

# The Mandelstam–Leibbrandt prescription and the Discretized Light Front Quantization.

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## Abstract

It is shown that the quantization of the unphysical degrees of freedom, which leads to the Mandelstam–Leibbrandt prescription for the infrared spurious singularities in the continuum light cone gauge, does indeed suggest some quite natural recipe to treat the zero modes in the Discretized Light Front Quantization of gauge theories.

## 1. Introduction

Light Front Dynamics (LFD) of field theories, in which  $x^+ = x^0 + x^3$  plays the role of the evolution parameter, has many appealing and useful features<sup>1</sup>. Among them, the maybe most important one concerning the quantum theory, is the occurrence of a nonperturbative vacuum simpler than in the ordinary time formulation. In the case of gauge theories, the LFD leads to the light–cone gauge as the most convenient choice for the subsidiary condition. However, owing to the need of defining the inverse of  $\partial_- \equiv \frac{\partial}{\partial x^-}$ ,  $x^- = x^0 - x^3$ , the difficult problem arises of a consistent handling for the infrared spurious singularities.

Since the very early attempts<sup>2,3</sup> to deal with the above matter, the attitude was the following: the zero modes, associated to  $\partial_-$ , are eliminated assuming suitable boundary conditions for all the fields at  $x^- = \pm\infty$  and, consequently, the spurious infrared singularities are defined, in the momentum space, through the Cauchy Principal Value (CPV) prescription (or some equivalent to it). It turns out that the ensuing Feynman perturbation theory does not fulfil any power counting criterion and eventually leads to inconsistent results, even at one loop<sup>4</sup> as in the SUSY N=4 model. As a consequence, the above mentioned philosophy is ruled out by explicit perturbative calculations.

In order to restore the agreement between light-cone gauge and covariant gauge perturbative calculations, in the SUSY N=4 model, S. Mandelstam proposed to define<sup>5</sup> the spurious infrared singularities as follows:

$$\frac{1}{[k_-]} \equiv \lim_{\epsilon \rightarrow 0^+} \frac{1}{k_- + i\epsilon \operatorname{sgn} k_+} \quad , \quad (1.1)$$

where the limit is understood in the sense of distribution (an alternative, but equivalent, form<sup>6</sup>

has been proposed by G. Leibbrandt). Shortly afterwards it has been shown that the Mandelstam–Leibbrandt (ML) prescription (1.1) originates from canonical **equal time** quantization<sup>7</sup> and, later on, that the corresponding Feynman perturbation theory lies on the same firm ground as in the covariant gauges. As a matter of fact, the ML prescription fulfils generalized power counting, it allows the Wick rotation in the Feynman integrals and, very remarkably, it leads to perturbative renormalizability and unitarity<sup>7,8</sup>, once in the effective action some non-local and non-covariant counterterms are introduced, which are **completely determined** to all order in the loop expansion.

## 2. The continuum Light Front formulation

As previously mentioned, the ML prescription naturally emerges from the ordinary equal time canonical quantization. Very recently, it has been shown<sup>9</sup> that actually the ML form of the propagator can be obtained from a Light Front formulation, provided some zero modes are properly taken into account and suitably quantized. Let me briefly recall the main points of the derivation.

The lagrangean density of the free radiation field in the light–cone gauge is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_k\lambda\partial_k A^+ \quad , \quad (2.1)$$

with  $x_\perp = (x^1, x^2)$ ,  $j, k, \dots = 1, 2$ ,  $A^\pm = A^0 \pm A^3$ , the evolution being along  $x^+$ . The subsidiary condition  $A^+ = 0$  immediately follows, if the boundary condition  $A^+(x^\pm, x_\perp) \rightarrow 0$ , when  $|x_\perp| \rightarrow 0$ , is assumed and the equations of motion read

$$\partial_-^2 A_+ + \partial_- \partial_k A^k = 0 \quad , \quad (2.2a)$$

$$(4\partial_+ \partial_- - \partial_\perp^2)A_k - \partial_k(\partial_- A_+ + \partial_j A^j) = 0 \quad , \quad (2.2b)$$

$$2\partial_+ \partial_- A_+ - \partial_\perp^2 A_+ - 2\partial_+ \partial_k A^k = 2\partial_\perp^2 \lambda \quad , \quad (2.2c)$$

