# A classification of integrable Weingarten surfaces

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The talk will discuss integrable PDE related to immersed surfaces in $\mathbb{R}^3$. Cassification of such PDE is a joint project with Hynek Baran.

Our expectations:

– obtaining lists of integrable classes of surfaces, as complete as possible

– identifying old cases (including well-forgotten ones);

– discovering new integrable classes/new integrable PDE.

## Motivations

**Soliton PDE.** Around 1970, soliton theory started to bring new and powerful integration methods. Multiple intersections with differential geometry exist.

A. Sym, Soliton surfaces and their applications. Soliton geometry from spectral problems, in: R. Martini, ed., *Geometric Aspects of the Einstein Equations and Integrable Systems,* Lecture Notes in Physics 239 (Springer, Berlin, 1985) 154–231.

C. Rogers and W.K. Schief, *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge Univ. Press, Cambridge, 2002).

**Main Question** (answer still pending). Is a given system of PDE (related to geometry or not) integrable in the sense of soliton theory?

## Definition

Given a system $\mathcal{E}$ of PDE in independent variables $x, y$, a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-valued zero curvature representation for $\mathcal{E}$ is a form $\alpha = A\,dx + B\,dy$ with $A, B \in \mathfrak{g}$ such that

$$D_y A - D_x B + [A, B] = 0$$

as a consequence of the system $\mathcal{E}$.

## Applications

– Zakharov–Shabat formulation of the inverse spectral transform,
– algebro–geometric solutions in terms of theta functions,
– Bäcklund/Darboux transformations,
– nonlocal symmetries,
– recursion operators and hierarchies of symmetries.

## Example

The mKdV equation $u_t + u_{xxx} - 6u^2 u_x = 0$ has an $\mathfrak{sl}_2$-valued zero curvature representation $A\,dx + B\,dt$ with

$$A = \begin{pmatrix} u & \lambda \\ 1 & -u \end{pmatrix},$$

$$B = \begin{pmatrix} -u_{xx} + 2u^3 - 4\lambda u & 2\lambda u_x + 2\lambda u^2 - 4\lambda^2 \\ -2u_x + 2u^2 - 4\lambda & u_{xx} - 2u^3 + 4\lambda u \end{pmatrix}.$$

Indeed, $D_t(A) - D_x(B) + [A, B] = (u_t + u_{xxx} - 6u^2 u_x) \cdot C$, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here $\lambda$ is a parameter (the spectral parameter).

**Problem.** How to tell whether a given nonlinear system has a zero curvature representation?

## The method

Resources:

M.M., A direct procedure to compute zero-curvature representations. The case $\mathfrak{sl}_2$, in: *Secondary Calculus and Cohomological Physics*, Proc. Conf. Moscow, 1997 (ELibEMS, 1998) pp. 10.

Normal forms:

P. Sebestyén, Normal forms of irreducible $\mathfrak{sl}_3$-valued zero curvature representations, *Rep. Math. Phys* 55 (2005) No. 3, 435–445.

P. Sebestyén, On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations, *Rep. Math. Phys* 62 (2008) No. 1.

## Description of the method

Supposing $A, B, C_l$ to be in a normal form, the determining system

$$(D_y A - D_x B + [A, B])|_{\mathcal{E}} = 0,$$

$$\sum_{I,l} (-\widehat{D})_I \left( \frac{\partial F^l}{\partial u^k_I} C_l \right) \bigg|_{\mathcal{E}} = 0$$

has the following properties:

– is a system of differential equations in total derivatives;

– has the same number of unknowns as equations;

– is quasilinear in $A, B$ and linear in $C_l$;

– impossible to solve without computer algebra;

– solution algorithms are resource demanding;

– computation splits into cases to avoid division by zero (a consequence of nonlinearity in $A, B$).

## The spectral parameter problem

**Example.** Gauss–Weingarten equations = a parameterless zero curvature representation of the Gauss–Mainardi–Codazzi equations.

**Problem.** When a parameter can be incorporated?

Solution exploiting a symmetry group parameter:

D. Levi, A. Sym and Tu Gui-Zhang, preprint 1990

J. Cieśliński, Lie symmetries as a tool to isolate integrable geometries, in: M. Boiti et al., eds., *Nonlinear Evolution Equations and Dynamical Systems* (World Scientific, Singapore, 1992).

