On Darboux's approach to R-separability

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Gaston Darboux
"Memoire sur la theorie des coordonnees curvilignes, et des systemes orthogonaux"
Annales Scientifiques de l' E.N.S. v. 7 (1878) pp. 101-150, 227-260, 275-348

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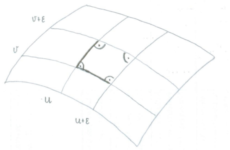
Key notions

1. **Isothermic surfaces in $\mathbb{E}^3$**
   (classical differential geometry and solitons)
2. **Isothermic coordinates in $\mathbb{E}^3$**
   (R-separability of variables in LPDEqs of mathematical physics)
3. **Isothermal coordinates in $\mathbb{E}^3$**
   (physics of stationary heat flow)

Isothermic surfaces in $\mathbb{E}^3$

Curvature net admits conformal parametrization

$$ds^2 = \lambda^2(u, v)\left(du^2 + dv^2\right)$$



- E. Bour (1862): *les surfaces a lignes de courbure isothermes*
- *G. Darboux (1866): surfaces divisible into infinitesimal squares by their curvature lines*
- *G. Darboux (1889): surfaces isothermiques*

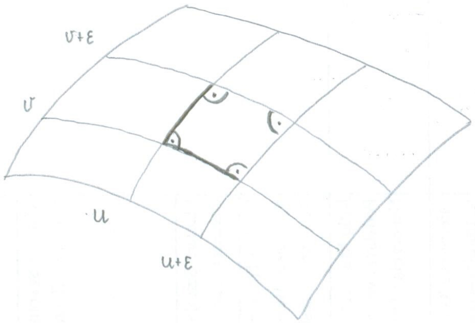
Examples

1. Darboux - Moutard cyclides
2. Dupin cyclides
3. Quadrics
4. Surfaces of rotation
5. $H = const$ surfaces

Isothermic surfaces in $\mathbb{E}^3$

Curvature net admits conformal parametrization

$$ds^2 = \lambda^2(u, v)\left(du^2 + dv^2\right)$$



Julius Weingarten, " Ueber die Differentialgleichungen der Oberflachen welche durch ihre Krummungslinien in unendlich kleine Quadrate getheilt werden konnen ", 1885

Isothermic coordinates in $\mathbb{E}^3$

Orthogonal coordinates x^i *such that all coordinate surfaces*

$$x^i = const$$

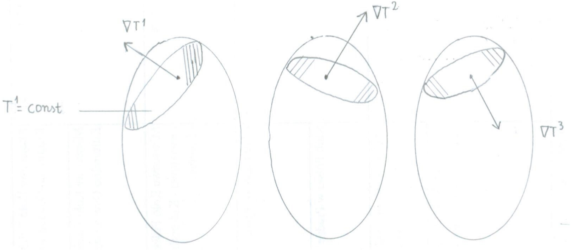
are isothermic surfaces.

Examples

1. *Elliptic coordinates*
2. *Paraboloidal coordinates*
3. *Conical coordinates*
4. *Spherical coordinates*
5. *Darboux's cyclidic coordinates (1865)*
6. *Roberts Dupin - cyclidic coordinates (W. Roberts 1861, J.C. Maxwell 1867, F.G. Friedlander 1946)*

Isothermal coordinates in $\mathbb{E}^3$

Homogeneous isotropic heat conductor in three different states of stationary flow



$T = T^1(x, y, z) \quad T = T^2(x, y, z) \quad T = T^3(x, y, z)$

$$\Delta T^i = 0, \quad \nabla T^i \cdot \nabla T^j = 0 \quad (i \neq j)$$

$$x^i = T^i(x, y, z) - \text{isothermal coordinates}$$

Theorem (Bertrand 1844)
Isothermal coordinates are isothermic coordinates but not vice versa

Isothermic metric
$\mathbb{E}^3$ *metric in terms of isothermic coordinates*

Darboux's isothermicity problem

To find all isothermic metrics

- In thesis (1866) **without** motivation
- In 3rd part of his celebrated (and today forgotten) memoir of 1878 **with** strong motivation:
  last 50 pages contain the first general treatment of R-separability of variables in Laplace eq. on $\mathbb{E}^3$

