Classical r-matrix like approach to Frobenius manifolds

Błażej Szablikowski

Department of Mathematics, University of Glasgow
(& A. Mickiewicz University, Poznań)

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Frobenius manifold

Smooth (or holomorphic) manifold M with (nondegenerate) metric $\eta : \Gamma(TM) \times \Gamma(TM) \to \mathcal{C}^{\infty}(M)$

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and

Frobenius algebra structure

associative commutative unital multiplication $\circ : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which is invariant:

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z).$$

Frobenius manifold II

Further requirements

1. η is flat;
2. ∇c is symmetric in all its four arguments, where $c(X, Y, Z) := \eta(X \circ Y, Z)$;
3. the unit e is flat, i.e. $\nabla e = 0$;
4. exists Euler field E (i.e. $\nabla\nabla E = 0$) s.t.

$$\mathcal{L}_E \circ = \circ \qquad \text{and} \qquad \mathcal{L}_E \eta = d\, \eta,$$

where d is a number. Normalisation condition $\mathcal{L}_E e = -e$.

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Intersection form

Second metric

$$\gamma^{-1}(\alpha, \beta) := \langle \alpha \circ \beta, E \rangle \qquad \alpha, \beta \in T^* M$$

WDVV associativity equations I

Prepotential

Let $\{t^\alpha\}$ be (local) flat coordinates of η s.t. $e = \partial_{t^1}$. Then, there exists (smooth) function $F(t)$ such that

$$c_{\alpha,\beta,\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \qquad \text{and} \qquad \eta_{\alpha,\beta} = \frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta}.$$

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WDVV equations

The structure constants: $\partial_\alpha \circ \partial_\beta = c^\gamma_{\alpha\beta}(t)\partial_\gamma$, where $c^\gamma_{\alpha\beta} = c_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}$. Then, the associativity equations on $F(t)$ are

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F(t)}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\alpha}.$$

Quasi-homogeneity condition

$$EF = (3 - d)F + quadratic\ terms,$$

where

$$E = \left((1 - q_\alpha)t^\alpha + r_\alpha\right)\partial_\alpha.$$

Dubrovin-Novikov bracket

Loop manifold: $\mathcal{L}M = \{\mathbb{S}^1 \to M\}$

Hydrodynamic Poisson bracket

has the form

$$\{h, f\} = \int_{\mathbb{S}^1} \frac{\partial f}{\partial u^\mu} \pi^{\mu\nu} \frac{\partial h}{\partial u^\nu}\, dx := \int_{\mathbb{S}^1} \left\langle df, \eta^{-1} \nabla_{u_x} dh \right\rangle dx, \qquad (1)$$

where

$$\pi^{\mu\nu} = \eta^{\mu\nu} \partial_x - \eta^{\mu\varepsilon} \Gamma^{\nu}_{\varepsilon\sigma} u^\sigma_x$$

and $u^\sigma : \mathbb{S}^1 \to M$ are (dynamical) coordinate fields.

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Recall that (1) is a Poisson bracket wrt nondegenerate metric η iff

(i) η is flat

(ii) and $\Gamma^\nu_{\varepsilon\sigma}$ is the Levi-Civita connection of η.

Further properties

Deformed flat connection

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Thus

$$\widetilde{\nabla}_i dh_k(\lambda) = 0 \qquad \Longleftrightarrow \qquad \partial_i \partial_j h_k = \lambda\, c_{ij}^l\, \partial_l h_k$$

Expanding, $h_k(\lambda) = \sum_{p=0}^{\infty} h_{k,p} \lambda^p$, the coefficients can be determined recursively from

$$\partial_i \partial_j h_{k,p} = c_{ij}^l\, \partial_l h_{k,p-1} \qquad p = 1, 2, \dots \; .$$

Principal hierarchy

Taking $h_{k,p}$ as Hamiltonian densities and applying D-N Poisson tensor wrt η, in flat coordinates the hierarchy takes the form

$$(t^{\mu})_{T^{k,p}} = \eta^{\mu\nu}\partial_x\partial_{t^{\nu}}h_{k,p}(t).$$

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Claim

Let $h_{\alpha,0} \equiv \eta_{\alpha\mu}t^\mu$, then the prepotential F is determined by $h_{\alpha,1}$:

$$F = \frac{1}{3-d}\sum_\alpha \left[(1-q_\alpha)t^\alpha + r_\alpha\right] h_{\alpha,1} + q.t. \qquad d \neq 3.$$

Construction of Frobenius algebras I

Proposition

Let $\circ_\ell$ be a second commutative associative multiplication on $\mathcal{A}$. Let $\mathfrak{g}$ be an associative algebra and second multiplication be

$$a \circ_\ell b := \ell(a)b + a\ell(b) \qquad a, b \in \mathfrak{g},$$

generated by a linear map $\ell : \mathfrak{g} \to \mathfrak{g}$. A sufficient condition for its associativity is the so-called (modified) Poincare-Bertrand formula

$$\ell(a \circ_\ell b) - \ell(a)\ell(b) = \delta\, ab, \tag{2}$$

where $\delta \in Center(\mathfrak{g})$.