Local symmetries can be insufficient (NHNLS example); extended symmetries operating in classes of equations are necessary:

J. Cieśliński, Non-local symmetries and a working algorithm to isolate integrable geometries, *J. Phys. A: Math. Gen.* **26** (1993) L267–L271.

## A cohomological solution

To solve the spectral parameter problem in a given Lie algebra:

1) compute cohomological obstructions, obtained when expanding the zero curvature representation in terms of the (prospective) spectral parameter $A = \sum_i A_i\lambda^i$, $B = \sum_i B_i\lambda^i$

$$D_y A_0 - D_x B_0 + [A_0, B_0] = 0,$$

$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0,$$

$$D_y A_2 - D_x B_2 + [A_2, B_0] + [A_1, B_1] + [A_0, B_2] = 0,$$

$$\dots;$$

2) compute the full zero curvature representation using the information obtained in the first step to cut off branches.

M.M., On the spectral parameter problem, *Acta Appl. Math.* **109** (2010) 239–255.

**Warning.** The solution could exist in a larger Lie algebra.

## The classification project

We consider geometrically determined classes of surfaces, meaning classes determined by a single condition

$$F(p_1, \ldots, p_k) = 0,$$

where $p_i$ are differential invariants with respect to reparameterizations and euclidean motions (principal curvatures, their gradients, etc.).

We classify relations $F = 0$ such that

– the associated Gauss–Mainardi–Codazzi equations possess a zero curvature representation depending on a nonremovable (spectral) parameter;

– the zero curvature representation has a prescribed order $r$ and takes values in a prescribed Lie algebra $\mathfrak{sl}(n)$.

## Weingarten surfaces

To start with, we focus on Weingarten surfaces, i.e., classes of immersed surfaces in $\mathbf{E}^3$ determined by a functional relation between the principal curvatures $k_1, k_2$.

**Examples.** All rotation surfaces; constant Gaussian curvature surfaces; constant mean curvature surfaces.

**Classification Problem.** Which functional relations $f(k_1, k_2) = 0$ determine an integrable class of Weingarten surfaces?

**Example.** Bonnet surfaces are surfaces that admit a nontrivial isometry preserving both principal curvatures. Bonnet surfaces are integrable, are Weingarten surfaces, but the functional relation $f(k_1, k_2) = 0$ is different for different Bonnet surfaces. Hence, Bonnet surfaces are not an integrable class of Weingarten surfaces.

## The Finkel–Wu conjecture

**Example.** Any linear relation between the mean curvature $\frac{1}{2}(k_1 + k_2)$ and the Gauss curvature $k_1 k_2$:

$$a k_1 k_2 + b(k_1 + k_2) + c = 0$$

determines an integrable class (linear Weingarten surfaces).

**Conjecture.** The only class of integrable Weingarten surfaces are the linear Weingarten surfaces.

Hongyou Wu, Weingarten surfaces and nonlinear partial differential equations, *Ann. Global Anal. Geom.* **11** (1993) 49–64.

F. Finkel, On the integrability of Weingarten surfaces, in: A. Coley et al., ed., *Bäcklund and Darboux Transformations. The Geometry of Solitons*, AARMS-CRM Workshop, June 4-9, 1999, Halifax, N.S., Canada, (Amer. Math. Soc., Providence, 2001) 199–205.

## Preliminaries

Parameterized by the lines of curvature, surfaces $\mathbf{r}(x, y)$ have the fundamental forms

$$\mathrm{I} = u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2, \qquad \mathrm{II} = \frac{u^2}{\rho}\,\mathrm{d}x^2 + \frac{v^2}{\sigma}\,\mathrm{d}y^2.$$

where $\rho, \sigma$ are the principal radii of curvature, $\rho = 1/k_1$, $\sigma = 1/k_2$.

In the Weingarten case, $\rho = \rho(\sigma)$, the Mainardi–Codazzi subsystem can be explicitly solved. The full GMC system then reduces to the Gauss equation alone.

**Proposition.** The Gauss equation of Weingarten surfaces can be written in the form

$$R_{xx} + S_{yy} + T = 0,$$

where $R, S, T$ are functions of $\sigma$.

