Time scales analogues of differential equations

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Plan of the talk

1. Preliminaries. Notation.
2. Exponential, hyperbolic and trigonometric functions on time scales. Short review of existing results.
3. New definition of the exponential function on $\mathbb{T}$, motivated by the Cayley transformation.
4. Cayley-hyperbolic and C-trigonometric functions on $\mathbb{T}$.
5. New approach to $\mathbb{T}$-analogues of ODE's. Application of geometric numerical schemes.
6. Short note on other developments and further directions:
   - modification of the q-calculus,
   - Padé-analogues of the exponential function on $\mathbb{T}$,
   - time scales analogue of the exact discretization,
   - dynamic systems on Lie groups.
   - Sine-Gordon equation on time scales.

1. Preliminaries. Notation

Time scale $\mathbb{T}$ is any (non-empty) closed subset of $\mathbb{R}$.

Forward jump operator $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} =: t^\sigma$

Backward jump operator $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$

Graininess $\mu(t) := \sigma(t) - t$.

Rd-continuous function is continuous at right-dense points ($\sigma(t) = t$) and has a finite limit at left-dense points ($\rho(t) = t$).

Graininess μ is always rd-continuous (but is not continuous at points which are left-dense and right-scattered).

Delta derivative: $$f^{\Delta}(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

Theorem (Hilger): Any rd-continuous function f has an anti-derivative F (i.e., $F^{\Delta} = f$).

Nabla derivative: $$f^{\nabla}(t) := \lim_{\substack{s \to t \\ s \neq \rho(t)}} \frac{f(\rho(t)) - f(s)}{\rho(t) - s} .$$

Hilger's exponential function, denoted by $e_\alpha(t,t_0)$, is the unique solution of the following initial value problem ($\alpha : \mathbb{T} \to \mathbb{C}$ is given):

$$x^\Delta = \alpha(t)\, x \ , \qquad x(t_0) = 1 \ .$$

Nabla exponential function, denoted by $\widehat{e}_\alpha(t,t_0)$, satisfies:

$$x^\nabla = \alpha(t)\, x \ , \qquad x(t_0) = 1 \ .$$

Continuous case ($\mathbb{T} = \mathbb{R}$): $\quad e_\alpha(t,t_0) = \widehat{e}_\alpha(t,t_0) = \exp \int_{t_0}^{t} \alpha(t) \Delta t.$

$$\mathbb{T} = \mathbb{R}, \ \ \alpha(t) = z \quad \Rightarrow \quad e_\alpha(t) = \widehat{e}_\alpha(t) = e^{zt}.$$

Discrete constant case ($\mathbb{T} = h\mathbb{Z}$, $\alpha(t) = z \in \mathbb{C}$):

$$e_z(t) = \left(1 + \frac{zt}{n}\right)^n, \qquad \widehat{e}_z(t) = \left(1 - \frac{zt}{n}\right)^{-n}, \qquad t = nh \ .$$

2. Hyperbolic and trigonometric functions

$$\mathbb{T}=\mathbb{R} \;\Rightarrow\; \cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right) \;, \quad e^{-ix} = (e^{ix})^{-1} \;(!)$$

Unfortunatelly: $\qquad e_{-\alpha}(t,t_0) \neq e_{\alpha}^{-1}(t,t_0).$

2.1. Hilger (1999): $\qquad \cosh_\alpha(t) = \dfrac{e_\alpha(t)+e_\alpha^{-1}(t)}{2}$ etc.

Advantages: $\cosh_\alpha^2(t) - \sinh_\alpha^2(t) = 1$,

Disadvantages: $\cosh_\alpha^\Delta(t)$ is not proportional to $\sinh_\alpha(t)$,

$\cosh_{i\omega}(t) \notin \mathbb{R}$ (for $\omega \in \mathbb{R}$) . How to define sine and cosine?

Hilger (1999): $\omega(t) = \text{const} \;\Rightarrow\; \cos_\omega(t) := \cos(\omega t)$ (**?!**)

Exact discretization!