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where $\delta \in Center(\mathfrak{g})$.

Besides, $\ell'(\cdot) = \ell(d\cdot)$ satisfies (2) for arbitrary $d \in Center(\mathfrak{g})$, with $\delta' = \delta d^2$, iff ℓ satisfies (2).

Construction of Frobenius algebras II

Let $\mathcal{A}$ be a commutative associative unital algebra, with a trace form $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ s.t. the pairing $(a, b)_{\mathcal{A}} := \mathrm{tr}(ab)$ is non-degenerate. Let $\circ_\ell$ be a second commutative associative multiplication on $\mathcal{A}$.

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Invariant metric

Then, the metric

$$\eta(a, b) := \mathrm{tr}\left(a \circ_\ell b\right) \qquad a, b \in \mathcal{A}$$

is naturally invariant

$$\eta\left(a \circ_\ell b, c\right) = \mathrm{tr}\left(a \circ_\ell b \circ_\ell c\right) = \eta\left(a, b \circ_\ell c\right).$$

Classical r-matrices

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra.

Classical r-matrix

is a linear map $r : \mathfrak{g} \to \mathfrak{g}$ such that

$$[a,b]_r := [r(a),b] + [a,r(b)] \qquad a,b \in \mathfrak{g}$$

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Modified Yang-Baxter equation

A sufficient condition for r to be a classical r-matrix is to satisfy:

$$[r(a), r(b)] - r\left([a, b]_r\right) + \alpha\,[a, b] = 0,$$

where α is a number.

Simplest solutions

Assume that $(\mathcal{A}, \cdot)$ can be decomposed into subalgebras, i.e.

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \qquad \mathcal{A}_\pm \mathcal{A}_\pm \subset \mathcal{A}_\pm \qquad \mathcal{A}_+ \cap \mathcal{A}_- = \emptyset.$$

Then

$$\ell = \frac{1}{2}\left(P_+ - P_-\right).$$

satisfies the Poincare-Bertrand equation for $\delta = \frac{1}{4}$.

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Then

$$r = \frac{1}{2}\left(P_+ - P_-\right)$$

satisfies the Yang-Baxter equation for $\alpha = \frac{1}{4}$.

Theorem by L.-C. Lie

Let $(\mathcal{A}, \{\cdot,\cdot\})$ be a Poisson algebra with a non-degenerate ad-invariant scalar product $(a,b)_{\mathcal{A}} = \mathrm{Tr}(ab)$. Assume that r is a classical r-matrix, then for each $n \geqslant 0$ the formula

$$\{h,f\}_n(\lambda) = (\lambda, \{r(\lambda^n df), dh\} + \{df, r(\lambda^n dh)\})_{\mathcal{A}},$$

where $h, f \in \mathcal{C}^\infty(\mathcal{A})$, defines a Poisson structure on $\mathcal{A}$. Moreover, all brackets are compatible.

Hydrodynamic Poisson structure I

Let the Poisson bracket on respective $\mathcal{A}$ be given in the form

$$\{\cdot,\cdot\} = \partial \wedge \partial_x \qquad \partial, \partial_x \in Der(\mathcal{A}),$$

and ∂_x is such that $\partial_x r = r\partial_x$.

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and ∂_x is such that $\partial_x r = r\partial_x$.

Linear metric

Then, the linear Poisson bracket takes the hydrodynamic form

$$\pi_0(\omega) = r^*(\lambda'\omega_x) - \lambda' r(\omega_x) - r^*(\lambda_x\omega') + \lambda_x \partial r(\omega)$$

for which the corresponding metric is

$$\eta^{-1}(\omega) = r^*(\lambda'\omega) - \lambda' r(\omega).$$

Multiplication

the corresponding product

$$a \circ_r b = r^*(\lambda' a)b + ar^*(\lambda' b)$$

is associative iff $\ell = r^*$ satisfies the Poincare-Bertrand formula.