## A non-parametric zero curvature representation

The Gauss–Mainardi–Codazzi equations always posses a non-parametric zero curvature representation

$$A_0 = \begin{pmatrix} \dfrac{\mathrm{i}u_y}{2v} & -\dfrac{u}{2\rho} \\ \dfrac{u}{2\rho} & -\dfrac{\mathrm{i}u_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\dfrac{\mathrm{i}v_x}{2u} & -\dfrac{\mathrm{i}v}{2\sigma} \\ -\dfrac{\mathrm{i}v}{2\sigma} & \dfrac{\mathrm{i}v_x}{2u} \end{pmatrix}$$

($x, y$ label the lines of curvature).

**Question.** Can we incorporate a parameter?

**Answer.** No, unless we impose a suitable additional condition.

**Problem.** Which *geometric* conditions $F(\rho, \sigma) = 0$ imply integrability?

## Results of the computation

Weingarten surfaces determined by an explicit dependence $\rho(\sigma)$ possess a one-parametric zero curvature representation if and only if the **determining equation**

$$\rho''' = \frac{3}{2\rho'}\rho''^2 + \frac{\rho' - 1}{\rho - \sigma}\rho'' + 2\frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2}$$

holds (the prime denotes $\mathrm{d}/\mathrm{d}\sigma$).

This equation has

– a general solution in terms of elliptic integrals;

– a number of special cases when the solution $\rho(\sigma)$ can be expressed in terms of elementary functions.

**Surprise.** All the special cases were known in the XIX century.

**Corollary.** The Finkel–Wu conjecture is false.

## Solving the determining ODE

Two geometric 1-parametric groups of symmetries:

- scaling (changing the ruler) $\rho \longmapsto \mathrm{e}^T \rho$, $\sigma \longmapsto \mathrm{e}^T \sigma$;
- translation (offsetting, normal shift) $\rho \longmapsto \rho + T$, $\sigma \longmapsto \sigma + T$.

They help us to reduce the order by two.

The resulting 1st order ODE is separable.

The general solution $\rho(\sigma)$ is

$$\rho + \sigma = \frac{1}{m} \int^{m(\rho-\sigma)} \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s.$$

Here $m$ is a scaling parameter, the integration constant is an offsetting parameter, and $c$ is a "true" parameter.

## Summary of the special cases

up to scaling and offsetting; $\rho, \sigma$ are the principal radii of curvature.

| relation | integrable equation |
|---|---|
| $\rho + \sigma = 0$ | $z_{xx} + z_{yy} + \mathrm{e}^z = 0$ |
| $\rho\sigma = 1$ | $z_{xx} + z_{yy} - \sinh z = 0$ |
| $\rho\sigma = -1$ | $z_{xx} - z_{yy} + \sin z = 0$ |
| $\rho - \sigma = \sinh(\rho + \sigma)$ | $(\tanh z - z)_{xx} + (\coth z - z)_{yy} + \operatorname{csch} 2z = 0$ |
| $\rho - \sigma = \sin(\rho + \sigma)$ | $(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$ |
| $\rho - \sigma = 1$ | $z_{xx} + (1/z)_{yy} + 2 = 0$ |
| $\rho - \sigma = \tanh \rho$ | $\frac{1}{4}(\sinh z - z)_{xx} + (\coth \frac{1}{2} z)_{yy} + \coth \frac{1}{2} z = 0$ |
| $\rho - \sigma = \tan \rho$ | $\frac{1}{4}(\sin z - z)_{xx} + (\cot \frac{1}{2} z)_{yy} + \cot \frac{1}{2} z = 0$ |
| $\rho - \sigma = \coth \rho$ | $\frac{1}{4}(\sinh z + z)_{xx} - (\tanh \frac{1}{2} z)_{yy} + \tanh \frac{1}{2} z = 0$ |
| $\rho - \sigma = -\cot \rho$ | $\frac{1}{4}(\sin z + z)_{xx} + (\tan \frac{1}{2} z)_{yy} + \tan \frac{1}{2} z = 0$ |

## Surfaces of constant astigmatism

The relation $\rho - \sigma = \text{const}$ was among the special solutions.

H. Baran and M.M., On integrability of Weingarten surfaces: a forgotten class, *J. Phys. A: Math. Theor.* **42** (2009) 404007.