2.2. Bohner, Peterson (2001):

$$\cosh_\alpha(t) = \frac{e_\alpha(t) + e_{-\alpha}(t)}{2} \ , \quad \cos_\omega(t) = \cosh_{i\omega}(t) \ , \ \text{etc.}$$

Advantages: $\omega \in \mathbb{R} \quad \Rightarrow \quad \cos_\omega(t) \in \mathbb{R}, \quad \sin_\omega(t) \in \mathbb{R},$

$$\sinh_\alpha^\Delta(t) = \alpha \cosh_\alpha(t) \ , \quad \sin_\omega^\Delta(t) = \omega \cos_\omega(t), \ \text{etc.}$$

Disadvantages: in place of Pythagorean identities we have qualitatively different equalities, i.e.,

$$\cosh_\alpha^2(t) - \sinh_\alpha^2(t) = e_{-\mu\alpha^2}(t), \quad \cos_\omega^2(t) + \sin_\omega^2(t) = e_{\mu\omega^2}(t) \ .$$

Sine and cosine are not bounded.

3. New definition of the exponential function

The Cayley-exponential function $E_\alpha(t, t_0)$ satisfies the following initial value problem:

$$x^{\Delta}(t) = \alpha(t) \langle x(t) \rangle \ , \qquad x(t_0) = 1 \ ,$$

where α is regressive (i.e., $\mu\alpha \neq \pm 2$) and rd-continuous on $\mathbb{T}$,

and $$\langle x(t) \rangle := \frac{x(t) + x(\sigma(t))}{2} \ .$$

J.L.Cieśliński (2010), "New definitions of exponential, hyperbolic and trigonometric functions on time scales", *preprint* **arXiv: 1003.0697 [math.CA]**.

Continuous case

$$\mathbb{T} = \mathbb{R} \quad \Rightarrow \quad E_\alpha(t) = \exp \int_0^t \alpha(\tau) d\tau.$$

Discrete case

$$\mathbb{T} = h\mathbb{Z},\ \alpha = \text{const}, \quad \Rightarrow \quad E_\alpha(t) = \left(\frac{1 + \frac{1}{2n} t\alpha}{1 - \frac{1}{2n} t\alpha} \right)^n, \quad t = nh.$$

Similar formulas (discrete case): Ferrand (1944), Duffin (1956), Zeilberger, Dym (1977), Date, Jimbo, Miwa (1982), Nijhoff, Quispel, Capel (1983), Iserles (2001), Mercat (2001).

Cayley-exponential (C-exponential) function E_α is given by

$$E_\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \zeta_{\mu(s)}(\alpha(s))\Delta s\right), \qquad E_\alpha(t) := E_\alpha(t,0),$$

where α $(\alpha : \mathbb{T} \to \mathbb{C})$ is *regressive* (i.e., $\mu\alpha \neq \pm 2$) and rd-continuous, and

$$\zeta_\mu(z) := \frac{1}{\mu} \log \frac{1 + \frac{1}{2} z\mu}{1 - \frac{1}{2} z\mu} \ , \qquad \zeta_0(z) := z \ , \qquad \left(z = \frac{2}{h} \tanh \frac{h\zeta}{2} \right).$$

Classical Cayley transformation: $z \to \text{cay}(z,a) = \dfrac{1+az}{1-az}$ maps the imaginary axis into the unit circle.

Properties of the Cayley-exponential function:

1. $E_\alpha(t^\sigma, t_0) = \dfrac{1 + \frac{1}{2}\mu(t)\alpha(t)}{1 - \frac{1}{2}\mu(t)\alpha(t)}\ E_\alpha(t, t_0)$

2. $\overline{E_\alpha(t, t_0)} = E_{\bar{\alpha}}(t, t_0)$, $\qquad (E_\alpha(t, t_0))^{-1} = E_{-\alpha}(t, t_0)$

3. $E_\alpha(t, t_0)\, E_\alpha(t_0, t_1) = E_\alpha(t, t_1)$

4. $E_\alpha(t, t_0)\ E_\beta(t, t_0) = E_{\alpha \oplus \beta}(t, t_0)$

where $t^\sigma \equiv \sigma(t)$ and $\alpha \oplus \beta := \dfrac{\alpha + \beta}{1 + \frac{1}{4}\mu^2 \alpha\beta}$.

