonical light-front Hamiltonian. As a next approximation we take into account the modes with $|p| \leq \Lambda$, where $\Lambda \gg \delta$, and Λ does not depend on η. The intermediate modes with $\Lambda < |p| < \eta^{-1}\delta$ are neglected. Then Hamiltonian can be decomposed in powers of η as follows:

$$H = \frac{H_0}{\eta} + H_1 + \eta H_2 + O(\eta^2), \tag{4}$$

where

$$H_0 \;=\; (\frac{1}{2}\eta\int dx(\Pi+\partial\phi)^2_{|p|\geq\eta^{-1}\delta})_{\eta\to 0} = \int_{-\infty}^{-\delta} dq|2q|\check{a}^\dagger(q)\check{a}(q), \tag{5}$$

$$H_1 \;=\; (H)_{|p|\leq\Lambda}, \tag{6}$$

$$H_2 \;=\; :\int dy\{\frac{m^2}{2}\check{\phi}^2 + 6\lambda\hat{\phi}^2(0)\check{\phi}^2 + 4\lambda\hat{\phi}(0)\check{\phi}^3 + \lambda\check{\phi}^4\}:. \tag{7}$$

Here

$$\check{\phi}(y) \;\equiv\; (\phi(x\equiv\eta y))_{|p|\geq\eta^{-1}\delta} = \int_{|q|\geq\delta}\frac{dq}{\sqrt{4\pi|q|}}(\check{a}(q)e^{iqy} + h.c.), \tag{8}$$

$$\hat{\phi}(x) \;\equiv\; (\phi(x))_{|p|\leq\Lambda}; \tag{9}$$

$(\ldots)_{|p|\leq\Lambda}$ means that only the modes with $|p| \leq \Lambda$ are included.

Now we apply the perturbation theory with respect to η:

$$H|f\rangle = E|f\rangle, \quad |f\rangle = \sum_{n=0}^{\infty} \eta^n |f_n\rangle, \quad E \;=\; \frac{1}{\eta}\sum_{n=0}^{\infty} \eta^n E_n, \tag{10}$$

or

$$(H_0 - E_0)|f_0\rangle = 0, \tag{11}$$

$$(H_1 - E_1)|f_0\rangle + (H_0 - E_0)|f_1\rangle = 0, \tag{12}$$

$$(H_2 - E_2)|f_0\rangle + (H_1 - E_1)|f_1\rangle + (H_0 - E_0)|f_2\rangle = 0, \tag{13}$$

The light-front "energies"are equal to $\lim_{\eta\to 0}(\eta^{-1}E(\eta))$. So the proper solution corresponds to $E_0 = E_1 = 0$. We assume that $a(p)|physical \;\; vacuum\rangle = 0$ if $|p| \geq \eta^{-1}\delta, \eta \to 0$ because at large $|p|$ the H_0 part of H dominate. Then each solution of eq.(11) at $E_0 = 0$ is a superposition of states

$$a^\dagger(p_1)\ldots a^\dagger(p_n)\hat{\phi}\ldots\hat{\phi}\hat{\Pi}\ldots\hat{\Pi}|physical \;\; vacuum\rangle, \tag{14}$$

where $p_1, \ldots, p_n \geq \eta^{-1}\delta, \; \hat{\phi} \equiv (\phi)_{|p|\leq\Lambda}, \hat{\Pi} \equiv (\Pi)_{|p|\leq\Lambda}$.

Let $\mathcal{P}_0$ be a projector onto the subspace of all such superpositions, so that $\mathcal{P}_0|f_0\rangle = |f_0\rangle, \mathcal{P}_0 H_0 = 0$. Then at $E_0 = E_1 = 0$ it follows from eq.(12) that

$$\mathcal{P}_0 H_1 \mathcal{P}_0|f_0\rangle = 0. \tag{15}$$

The operator H_1 depends only on "soft" fields $\hat{\phi}, \hat{\Pi}$. Between soft states we have $H_1 = H$, which is positive definite. Hence the eq.(15) can be fulfiled only under absence of $\hat{\phi}, \hat{\Pi}$ in the states (14). Thus each solution $|f_0\rangle$ of eqs (11) and (15) is a superposition of states

$$a^\dagger(p_1)\ldots a^\dagger(p_n)|physical \;\; vacuum\rangle = 0, \; p_1, \ldots, p_n \geq \eta^{-1}\delta. \tag{16}$$

According to eq.(12) at $E_0 = E_1 = 0$ we can than put $|f_1\rangle = 0$ without destroying the generality.

Let $\mathcal{P}_0'$ be the projector onto the subspace of all superpositions of states (16), so that $\mathcal{P}_0' H_0 = 0, \mathcal{P}_0'|f_0\rangle = |f_0\rangle$. Then from eq.(13) one gets

$$\mathcal{P}_0' H_2 \mathcal{P}_0'|f_0\rangle = E_2|f_0\rangle. \tag{17}$$

The eq.(17) permits to find E_2. Some terms of H_2 contains, beside hard fields, also products of "soft" fields $\hat{\phi}$. Between states $|f_0\rangle$ this products reduce to vacuum averages of them, because $\hat{\phi}$ comute with hard fields. Puting

$$\sqrt{2}\, q \equiv q^+, \; (\frac{1}{\sqrt{2}})y \equiv x^-,$$

$$\check{\phi}_+(y) \equiv \int_{\delta>0}^{\infty} \frac{dq}{\sqrt{4\pi q}} (\check{a}(q) exp(iqy) + h.c.) \equiv \Phi(x^+ = 0, x^-), \tag{18}$$

we reduce the operator $\frac{1}{\sqrt{2}}\mathcal{P}_0' H_2 \mathcal{P}_0'$ to the form

$$(P^-)_{x^+=0} \equiv \frac{1}{\sqrt{2}}\mathcal{P}_0' H_2 \mathcal{P}_0' =: \int dx^- \{\frac{1}{2}(m^2 + 12\lambda\langle\hat{\phi}^2\rangle_v)\Phi^2 + 4\lambda\langle\phi\rangle_v \Phi^3 + \lambda\Phi^4\} : . \tag{19}$$