Useful formulae

Proposition

Rewriting the recurrence formula on the cotangent bundle of Frobenius manifold one finds that

$$\widetilde{\nabla}_X dh = 0 \quad \Longleftrightarrow \quad \eta^{-1}\nabla_X dh(\lambda) = \lambda dh(\lambda) \circ^{\dagger} X,$$

where $\left\langle \beta, \gamma \circ^{\dagger} X \right\rangle := \langle \gamma \circ \beta, X \rangle$.

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Proposition

On $(\mathcal{A}, \cdot, \circ_\ell)$ with $\ell = r^*$ we have

$$\gamma \circ_r^\dagger \lambda_t = \lambda' r(\gamma\lambda_t) + r^*(\lambda'\gamma)\lambda_t.$$

Algebra

Let

$$\mathcal{A} = \left\{ \sum_i u_i z^i | u_i \in \mathbb{C} \right\} = \mathcal{A}_{\geqslant k} \oplus \mathcal{A}_{<k}.$$

Then $\ell = P_{\geqslant k} - \frac{1}{2}$ only for $k = 0$ or 1.

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Multiplication

We consider only the case $k = 0$. Let $\lambda \in \mathcal{A}$, then the multiplication is given by

$$\begin{aligned} a \circ b &:= \ell(\lambda_z a)b + a\ell(\lambda_z b) \\ &= (\lambda_z a)_{\geqslant 0} b + a(\lambda_z b)_{\geqslant 0} + \lambda_z ab. \end{aligned}$$

Poisson algebras of Laurent series near ∞ II

Trace form

The respective trace form is

$$\operatorname{tr}(\cdot) := -\operatorname{res}_{z=\infty}(z^{-r}\cdot) \quad r = 0, 1$$

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One finds the following corresponding metrics

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Proposition

The recurrence formula takes the form:

$$\frac{\partial h_p(\lambda)}{\partial \lambda} = -h_{p-1}(\lambda) \qquad p = 0, 1, 2, \dots .$$

Poisson algebras of Laurent series near ∞ III

Theorem

Respective Frobenius manifolds are:

(i) for $r = 0$ when

$$M = \left\{ \lambda = z^N + u_{N-2} z^{N-2} + \ldots + u_0 \right\} \subset \mathcal{A},$$

corresponding to dKdV (equivalent with A_N model);

(ii) for $r = 1$ when

$$M = \left\{ \lambda = z^N + u_{N-1} z^{N-1} + \ldots + u_{-m} z^{-m} \right\} \subset \mathcal{A} \qquad N, m \geqslant 0,$$

corresponding to dToda.

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Euler vector field

$$E(\lambda) = \lambda - \frac{1}{N} z \lambda_z .$$

Example, the case $r = 0$ (A_3)

Superpotential

$$\lambda = z^4 + uz^2 + vz + w = z^4 + t^3 z^2 + t^2 z + t^1 + \frac{1}{8}(t^3)^2.$$

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Linear Poisson tensor $\pi_0(\omega) = \{(\omega)_{\geqslant 0}, \lambda\} - (\{\omega, \lambda\})_{\geqslant 0}$. Then, Casimirs are $c_i = \int_{\mathbb{S}^1} t^i \, dx$, where

$$t^i = \frac{4}{i-4} \mathrm{res}_{z=\infty} \lambda^{-\frac{i}{4}+1} \qquad dt^i = \lambda^{-\frac{i}{4}}$$

s.t. $(dt^i)^{\infty}_{\geqslant 0} = 0$.

Flat coordinates

$t^1 = -\frac{1}{8}u^2 + w$, $t^2 = v$, $t^1 = u$.

Euler vector field

$$E(\lambda) = \lambda - \frac{1}{4} z\lambda_z = \frac{1}{2} t^3 z^2 + \frac{3}{4} t^2 z + t^1 + \frac{1}{8}(t^3)^2.$$

Hence,

$$E = \frac{1}{2} t^3 \partial_{t^3} + \frac{3}{4} t^2 \partial_{t^2} + t^1 \partial_{t^1}.$$

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One finds: $h_{i,1} = \frac{16}{i(i+4)}\mathrm{res}_{z=\infty}\lambda^{\frac{i}{4}+1}$ for $i = 1, 2, 3$.

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Prepotential

$$F = -\frac{5}{4} t^1 (t^2)^2 - \frac{5}{4} (t_1)^2 t^3 + \frac{5}{32} (t^2)^2 (t^3)^2 - \frac{1}{384} (t^3)^6$$