"Lorentz velocity transformation", $\dfrac{2}{\mu}$ is an analogue of the speed of light.

4. C-hyperbolic and C-trigonometric functions.

$$\mathrm{Cosh}_\alpha(t) := \frac{E_\alpha(t) + E_{-\alpha}(t)}{2} , \quad \mathrm{Sinh}_\alpha(t) := \frac{E_\alpha(t) - E_{-\alpha}(t)}{2} ,$$

$$\mathrm{Cos}_\omega(t) := \frac{E_{i\omega}(t) + E_{-i\omega}(t)}{2} , \quad \mathrm{Sin}_\omega(t) := \frac{E_{i\omega}(t) - E_{-i\omega}(t)}{2i} .$$

C-exponential function satisfies;

$$(E_\alpha(t))^{-1} = E_{-\alpha}(t) , \quad E_{\bar{\alpha}}(t) = \overline{E_\alpha(t)}.$$

Therefore, $\mathrm{Re}\,\alpha(t) \equiv 0 \;\Rightarrow\; |E_\alpha(t)| \equiv 1,$

which implies good properties of C-trigonometric functions.

Theorem.

$$\text{Cosh}^2_\alpha(t) - \text{Sinh}^2_\alpha(t) = 1 \ ,$$

$$\text{Cosh}^\Delta_\alpha(t) = \alpha(t)\ \langle \text{Sinh}_\alpha(t) \rangle \ , \quad \text{Sinh}^\Delta_\alpha(t) = \alpha(t)\ \langle \text{Cosh}_\alpha(t) \rangle \ ,$$

$$\text{Cos}^2_\omega(t) + \text{Sin}^2_\omega(t) = 1 \ ,$$

$$\text{Cos}^\Delta_\omega(t) = -\omega(t)\ \langle \text{Sin}_\omega(t) \rangle \ , \quad \text{Sin}^\Delta_\omega(t) = \omega(t)\ \langle \text{Cos}_\omega(t) \rangle \ ,$$

Theorem. If $\omega(t) = \text{const}$, then Cayley-sine and Cayley-cosine functions satisfy the equation ("harmonic oscillator on time scales"):

$$x^{\Delta\Delta} + \omega^2 \langle\langle x(t) \rangle\rangle = 0 \ , \qquad \left(\langle\langle x(t) \rangle\rangle \equiv \frac{x^{\sigma\sigma} + 2x^\sigma + x}{4} \right).$$

5. $\mathbb{T}$-analogues of ODE motivated by numerical schemes

We consider a general ODE:

$$\dot{x} = f(x,t) \ , \qquad t \in \mathbb{T}, \qquad x(t) \in \mathbb{C}^N, \qquad f(x(t),t) \in \mathbb{C}^N$$

Standard time scales analogues:

Forward (explicit) Euler scheme $\quad x^{\Delta}(t) = f(x(t),t)$

Backward (implicit) Euler scheme $\quad x^{\nabla}(t) = f(x(t),t)$

What about other numerical schemes? ...

Trapezoidal rule (notation: $x = x(t), \quad x^\sigma = x(t^\sigma)$)

autonomous case: $x^\Delta = \frac{1}{2}\left(f(x) + f(x^\sigma)\right)$

general case: $x^\Delta = \dfrac{1}{4}\left(f(x,t) + f(x^\sigma,t) + f(x,t^\sigma) + f(x^\sigma,t^\sigma)\right)$

Remark. $f(x,t) = \alpha(t)\, x \quad \Rightarrow \quad x^\Delta = \langle\alpha\rangle\, \langle x\rangle$

Yet more symmetric definition of the exponential function!

Implicit midpoint rule

autonomous case $\quad x^\Delta = f\left(\dfrac{x + x^\sigma}{2}\right)$

general case: $x^\Delta = \dfrac{1}{2}\left(f\left(\dfrac{x + x^\sigma}{2}, t\right) + f\left(\dfrac{x + x^\sigma}{2}, t^\sigma\right)\right)$ (?)