Popular among nineteenth-century geometers:

A. Ribaucour, Note sur les développées des surfaces, *C. R. Acad. Sci. Paris* 74 (1872) 1399–1403.

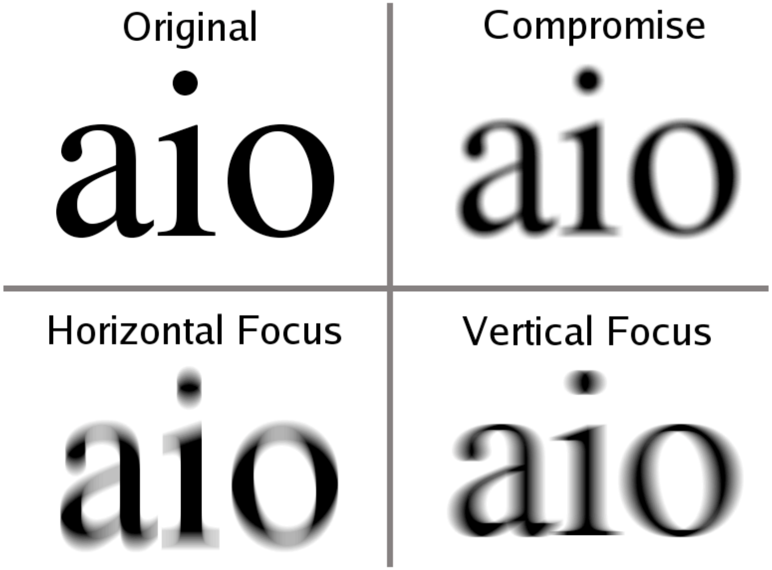
A. Mannheim, Sur les surfaces dont les rayons de courbure principaux sont fonctions l'un de l'autre, *Bull. S.M.F.* 5 (1877) 163–166.

R. Lipschitz, Zur Theorie der krummen Oberflächen, *Acta Math.* 10 (1887) 131–136.

R. von Lilienthal, Bemerkung über diejenigen Flächen bei denen die Differenz der Hauptkrümmungsradien constant ist, *Acta Math.* 11 (1887) 391–394.

## Astigmatism

A general reflecting or refracting surface exhibits two focuses in perpendicular directions at distances equal to $\rho$ and $\sigma$.




Tallfred, `http://en.wikipedia.org/wiki/Astigmatism_(eye)`

The difference $\rho - \sigma$ is known as the *interval of Sturm* or the *astigmatic interval* or the *amplitude of astigmatism* or the *astigmatism*.

## The constant astigmatism equation

The constant $\rho - \sigma$ can be always reduced to 1 by rescaling the ambient metric. Then the Gauss equation can be put in the form

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0,$$

which we call the constant astigmatism equation.

The equation has obvious translational symmetries (reparameterization) $\partial_x, \partial_y$, the scaling symmetry

$$2z\frac{\partial}{\partial_z} - x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y},$$

which corresponds to offsetting, and a discrete symmetry

$$x \longrightarrow y, \quad y \longrightarrow x, \quad z \longrightarrow \frac{1}{z},$$

which corresponds to swapping the orientation & taking the parallel surface at the unit distance.

## Two third-order symmetries

One of them has the generator

$$
\begin{aligned}
&\frac{z^3}{K^3}(z_{xxx} - z z_{xxy})\\
&\quad - \frac{3}{K^5} z^3 (z_x - z z_y)(z_{xx} - z z_{xy})^2 - \frac{2}{K^5} z^5 (9 z_x - z z_y) z_{xx}\\
&\quad + \frac{1}{2K^5} z^2 (9 z_x^2 + 4 z z_x z_y - z^2 z_y^2)(z_x - z z_y) z_{xx}\\
&\quad - \frac{2}{K^5} z^3 z_x (z_x - z z_y)(4 z_x - z z_y) z_{xy} + \frac{4}{K^5} z^6 z_x z_{xy}\\
&\quad + \frac{3}{K^5} z^4 (5 z_x - z z_y) z_x^2 - \frac{3}{K^5} z (z_x - z z_y) z_x^4,
\end{aligned}
$$

where $K = \sqrt{(z_x - z z_y)^2 + 4 z^3}$.

The other symmetry is obtained by conjugation with the discrete symmetry above.

## A recursion operator

due to A. Sergyeyev (private communication).