leading to  $\partial_- \lambda = 0$ . If we impose the boundary condition  $\lambda \rightarrow 0$  when  $x^- \rightarrow \pm\infty$ , then  $\lambda \equiv 0$  and no zero modes are present. However, as previously emphasized, this eventually yields the CPV prescription for the spurious singularity in the Feynman propagator and, therefore, to the inconsistent perturbation theory. To be consistent we have to keep  $\lambda = \lambda(x^+, x_\perp) \neq 0$ , which has to be correctly determined within the Light Front formalism. To this aim let us define

$$A_k(x) \equiv T_k(x) + \frac{\partial_k}{\partial_\perp^2} \varphi(x^+, x_\perp) \quad ; \quad (2.3)$$

from the equations of motion we obtain

$$(4\partial_- \partial_+ - \partial_\perp^2)T_k = 0 \quad , \quad (2.4a)$$

$$A_+ = \partial_-^{-1} \partial_k T_k - 2 \left( \lambda + \frac{\partial_+}{\partial_\perp^2} \varphi \right) = \frac{4\partial_+}{\partial_\perp^2} \partial_k T_k - 2 \left( \lambda + \frac{\partial_+}{\partial_\perp^2} \varphi \right) \quad , \quad (2.4b)$$

since we are working with *on shell* free fields  $T_k$ .

Now, in order to find some Light Front operator algebra **isomorphic** to the canonical equal time operator algebra, we have to impose ( $\tilde{v} \equiv (v^-, v_\perp)$ )

$$[T^j(x), \partial^+ T^k(y)]_{x^+=y^+} = i\delta^{jk} \delta^{(3)}(\tilde{x} - \tilde{y}) \quad ; \quad (2.5)$$

$$[\varphi(\tilde{x}), \lambda(\tilde{y})] = i\delta(x^+ - y^+) \delta^{(2)}(x_\perp - y_\perp) \quad ; \quad (2.6)$$

$$[T^k(x), \varphi(\tilde{y})] = [T^k(x), \lambda(\tilde{y})] = [\varphi(\tilde{x}), \varphi(\tilde{y})] = [\lambda(\tilde{x}), \lambda(\tilde{y})] = 0 \quad . \quad (2.7)$$

Some key remarks are in place concerning the above operator algebra: namely,

i) the commutator  $[\varphi(\tilde{x}), \lambda(\tilde{y})]_{x^+=y^+}$  does not make sense. This means that it is not possible to simultaneously specify all the fields on the same “initial” hyperplane  $x^+ = \text{constant}$ , but one has to specify the zero mode commutators for different (not coincident) Light Front “times”. A related feature is that the canonical Light Front Hamiltonian  $P_+$  does not provide the evolution of the zero mode fields (see also below).

ii) The space of the state vectors is an indefinite metric linear space, as we already know from canonical equal time quantization<sup>7</sup>.

iii) In the Light Front dynamics, we have to require<sup>3,7</sup> that all the components of  $T_\mu(x)$  are operator valued distributions acting on the Besov space  $\mathcal{T}(\mathbf{R}^3)$ , at fixed  $x^+$ ; this is the space of the rapidly decreasing functions, whose integral over  $x^-$  does indeed vanish. Consequently one might attempt to formulate the theory on a compact domain along  $x^-$  in the presence of periodic boundary conditions.

### 3. Discretized Light Front Quantization

The Discretized Light Front Quantization (DLFQ) has been proposed<sup>10</sup> to provide an infrared cut-off for the spurious singularities and some alternative non-perturbative computer algorithm other than Euclidean lattice QCD; moreover one can easily appreciate the non trivial features associated with the onset of the zero modes. Let us define our theory on the hypercylinder  $\Omega^- = \{x^\mu | x^+ \in \mathbf{R}, x_\perp \in \mathbf{R}^2; x^- \in [-L, L]\}$  and impose to  $A_\mu$  periodic boundary conditions: namely,

$$A_\mu(x) = A_\mu^\circ(x^+, x_\perp) + \sum_{n \neq 0} A_\mu^n(x^+, x_\perp) \exp \left\{ i \frac{\pi n}{L} x^- \right\} \quad , \quad (3.1)$$

the zero modes  $A_\mu^{n=0} \equiv A_\mu^\circ$  being now independent fields in the LFD.