Discrete gradient method (a simplest case)

Hamiltonian $H(p,q) = T(p) + V(q)$ yields $\dot{q} = \dfrac{\partial T}{\partial p}$, $\dot{p} = -\dfrac{\partial V}{\partial q}$.

$\mathbb{T}$-analogue:

$$q^{\Delta} = \frac{\Delta T}{\Delta p} \ , \qquad p^{\Delta} = -\frac{\Delta V}{\Delta q} \ .$$

where the "discrete gradient" is defined as

$$\frac{\Delta T}{\Delta p}(p) := \lim_{P \to p} \frac{T(p^{\sigma}) - T(P)}{p^{\sigma} - P}, \qquad \frac{\Delta V}{\Delta q}(q) := \lim_{Q \to q} \frac{V(q^{\sigma}) - V(Q)}{q^{\sigma} - Q}.$$

Theorem. On any time scale: $T(p) + V(q) = \text{const}$.

Classical harmonic oscillator $\ddot{q}+\omega_0^2 q = 0, \qquad q(t) \in \mathbb{R},$

Implicit midpoint, trapezoidal and discrete gradient schemes yield the following $\mathbb{T}$-analogue of harmonic oscillator:

$$q^{\Delta\Delta} + \omega_0^2 \ \langle\langle q \rangle\rangle = 0 \qquad \langle\langle q \rangle\rangle := \frac{q^{\sigma\sigma} + 2q^{\sigma} + q}{4} \ .$$

Solutions: C-sine and C-cosine functions (very good qualitative properties: **bounded**, often oscillatory-like).

Of course, all these schemes yield different results for other, nonlinear, equations.

6.1. New q-exponential function $\mathcal{E}_q^x$ is defined as

$$\mathcal{E}_q^x := e_q^{\frac{x}{2}}\, E_q^{\frac{x}{2}} = \prod_{k=0}^{\infty} \frac{1 + q^k(1-q)\frac{x}{2}}{1 - q^k(1-q)\frac{x}{2}} \ ,$$

where e_q^x, E_q^x are standard q-exponential functions

Theorem:

$$\mathcal{E}_q^x = \sum_{n=0}^{\infty} \frac{x^n}{\{n\}!} \ ,$$

$$\{n\} := \frac{1 + q + \ldots + q^{n-1}}{\frac{1}{2}(1+q^{n-1})} \equiv \frac{[n]}{\frac{1}{2}(1+q^{n-1})}.$$

New q-trigonometric functions motivated by the Cayley transformation

$$\mathcal{S}in_q x = \frac{\mathcal{E}_q^{ix} - \mathcal{E}_q^{-ix}}{2i} \ , \quad \mathcal{C}os_q x = \frac{\mathcal{E}_q^{ix} + \mathcal{E}_q^{-ix}}{2} \ .$$

Properties:

$$\mathcal{C}os_q^2 x + \mathcal{S}in_q^2 x = 1 \ ,$$

$$D_q \mathcal{S}in_q x = \langle \mathcal{C}os_q x \rangle \ ,$$

$$D_q \mathcal{C}os_q x = -\langle \mathcal{S}in_q x \rangle \ ,$$

where D_q (q-derivative) is defined by $D_q f(x) := \dfrac{f(qx) - f(q)}{qx - x}$, and $\langle f(x) \rangle := \dfrac{f(x) + f(qx)}{2}$.

6.2. Padé-analogues of the exponential function on $\mathbb{T}$

Padé approximant of e^x is a rational function $R_{j,k}(x) = \dfrac{P_j(x)}{Q_k(x)}$, where orders j, k are given, which agrees with e^x (at $x = 0$) to the highest possible order.