If $Z$ is a generating function of a symmetry, then so is

$$Z' = -z_y U + z_x V + 2zW,$$

where $U, V, W$ satisfy

$$D_x U = Z, \qquad D_x V = W, \qquad D_x W = D_y Z,$$
$$D_y U = W, \qquad D_y V = \frac{Z}{z^2}, \qquad D_y W = D_x \frac{Z}{z^2}.$$

In the pseudodifferential form:

$$Z' = -z_y D_x^{-1} + z_x D_x^{-2} D_y + 2z D_x^{-1} D_y.$$

Takes local symmetries to nonlocal ones.

## Relation to the sine–Gordon equation

A. Ribaucour, Note sur les développées des surfaces, *C. R. Acad. Sci. Paris* 74 (1872) 1399–1403.

The focal surfaces of surfaces satisfying $\rho - \sigma = \text{const}$ are pseudospherical. Hence a relation to the sine-Gordon equation.

Let $w = \frac{1}{2} \ln z$. Determine function $\phi'$ and coordinates $\xi, \eta$ from

$$\cos\phi' = \frac{w_x^2 - \mathrm{e}^{2w} - \mathrm{e}^{4w} w_y^2}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}},$$

$$\sin\phi' = -\frac{2\mathrm{e}^{w} w_x}{\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}},$$

$$d\xi = \tfrac{1}{2}\sqrt{(w_x + \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,dx + \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x + w_y)^2 + \mathrm{e}^{-2w}}\,dy,$$

$$d\eta = \tfrac{1}{2}\sqrt{(w_x - \mathrm{e}^{2w} w_y)^2 + \mathrm{e}^{2w}}\,dx - \tfrac{1}{2}\sqrt{(\mathrm{e}^{-2w} w_x - w_y)^2 + \mathrm{e}^{-2w}}\,dy.$$

Then $\phi'(\xi, \eta)$ is a solution to the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$.

## The Bianchi transformation

Another solution of the sine-Gordon equation can be obtained from the other focal surface.

The two focal surfaces are related by the classical Bianchi transformation:

– Corresponding points have a constant distance equal to $\rho - \sigma$;

– Corresponding normals are orthogonal;

– The line joining the corresponding points is tangent to both focal surfaces.

The Bianchi transformation is, however, superseded by the classical Bäcklund transformation, where the condition on the angle between the normals is relaxed from being right to being constant.

This probably explains why surfaces of constant curvature fell into oblivion.

## Inverse relation to the sine–Gordon equation

An arbitrary pseudospherical surface can be equipped with a parabolic geodesic net. Involutes of the geodesics along the same starting line form a surface of constant astigmatism.

Let $\phi(\xi,\eta)$ be a solution of the sine-Gordon equation $\phi_{\xi\eta} = \sin\phi$. Let $\alpha, \beta$ be solutions of the compatible equations

$$\beta_\xi = -\sin\alpha, \qquad \alpha_\eta = -\sin\beta, \qquad \alpha - \beta = \phi.$$

Compute functions $X, x, y$ from

$$dX = \cos\alpha\, d\xi + \cos\beta\, d\eta,$$

$$dx = \mathrm{e}^{-X}(\sin\alpha\, d\xi + \sin\beta\, d\eta),$$

$$dy = \mathrm{e}^{X}(\sin\alpha\, d\xi + \sin\beta\, d\eta).$$

Then $\mathrm{e}^{-2X(x,y)}$ is a solution of the constant astigmatism equation.

## Von Lilienthal surfaces

R. von Lilienthal, Bemerkung über diejenigen Flächen bei denen die Differenz der Hauptkrümmungsradien constant ist, *Acta Math.* 11 (1887) 391–394.

A special case of the Lipschitz solution

R. Lipschitz, Zur Theorie der krummen Oberflächen, *Acta Math.* 10 (1887) 131–136.

Von Lilienthal surfaces are (made of) involutes of meridians of the pseudosphere starting at the same 'parallel'.

The pseudosphere itself is the involute of the catenoid.

All they are rotation surfaces:

– Catenoid = rotation of the catenary.

– Pseudosphere = rotation of the tractrix.

– Von Lilienthal surfaces = see the picture.

## Weingarten's 'new class of surfaces'

Surfaces satisfying relation $\rho - \sigma = \sin(\rho + \sigma)$.

J. Weingarten, Über die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine function des anderen ist, *J. Reine Angew. Math.* **62** (1863) 160–173.