Analytic reformulation

To find all metrics $ds^2 = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2$ *where*

$$H_1 = \frac{G_2(x^1,x^3)G_3(x^1,x^2)}{M(x^1,x^2,x^3)f_1(x^1)}, \quad H_2 = \frac{G_1(x^2,x^3)G_3(x^1,x^2)}{M(x^1,x^2,x^3)f_2(x^2)}, \quad H_3 = \frac{G_1(x^2,x^3)G_2(x^1,x^3)}{M(x^1,x^2,x^3)f_3(x^3)}$$

and satisfying

$$R_{ijkl} = 0.$$

3-dim. Riemann spaces

Theorem(É. Cartan 1945, D.M. DeTurck and D.Yang 1984)
Any 3-dim. Riemann space $\mathcal{R}^3$ admits orthogonal coordinates

$$ds^2 = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2 \ .$$

A metric in $\mathcal{R}^3$ of the form

$$ds^2 = \frac{1}{M^2}\left[\frac{G_2^2 G_3^2}{f_1^2}(dx^1)^2 + \frac{G_1^2 G_3^2}{f_2^2}(dx^2)^2 + \frac{G_1^2 G_2^2}{f_3^2}(dx^3)^2\right]$$

is called isothermic. Any coordinate surface of isothermic metric is isothermic, i.e. its curvature net allows conformal parametrization.

Modified isothermicity problems

Six rotation coefficients (Darboux 1878)

$$\beta_{ik} = \frac{1}{H_i} \frac{\partial H_k}{\partial x^i} \qquad (i \neq k)$$

Flatness conditions in terms of rotation coefficients (**nine** eqs)

$$\frac{\partial \beta_{23}}{\partial x^1} = \beta_{21}\beta_{13} \qquad \dots \qquad \dots$$

$$\dots \qquad \dots \qquad \dots$$

$$\frac{\partial \beta_{12}}{\partial x^1} + \frac{\partial \beta_{21}}{\partial x^2} + \beta_{31}\beta_{32} = 0 \qquad \dots \qquad \dots$$

Idea: to replace flatness conditions by weaker ones.

Why?

See next slide devoted to R-separability of variables in Helmholtz eq. on 3-dim. Riemann space.

R-separable coordinates (metrics) in Helmholtz eq. on $\mathcal{R}^3$

$$ds^2 = H_1^2(dx^1)^2 + H_2^2(dx^2)^2 + H_3^2(dx^3)^2 \tag{1}$$

Helmholtz eq. on $\mathcal{R}^3$

$$\Delta\psi + k^2\psi = 0 \qquad (k = const)$$

$$\frac{1}{H_1H_2H_3}\left(\partial_1\frac{H_2H_3}{H_1}\partial_1 + \partial_2\frac{H_1H_3}{H_2}\partial_2 + \partial_3\frac{H_1H_2}{H_3}\partial_3\right)\psi + k^2\psi = 0 \tag{2}$$

Coordinates x^i are R-separable in Helmholtz eq. (metric (1) is R-separable in Helmholtz eq.) if there exist seven functions $R(x^1, x^2, x^3)$, $p_1(x^1)$, $p_2(x^2)$, $p_3(x^3)$, $q_1(x^1)$, $q_2(x^2)$, $q_3(x^3)$ *such that*

$$\varphi_i'' + p_i\varphi_i' + q_i\varphi_i = 0 \quad (i = 1, 2, 3) \implies \psi = R\varphi_1\varphi_2\varphi_3 \quad \text{solves (2)}$$

R-separable coordinates (metrics) in Helmholtz eq. on $\mathcal{R}^3$

Theorem

Metric (1) is R-separable in Helmholtz eq. if and only if

a) *metric is isothermic*

b)

$$\frac{M^2 f_1^2}{G_2^2 G_3^2}\left(\frac{1}{f_1}\partial_1 f_1 \partial_1 \frac{1}{R} + \frac{q_1}{R}\right) + \frac{M^2 f_2^2}{G_1^2 G_3^2}\left(\frac{1}{f_2}\partial_2 f_2 \partial_2 \frac{1}{R} + \frac{q_2}{R}\right) + \frac{M^2 f_3^2}{G_1^2 G_2^2}\left(\frac{1}{f_3}\partial_3 f_3 \partial_3 \frac{1}{R} + \frac{q_3}{R}\right) = \frac{k^2}{R}$$