- $E^{\alpha}_{1,0}(t, t_0) = e_{\alpha}(t, t_0)$ delta exponential function
- $E^{\alpha}_{0,1}(t, t_0) = \widehat{e}_{\alpha}(t, t_0)$ nabla exponential function
- $E^{\alpha}_{1,1}(t, t_0) = E_{\alpha}(t, t_0)$ Cayley-exponential function
- $E^{\alpha}_{2,2}(t^{\sigma}, t_0) = \dfrac{1 + \frac{1}{2}\alpha\mu + \frac{1}{12}(\alpha\mu)^2}{1 - \frac{1}{2}\alpha\mu + \frac{1}{12}(\alpha\mu)^2}\ E^{\alpha}_{2,2}(t, t_0)$, which satisfies:

  $$x^{\Delta} = \frac{\alpha}{1 + \frac{1}{12}(\alpha\mu)^2}\ \langle x \rangle\ , \qquad x^{\Delta} = \frac{\alpha}{1 - \frac{1}{2}\alpha\mu + \frac{1}{12}(\alpha\mu)^2}\ x, \qquad (x = E^{\alpha}_{2,2}).$$

- Similarly: $E^{\alpha}_{j,k}(t^{\sigma}, t_0)$ (with "good" trigonometry for $k = j$).

6.3. **Exact analogues** of elementary/special functions on $\mathbb{T}$

Given $f : \mathbb{R} \to \mathbb{C}$, we define its **exact analogue** $\tilde{f} : \mathbb{T} \to \mathbb{C}$ as $\tilde{f} := f|_{\mathbb{T}}$, i.e.,

$$\tilde{f}(t) := f(t) \quad (\text{for } t \in \mathbb{T}) \ .$$

The path $f \to \tilde{f}$ is obvious and unique, but how to find f corresponding to a given $\tilde{f}$? For example, which function $a : \mathbb{R} \to \mathbb{C}$ corresponds to a given function $\alpha : \mathbb{T} \to \mathbb{C}$? The answer is obvious if $\alpha = \text{const}$. What about other cases?

Exact exponential function on $\mathbb{T}$

Assumption: $\alpha = \text{const} \in \mathbb{C}$.

Definition. $E_\alpha^{ex}(t, t_0) := e^{\alpha(t-t_0)}$.

Theorem. The exact exponential function $E_\alpha^{ex}(t, t_0)$ satisifies

$$x^\Delta(t) = \alpha \ \psi_\alpha(t) \ \langle x(t) \rangle \ , \qquad x(t_0) = 1 \ ,$$

where $\psi_\alpha(t) = 1$ for right-dense points and

$$\psi_\alpha(t) = \frac{2}{\alpha\mu(t)} \tanh \frac{\alpha\mu(t)}{2}$$

for right-scattered points.

6.4. Dynamic systems on Lie groups

Natural generalizations of the Cayley transform:

- Lie algebra $\mathfrak{g} \to$ ("quadratic") Lie group G,
- anti-Hermitean operators $\to$ unitary operators.

Lemma. $A \in \mathfrak{g} \quad \Rightarrow \quad (I - A)^{-1}(I + A) \in G$.

The dynamic system $\Phi^{\Delta} = A\langle\Phi\rangle$, i.e., $\Phi^{\sigma} = \dfrac{I + \frac{1}{2}\mu A}{I - \frac{1}{2}\mu A}\Phi$

is a natural $\mathbb{T}$-analogue of $\dfrac{d}{dt}\Phi = A\Phi$ (here $A \in \mathfrak{g}$, $\Phi \in G$)

Another approach: J.L.Cieśliński (2007), "Pseudospherical surfaces on time scales: a geometric definition and the spectral approach", **J. Phys. A: Math. Theor. 40** (2007) 12525-12538.

6.5. Sine-Gordon equation on time scales

Discrete case: $$\frac{\sin(\frac{1}{4}\mu_x\mu_y\Phi^{\Delta_x\Delta_y})}{\frac{1}{4}\mu_x\mu_y} = \sin\langle\Phi\rangle \ ,$$

$$\langle\Phi\rangle := \frac{\Phi^{\sigma_x\sigma_y} + \Phi^{\sigma_x} + \Phi^{\sigma_y} + \Phi}{4} \ .$$

Extension on any $\mathbb{T}$: soon.

Lax pair: $\Psi^{\Delta_x} = U\Psi \ , \quad \Psi^{\Delta_y} = V\Psi \ ,$
where U is linear in λ, V is linear in λ^{-1}.

J.L.Cieśliński, "Pseudospherical surfaces on time scales...",
J. Phys. A: Math. Theor. **40** (2007) 12525-12538.