This operator is the light-front Hamiltonian which differs from the naive one by terms with unknown "condensate" parameters $\langle\hat{\phi}^2\rangle_v, \langle\phi\rangle_v$. The same form is obtained in a better approximation, when one uses the "Gauss" vacuum and corresponding operators $a, a^\dagger$ instead of bare ones[2].

To apply this procedure to (3+1) - dimensional $\lambda\phi^4$ - model we need only to introduce a transversal cutoff $|p^\perp| \leq \Lambda_\perp$, where $p^\perp \equiv (p^1, p^2)$. In (3+1) - models containing fermions, scalar and vector particles we can also introduce the transversal cutoff $|p^\perp| \leq \Lambda_\perp$. Then only logarithmic divergences of bosonic masses remain. In general they cannot be removed by normal ordering. So, infinite mass counterterms must be added to the Hamiltonian from the beginning. No divergences specific to Hamiltonian approach and absent in the Feynman formalism arise at this stage because the counterterms do not contain time derivatives[4]. To make the analysis rigorous one has to transform the Hamiltonian contaning counterterms to a finite form before the transition $\eta \to 0$. (by "similarity transformation"[3]). Here we describe shortly only the formal result for Yukawa model, using as before the approximate decomposition for fields, which formally escapes the ultraviolet divergences (at fixed $\Lambda_\perp$).

The model contains one scalar field $\phi(x)$ and one bispinor fermionic field

$$\Psi(x) \equiv \begin{pmatrix} \Psi_{+,\alpha}(x) \\ \Psi_{-,\alpha}(x) \end{pmatrix},$$

$\alpha = 1, 2$. The "Hamiltonian" $H \equiv (P^0 - P^3)_{x^0=0}$ is

$$H = H_\phi + i\int d^3x \Psi^\dagger \gamma^0 \left((\gamma^0 - \gamma^3)\partial_3 - \gamma^\perp \partial_\perp - i(M + g\phi)\right)\Psi. \tag{20}$$

As before one obtains the decomposition

$$H = \frac{1}{\eta}\left(H_0 + \eta H_1 + \eta^{3/2} H_{3/2} + \eta^2 H_2\right) + O(\eta^{3/2}). \tag{21}$$

The $\eta^{3/2}$ appears because terms with three field operators exist in H. Consequently the perturbation parameter is $\eta^{1/2}$. It must be $E_0 = E_{1/2} = E_1 = E_{3/2} = 0$. The E_2 is connected with light-front Lagrangian. The result is

$$(P^-)_{x^+=0} \;=\; :\int dx^- \int d^2x^\perp \{\frac{1}{2}(m^2 + 12\lambda\langle\hat{\phi}^2\rangle_v)\Phi^2 + 4\lambda\langle\phi\rangle_v\Phi^3 + \lambda\Phi^4 +$$

$$
\begin{aligned}
&+\frac{i}{\sqrt{2}}\check{\Psi}_+^\dagger[\not{\partial}_\perp + M + g\Phi + g\langle\phi\rangle_v]\partial_-^{-1}[\not{\partial}_\perp - M - g\Phi - g\langle\phi\rangle_v]\check{\Psi}_+ - \\
&-\frac{ig^2}{\sqrt{2}}(\langle\hat{\phi}^2\rangle_v - \langle\phi\rangle_v^2)\check{\Psi}_+^\dagger\partial_-^{-1}\check{\Psi}_+ - \\
&-g^2\int dx'^- \int d^2x'^\perp \check{\Psi}_+^\dagger(\vec{x}')F(x'^\perp - x^\perp)\check{\Psi}_+(\vec{x})\Phi(\vec{x}')\Phi(\vec{x}) - \\
&-g^2\int dx'^-[\Delta_1(x'^- - x^-)\Phi(x^\perp, x'^-)\Phi(x^\perp, x^-) + \\
&+\Delta_2(x'^- - x^-)\check{\Psi}_+^\dagger(x^\perp, x'^-)F(0)\check{\Psi}_+(x^\perp, x^-)] - (4\pi\sqrt{2})^{-1}\Lambda_\perp^2 \times \\
&\times \int dx'^-\Delta_3(x'^- - x^-)(M + g\langle\phi\rangle_v + g\Phi)_{x'^-}\,(M + g\langle\phi\rangle_v + g\Phi)_{x^-}\} :,
\end{aligned}
$$

where Φ is defined as before,$\check{\Psi}_\pm$ are hard components of spinors (defined as for scalar field), $\Delta_i(x'^- - x^-)$ (with $i = 1, 2, 3$) and $F(x'^\perp - x^\perp)$ are some functions depending on vacuum averages of products of soft fields $\hat{\phi}, \hat{\Psi}_{\pm,\alpha}, \hat{\Psi}_{\pm,\alpha}^\dagger$ taken at different points of space. The appearence of such functions can be compared with the approach of the work[3] where also entire functions are introduced into the light-front Hamiltonian from other reasons. The described considerations may help to reduce the arbitrarines contained in these functions.

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