Diagonal Ricci tensor

$$G_i = e^{g_i}$$

Theorem

1. Ricci tensor of isothermic metric is diagonal if and only if the following system of eqs is compatible

$$\begin{aligned} M_{,12} - g_{3,2}M_{,1} - g_{3,1}M_{,2} + (g_{1,2}g_{3,1} - g_{1,2}g_{2,1} + g_{2,1}g_{3,2})\, M = 0 \\ M_{,13} - g_{2,3}M_{,1} - g_{2,1}M_{,3} + (g_{1,3}g_{2,1} - g_{1,3}g_{3,1} + g_{2,3}g_{3,1})\, M = 0 \\ M_{,23} - g_{1,3}M_{,2} - g_{1,2}M_{,3} + (g_{1,2}g_{2,3} - g_{2,3}g_{3,2} + g_{1,3}g_{3,2})\, M = 0 \end{aligned} \tag{3}$$

2. The system (3) is compatible only in three cases

$$U_i = U_i(x^i) \;-\; \text{arbitrary} \quad (i = 1, 2, 3)$$

A) $g_1 = n\ln(U_2 - U_3), \quad g_2 = n\ln(U_3 - U_1), \quad g_3 = n\ln(U_1 - U_2)$
$n = const \neq 0, \quad U_i' \neq 0$

B) g_1 – arbitrary, $g_2 = U_1 + U_3$, $g_3 = U_1 + U_2$
g_2 – arbitrary, $g_1 = U_2 + U_3$, $g_3 = U_1 + U_2$
g_3 – arbitrary, $g_1 = U_2 + U_3$, $g_2 = U_1 + U_3$

C) $g_1 = U_2 - U_3, \; g_2 = U_3 - U_1, \; g_3 = U_1 - U_2, \quad U_i' \neq 0$

Class A

$$g_1 = n\ln(x^2 - x^3), \quad g_2 = n\ln(x^3 - x^1), \quad g_3 = n\ln(x^1 - x^2)$$

$$ds^2 = \frac{1}{M^2}\Big[(x^1 - x^2)^{2n}(x^1 - x^3)^{2n}\frac{1}{f_1^2}(dx^1)^2 + (x^2 - x^1)^{2n}(x^2 - x^3)^{2n}\frac{1}{f_2^2}(dx^2)^2 + (x^3 - x^1)^{2n}(x^3 - x^2)^{2n}\frac{1}{f_3^2}(dx^3)^2\Big]$$

where M is a solution to a system of Euler-Poisson-Darboux eqs

$$M_{,12} + \frac{n}{x^1 - x^2}M_{,1} - \frac{n}{x^1 - x^2}M_{,2} = 0$$
$$M_{,13} + \frac{n}{x^1 - x^3}M_{,1} - \frac{n}{x^1 - x^3}M_{,3} = 0$$
$$M_{,23} + \frac{n}{x^2 - x^3}M_{,2} - \frac{n}{x^2 - x^3}M_{,3} = 0$$

$$M = c + \sum_{i=1}^{m} c_i \left[(x^1 - d_i)(x^2 - d_i)(x^3 - d_i)\right]^n$$

Subclass $n = \frac{1}{2}$

$$ds^2 = \frac{1}{M^2}\left[\frac{(x^1-x^2)(x^1-x^3)}{f_1^2}(dx^1)^2 + \frac{(x^2-x^1)(x^2-x^3)}{f_2^2}(dx^2)^2 + \frac{(x^3-x^1)(x^3-x^2)}{f_3^2}(dx^3)^2\right]$$

$$M = c + \sum_{i=1}^{m} c_i\sqrt{(x^1-d_i)(x^2-d_i)(x^3-d_i)}$$

- elliptic metric
- paraboidal metric
- Darboux's cyclidic metric

Subclass $n = -1$

$$ds^2 = \frac{1}{M^2}\left[\frac{(dx^1)^2}{(x^1-x^2)^2(x^1-x^3)^2 f_1^2} + \frac{(dx^2)^2}{(x^2-x^1)^2(x^2-x^3)^2 f_2^2} + \frac{(dx^3)^2}{(x^3-x^1)^2(x^3-x^2)^2 f_3^2}\right]$$

$$M = \frac{b_1(x^1)}{(x^1-x^2)(x^1-x^3)} + \frac{b_2(x^2)}{(x^2-x^1)(x^2-x^3)} + \frac{b_3(x^3)}{(x^3-x^1)(x^3-x^2)}$$

Theorem

All coordinate surfaces of any metric of subclass $n = -1$ are Riemannian Dupin's cyclides, i.e. their principal curvatures are constant along the corresponding curvature lines.