Open problem: to extend the Ablowitz-Ladik spectral problem on time scales.

7. Conclusions and future directions

- Differential equations have no unique 'natural' time scales analogues. It is worthwhile to consider different numerical schemes in this context.

- Dynamic systems preserving integrals of motion and Lyapunov functions (discrete gradient method).

- Inequalities of Gronwall type.

- New developments in the q-calculus (e.g., modifications of q-gamma function and of the Jackson integral).

- Laplace and Fourier transformations.

- Locally exact $\mathbb{T}$-analogues of elementary functions.

Exponential function

Exponential function on $\mathbb{T}$ (Hilger, 1990)

$$e_\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \xi_{\mu(s)}(\alpha(s))\Delta s\right) ,$$

$$e_\alpha(t) := e_\alpha(t,0) ,$$

where

$$\xi_\mu(z) := \frac{1}{\mu}\log(1+z\mu) \quad (\text{for} \quad \mu > 0)$$

$$\xi_0(z) := z .$$

Assumption: α is μ-**regressive** (i.e., $\mu\alpha \neq -1$) and rd-continuous.

Properties of Hilger's exponential function:

1. $e_\alpha(t^\sigma, t_0) = (1+\mu(t)\alpha(t))\ e_\alpha(t, t_0)$,

2. $(e_\alpha(t, t_0))^{-1} = e_{\ominus^\mu \alpha}(t, t_0)$,

3. $e_\alpha(t, t_0)\ e_\alpha(t_0, t_1) = e_\alpha(t, t_1)$,

4. $e_\alpha(t, t_0)\ e_\beta(t, t_0) = e_{\alpha \oplus^\mu \beta}(t, t_0)$,

where α, β rd-continuous and μ-regressive, $t^\sigma \equiv \sigma(t)$,

$\alpha \oplus^\mu \beta := \alpha + \beta + \mu\alpha\beta$ and $\ominus^\mu \alpha := \frac{-\alpha}{1+\mu\alpha}$.

Theorem: $E_\alpha(t,t_0) = e_\beta(t,t_0)$, if

$$\alpha(t) = \frac{\beta(t)}{1+\frac{1}{2}\mu(t)\beta(t)}\ , \quad \beta(t) = \frac{\alpha(t)}{1-\frac{1}{2}\mu(t)\alpha(t)}\ .$$

Proof (sketch). We denote $x(t) = E_\alpha(t,t_0)$ and consider a right-scattered t (i.e., $t^\sigma \neq t$). Then:

$$x^\Delta(t) = \alpha(t)\langle x(t)\rangle \quad \Longleftrightarrow \quad \frac{x(t^\sigma)-x(t)}{t^\sigma - t} = \alpha(t)\ \frac{x(t^\sigma)+x(t)}{2}.$$

Hence (using $\mu(t) = t^\sigma - t$): $\quad x(t^\sigma) = \dfrac{1+\frac{1}{2}\mu(t)\alpha(t)}{1-\frac{1}{2}\mu(t)\alpha(t)}\ x(t)\ .$

Therefore, $\quad x^\Delta = \dfrac{x(t^\sigma)-x(t)}{\mu(t)} = \dfrac{\alpha(t)x(t)}{1-\frac{1}{2}\mu(t)\alpha(t)} = \beta(t)x(t),$

which means that $x(t) = e_\beta(t,t_0)$.

Numerical advantages of E_α

$$E_\alpha(t^\sigma, t) = \frac{1 + \frac{1}{2}\mu\alpha}{1 - \frac{1}{2}\mu\alpha} = 1 + \alpha\mu + \frac{1}{2}(\alpha\mu)^2 + \frac{1}{4}(\alpha\mu)^3 + \dots \ ,$$

$$e_\alpha(t^\sigma, t) = 1 + \alpha\mu \ ,$$

$$\hat{e}_\alpha(t^\sigma, t) = \frac{1}{1 - \alpha\mu} = 1 + \alpha\mu + (\alpha\mu)^2 + \dots$$

Continuous case:

$$\exp(\alpha\mu) = 1 + \alpha\mu + \frac{1}{2}(\alpha\mu)^2 + \frac{1}{6}(\alpha\mu)^3 + \dots$$

Therefore, for $\mu \neq 0$,

$E_\alpha(t^\sigma, t)$ is a **second-order approximation** of $\exp(\alpha\mu)$, while $e_\alpha(t^\sigma, t)$ and $\hat{e}_\alpha(t^\sigma, t)$ are of the first order only.

q-Calculus

Standard q-exponential functions e_q^x, E_q^x

(definitions and notation are much older than the time scales!)