Let us first discuss the free radiation field. The normal mode sector,  $n \neq 0$ , can be treated according to the usual Light Front formulation, since the derivative  $\partial_-$  can be inverted as

$$(\partial_-^{-1} * \Phi)(x) = \sum_{n \neq 0} \frac{L}{i\pi n} \Phi^n(x^+, x_\perp) \exp \left\{ i \frac{\pi n}{L} x^- \right\} \quad , \quad (3.2)$$

where  $\Phi$  is any of the normal field components. Among the zero modes, the component  $A_-^\circ$  is gauge invariant, since the infinitesimal gauge transformation  $\delta A^+(x) = \partial_- \Lambda(x)$  involves a periodic function  $\Lambda$ . The lagrangean density for the zero modes can be written as

$$\mathcal{L}_{ZM} = \frac{1}{2}(\partial A_-^\circ)^2 - \frac{1}{2}(F_{12}^\circ)^2 - F_{+k}^\circ \partial_k A_-^\circ - A_-^\circ \partial_\perp^2 \lambda \quad , \quad (3.3)$$

which is singular and leads to the primary first class constraints

$$\pi_-^\circ \approx 0 \quad ; \quad \rho_k^\circ \equiv \pi_k^\circ - \partial_k A_-^\circ \approx 0 \quad . \quad (3.4)$$

It should be stressed that, since the constraints  $\rho_k^\circ \approx 0$  are first class, at variance with the corresponding ones in the normal mode sector which are second class, there is an additional “transverse” gauge invariance in the zero mode sector. As a matter of fact, the equations of motion for the zero modes: namely,

$$\partial_\perp^2 A_-^\circ = 0 \quad \implies \quad A_-^\circ = 0 \quad ; \quad (3.5a)$$

$$\partial_\perp^2 A_+^\circ - 2\partial_- \partial_k A_k^\circ + 2\partial_\perp^2 \lambda = 0 \quad ; \quad (3.5b)$$

$$(\partial_\perp^2 \delta_{jk} - \partial_j \partial_k) A_k^\circ = 0 \quad , \quad (3.5c)$$

do indeed explicitly exhibit the “transverse” gauge invariance (notice that the canonical Light Front zero mode Hamiltonian  $P_+^\circ$  is weakly vanishing, thereby preventing the ordinary  $x^+$  evolution for the zero modes, as already mentioned). There are infinitely many ways, of course, to remove the above residual local gauge freedom<sup>11</sup>. However, at the quantum level, this entails the pathological CPV propagator in the continuum limit  $L \rightarrow \infty$ . On the contrary, the requirement of a smooth transition to the consistent continuum formulation actually suggests to keep that freedom and, instead of removing the gauge degrees of freedom, one is led to impose the following zero mode commutation relations: namely,

$$[\varphi(x^+, x_\perp), \lambda(y^+, y_\perp)] = i\delta(x^+ - y^+) \delta^{(2)}(x_\perp - y_\perp) \quad , \quad (3.6)$$

where  $A_j^\circ(x^+, x_\perp) = \partial_j (\partial_\perp^2)^{-1} \varphi(x^+, x_\perp)$ , in perfect analogy with eq. (2.6). The above recipe ensures that, in the continuum limit, the correct ML quantization scheme is indeed recovered.

The interaction with spinorial matter requires the introduction of the two Light Front components of the Dirac field  $\psi_\pm = \frac{1}{2} \gamma^0 \gamma^\pm \psi$ , satisfying antiperiodic boundary conditions (*i.e.* no fermion zero modes), in such a way to get a periodic fermion current  $J_\mu(x)$ . Among the Maxwell equations for the zero modes we have

$$\partial_\perp^2 A_-^\circ + J_-^\circ = 0 \quad , \quad (3.7)$$

involving the component of the potential to be gauged away. At first sight, eq. (3.7) seems to prevent<sup>11</sup> the usual light–cone subsidiary condition  $A_-^\circ = 0$ ; nonetheless, one can fulfil

the **strong** light–cone gauge, provided one requires  $J_-^\circ = 0$ . This constraint on the fermion field component  $\psi_+$ , which is the independent one, is quite acceptable in the discretized (*i.e.* regularized) formulation. As a matter of fact, the physical fermion current in the continuum can not contain zero modes, if we ask the charge  $Q_-$  to be finite after the removal of the infrared regularization along  $x^-$ . Once again, in order to solve the dynamics of the gauge potential zero modes, we still do not eliminate the redundant degrees of freedom and, after setting  $A_k^\circ = T_k^\circ + \partial_k(\partial_\perp^2)^{-1}\varphi$ , we impose the commutation relations (3.6), the quantities  $T_k^\circ$  being determined by equations of motion in the zero mode sector. In this way, step by step in perturbation theory, the consistent formulation including the ML zero modes  $\lambda$  and  $\varphi$  should be recovered in the continuum limit  $L \rightarrow \infty$ .

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