Roberts Dupin - cyclidic metrics

Class B

Most of eleven Eisenhart metrics

- Spherical metric

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$
$$g_1 = \ln\sin\theta, \quad g_2 = \ln r, \quad g_3 = \ln r$$

- Oblate spheroidal metric

$$ds^2 = \left(\cosh^2\eta - \cos^2\theta\right)\left[d\eta^2 + d\theta^2\right] + \cosh^2\eta\cos^2\theta d\varphi^2$$
$$g_1 = \ln\cos\theta, \quad g_2 = \ln\cosh\eta, \quad g_3 = \frac{1}{2}\ln\left(\cosh^2\eta - \cos^2\theta\right)$$

- Conical metric

Diagonal Ricci tensor & conformal flatness

Theorem

a) *Isothermic metric of subclass* $n = \frac{1}{2}$

$$ds^2 = \frac{1}{M^2}\left[\frac{(x^1-x^2)(x^1-x^3)}{f_1^2}(dx^1)^2 + \frac{(x^2-x^1)(x^2-x^3)}{f_2^2}(dx^2)^2 + \frac{(x^3-x^1)(x^3-x^2)}{f_3^2}(dx^3)^2\right]$$

is conformally flat if and only if

$$f_i^2 = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

b) Isothermic metric of subclass $n = -1$

$$ds^2 = \frac{1}{M^2}\left[\frac{(dx^1)^2}{(x^1-x^2)^2(x^1-x^3)^2 f_1^2} + \frac{(dx^2)^2}{(x^2-x^1)^2(x^2-x^3)^2 f_2^2} + \frac{(dx^3)^2}{(x^3-x^1)^2(x^3-x^2)^2 f_3^2}\right]$$

is conformally flat if and only if

$$f_i^2 = m_i x^2 + 2n_i x + p_i, \qquad \sum m_i = \sum n_i = \sum p_i = 0$$

These two conditions are complementary!

Tool: Cotton's tensor C_{ijk}

Cotton tensor in $\mathcal{R}^3$ is given by

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}(g_{ik}\nabla_j R - g_{ij}\nabla_k R)$$

where $R_{ij} = g^{kl}R_{ikjl}$ and $R = g^{ij}R_{ij}$ are Ricci tensor and scalar curvature of metric g_{ij}.

Theorem

Three dimensional Riemann space $\mathcal{R}^3$ is conformally flat if and only if Cotton tensor $C_{ijk} = 0$.

Flat isothermic metrics

Subclass $n = \frac{1}{2}$

- paraboloidal metric: $M = 1, \quad f_i^2(x) = 4(a-x)(b-x)$
- elliptic metric: $M = 1, \quad f_i^2(x) = 4(a-x)(b-x)(c-x)$
- Darboux's cyclidic metric

$$M = \sqrt{m} \pm \sqrt{n}\,, \quad m = \frac{(x^1+e)(x^2+e)(x^3+e)}{(e-a)(e-b)(e-c)}\,, \quad n = \frac{(x^1+d)(x^2+d)(x^3+d)}{(d-a)(d-b)(d-c)}$$
$$f_i^2(x) = (x+a)(x+b)(x+c)(x+d)(x+e)$$

- cyclidic metric: $M = 1 + \sqrt{\frac{x^1x^2x^3}{abc}}\,, \quad f_i^2(x) = (x-a)(x-b)(x-c)(x-d)$

Theorem (Darboux 1878)
Isothermic metric of subclass $n = -1$ is flat if and only if

$$(n_i^2 - m_ip_i)b_i^2 + 2b_i\left[\alpha_i(n_ix^i + p_i) - \beta_i(n_i + m_ix^i)\right] - \alpha_i\gamma_i + (\alpha_ix^i + \beta_i)^2 = 0$$

$$\sum_{i=1}^{3}\alpha_i = \sum_{i=1}^{3}\beta_i = \sum_{i=1}^{3}\gamma_i = 0$$

R-separability of flat isothermic metrics of subclass $n = -1$

Theorem (Darboux 1910)
Each flat isothermic metric of subclass $n = -1$ is R-separable in **Laplace** eq. on $\mathbb{E}^3$

$$\varphi_i'' + \frac{1}{2}\frac{a_i'}{a_i}\varphi_i' - \frac{m_i}{4a_i}\varphi_i = 0$$

$$\Downarrow$$

$$\sqrt{\frac{b_1}{(x^1 - x^2)(x^1 - x^3)} + \frac{b_2}{(x^2 - x^1)(x^2 - x^3)} + \frac{b_3}{(x^3 - x^1)(x^3 - x^2)}}\,\varphi_1\varphi_2\varphi_3$$

solves Laplace eq.

Theorem (Prus & Sym 2005)
Each Roberts Dupin - cyclidic metric is R-separable in **Helmholtz** eq. on $\mathbb{E}^3$.