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!} = \prod_{k=0}^{\infty} (1-(1-q)q^k x)^{-1},$$

$$E_q^x = \sum_{j=0}^{\infty} q^{\frac{1}{2}j(j-1)} \frac{x^j}{[j]!} = \prod_{k=0}^{\infty} (1+(1-q)q^k x) = (e_q^{-x})^{-1} = e_{1/q}^x.$$

where $[j]! = [1][2]\dots[j]$, $[j] = 1+q+\dots+q^{j-1}$, i.e.,

$$[j]! = 1\cdot(1+q)\cdot(1+q+q^2)\cdot\ldots\cdot(1+q+\dots+q^{j-1}) \ ,$$

$$q^{\frac{1}{2}j(1-j)}[j]! = 1\cdot\left(1+\frac{1}{q}\right)\cdot\ldots\cdot\left(1+\frac{1}{q}+\dots+\frac{1}{q^{j-1}}\right).$$

q-**Exponentials** in terms of exponential functions on $\mathbb{T} = \overline{q^{\mathbb{N}_0}}$

$$q < 1 \quad \Rightarrow \quad e_q^x = \widehat{e}_x(1,0), \quad E_q^x = e_x(1,0),$$

$$q > 1 \quad \Rightarrow \quad e_q^x = e_x(1,0), \quad E_q^x = \widehat{e}_x(1,0),$$

$$\mathcal{E}_q^x = E_x(1,0), \qquad \mathcal{E}_{1/q}^x = E_x(1,0),$$

Note that $$\mathcal{E}_{1/q}^x = \mathcal{E}_q^x, \qquad E_q^x = e_{1/q}^x.$$

Let $\mathbb{T} = q^{\mathbb{N}_0}$ $(0 < q < 1)$, and $\alpha(t) = x$. Then

$$e_x(t) = \prod_{j=k}^{\infty} \left(1 + (1-q)q^j x\right), \qquad t = q^k.$$

Standard q-trigonometric functions

$$\mathrm{sin}_q\, x = \frac{e_q^{ix} - e_q^{-ix}}{2i}\ , \quad \mathrm{Sin}_q x = \frac{E_q^{ix} - E_q^{-ix}}{2i}\ ,$$

$$\mathrm{cos}_q\, x = \frac{e_q^{ix} + e_q^{-ix}}{2}\ , \quad \mathrm{Cos}_q x = \frac{E_q^{ix} + E_q^{-ix}}{2}\ ,$$

Properties:

$$\mathrm{cos}_q\, x\ \mathrm{Cos}_q x + \mathrm{sin}_q\, x\ \mathrm{Sin}_q x = 1$$

$$D_q \,\mathrm{sin}_q\, x = \mathrm{cos}_q\, x\ , \quad D_q \,\mathrm{cos}_q\, x = -\,\mathrm{sin}_q\, x\ ,$$

$$D_q \mathrm{Sin}_q x = \mathrm{Cos}_q(qx)\ , \quad D_q \mathrm{Cos}_q x = -\mathrm{Sin}_q(qx)\ .$$

where D_q (q-derivative) is defined by $D_q f(x) := \dfrac{f(qx) - f(q)}{qx - x}$.

Positively regressive functions

Definition. Function $\alpha : \mathbb{T} \to \mathbb{R}$ is called *positively regressive*, if for all $t \in \mathbb{T}^\kappa$ we have $|\alpha(t)\mu(t)| < 2$.

Theorem. If $\alpha : \mathbb{T} \to \mathbb{R}$ is *rd-continuous* and *positively regressive*, then the Cayley-exponential function E_α is positive (i.e., $E_\alpha(t) > 0$ for all $t \in \mathbb{T}$).

Theorem. The set of real *positively regressive* functions is an abelian group with respect to the addition $\oplus$.

Attention. The set of all regressive functions is not closed with respect to the addition $\oplus$. In order to show this fact it is enough to take α, β such that $\mu^2\alpha\beta = -4$. Then $\alpha \oplus \beta$ is infinite.

Exact discretization

Modified delta derivative

$$x^{\Delta'_\alpha}(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{x(t^\sigma) - x(s)}{\delta_\alpha(t^\sigma - s)}$$

where $\delta_\alpha(\mu) := \dfrac{2}{\alpha} \tanh \dfrac{\alpha\mu}{2}$.

Lemma. $x^{\Delta}(t) = \psi_\alpha(t) x^{\Delta'_\alpha}(t)$.

Lemma. Exact exponential function $E_\alpha^{ex}(t, t_0)$ satisfies

$$x^{\Delta'_\alpha}(t) = \alpha\langle x(t)\rangle \ , \qquad x(t_0) = 1 \ .$$

This is the **exact discretization** of the equation $\dot{x} = \alpha x$.

Exact analogues of **hyperbolic** and **trigonometric** functions on $\mathbb{T}$.

$$\cosh^{ex}_{\alpha}(t) = \frac{E^{ex}_{\alpha}(t)+E^{ex}_{-\alpha}(t)}{2} = \cosh \alpha t \ ,$$

$$\sinh^{ex}_{\alpha}(t) = \frac{E^{ex}_{\alpha}(t) - E^{ex}_{-\alpha}(t)}{2} = \sinh \alpha t \ .$$

$$\cos^{ex}_{\omega}(t) = \frac{E^{ex}_{i\omega}(t)+E^{ex}_{-i\omega}(t)}{2} = \cos \omega t \ ,$$

$$\sin^{ex}_{\omega}(t) = \frac{E^{ex}_{i\omega}(t) - E^{ex}_{-i\omega}(t)}{2i} = \sin \omega t \ .$$

The last two definitions coincide with Hilger's definitions.

These functions satisfy rather complicated dynamic equations which simplify greatly in the case $\mu = \text{const}$.

Exact harmonic oscillator on $\mathbb{T}$.

If $\mu(t) = \text{const}$, $\omega(t) = \text{const}$, then $\cos_\omega^{ex}$ and $\sin_\omega^{ex}$ satisfy

$$x^{\Delta\Delta}(t) + \omega^2 \phi^2(\omega\mu) \langle\langle x(t) \rangle\rangle = 0 \ ,$$

or, equivalently,

$$x^{\Delta\Delta}(t) + \omega^2 \left(\text{sinc} \frac{\omega\mu}{2} \right)^2 x(t^\sigma) = 0 \ ,$$

where $\text{sinc}(x) := \frac{\sin x}{x}$ (for $x \neq 0$) , $\text{sinc}(0) := 1$.

Another equivalent form of this equation reads

$$x^{\Delta''_\omega \Delta''_\omega}(t) + \omega^2 x(t) = 0.$$

$x^{\Delta''_\omega}$ is another **modification of the delta derivative**.

$$x^{\Delta''_\omega}(t) = \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{x(t^\sigma) - x(s)\cos(\omega t^\sigma - \omega s)}{\omega^{-1}\sin(\omega t^\sigma - \omega s)} \ .$$

In order to avoid infinite values of $x^{\Delta''_\omega}$ we assume $|\omega\mu(t)| < \pi$. All positively regressive constant functions ω obviously satisfy this requirement.

Lemma. If $x = x(t)$ solves the equation $\ddot{x} + \omega^2 x = 0$ (defined for $t \in \mathbb{R}$), then

$$(x(t)|_{t\in\mathbb{T}})^{\Delta''_\omega} = \dot{x}(t)|_{t\in\mathbb{T}} \ .$$

Lemma.

$$x^{\Delta}(t) = \text{sinc}(\omega\mu)\ x^{\Delta''_\omega}(t) - \frac{1}{2}\mu\omega^2 \left(\text{sinc}\frac{\omega\mu}{2}\right)^2 x(t) \ .$$