Covered in §§ 745, 746, 766, 769, 770 of

G. Darboux, *"Leçons sur la théorie générale des surface et les applications géométriques du calcul infinitésimal,"* Vol. I–IV.

and §§ 135, 245, 246 of

L. Bianchi, *"Lezioni di Geometria Differenziale,"* Vol. I, II.

Darboux gave a general solution of the associated equation $(\tan z - z)_{xx} + (\cot z + z)_{yy} + \csc 2z = 0$. He also gave a remarkable geometric construction, further developed by Bianchi.

## Darboux correspondence

Darboux discovered a relationship with translation surfaces, further developed by Bianchi.

A *translation surface* is a surface that admits a parameterization $\tilde{\mathbf{r}}(\xi, \eta)$ such that

$$\tilde{\mathbf{r}}_{\xi\eta} = 0.$$

Equivalently, $\tilde{\mathbf{r}}(\xi, \eta) = \tilde{\mathbf{r}}_1(\xi) + \tilde{\mathbf{r}}_2(\eta)$. The curves $\tilde{\mathbf{r}}_1(\xi)$ and $\tilde{\mathbf{r}}_2(\eta)$ are called the *generating curves*.

Otherwise said, a translation surface is obtained when translating a curve along another curve. Translation surfaces are manifestly integrable if the curves are given by integrable systems of ODE.

A *middle evolute* of a surface consists of mid-points between the two focal surfaces.

## Darboux–Bianchi theorem I

**Proposition.** Let $\mathbf{r}$ satisfy

$$\rho - \sigma = \sin(\rho + \sigma),$$

let $\xi, \eta$ be the common asymptotic coordinates of its focal surfaces. Then

(i) the coordinates $\xi, \eta$ render the middle evolute $\tilde{\mathbf{r}}$ as a translation surface, i.e., $\tilde{\mathbf{r}}(\xi, \eta) = \tilde{\mathbf{r}}_1(\xi) + \tilde{\mathbf{r}}_2(\eta)$;

(ii) the generating curves $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$ have opposite nonzero constant torsion;

(iii) the normal vector $\mathbf{n}$ to the surface $\mathbf{r}$ at a point belongs to the intersection of the osculating planes of the generating curves $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$ through the corresponding point.

## Darboux–Bianchi theorem II

**Proposition.** Let $\mathbf{s}(\xi, \eta) = \mathbf{s}_1(\xi) + \mathbf{s}_2(\eta)$ be a nonplanar translation surface. Assume that the generating curves $\mathbf{s}_1(\xi)$ and $\mathbf{s}_2(\eta)$ are of opposite nonzero constant torsion $\tau$ and $-\tau$, respectively. Denote by $\mathbf{b}_1$ and $\mathbf{b}_2$ the respective binormal vectors of the generating curves $\mathbf{s}_1(\xi)$ and $\mathbf{s}_2(\eta)$ and by $\Theta = \arccos(\mathbf{b}_1, \mathbf{b}_2)$ the angle between them, $0 < \Theta < \pi$. Then the surface

$$\mathbf{r} = \mathbf{s} + \frac{\Theta + c_0}{\tau \sin \Theta} \mathbf{b}_1 \times \mathbf{b}_2$$

satisfies Weingarten's relation

$$\frac{\rho - \sigma}{c_1} = \sin\left(\frac{\rho + \sigma}{c_1} - c_0\right). \tag{1}$$

with $c_1 = 2/\tau$.

## Geometric characterization

The invertible offsetting transformation $\mathbf{r} \longmapsto \mathbf{r} + T\mathbf{n}$ preserves integrability in every reasonable sense of the word. Surfaces related by this transformation are said to be parallel. Either all are integrable or none is.

Parallel surfaces = normal surfaces to the same line congruence. Consequently, integrability is a property of this congruence and, therefore, must have an expression in terms of congruence invariants.

Normal congruences of Weingarten surfaces are known as $W$-congruences.

Recall that a generic surface has two focal surfaces

$$\mathbf{r}^{(1)} = \mathbf{r} + \sigma\mathbf{n}, \qquad \mathbf{r}^{(2)} = \mathbf{r} + \rho\mathbf{n}.$$

each of which is formed by the evolutes of one family of the curvature lines.

## Invariant characterization

The Gaussian curvatures are $K^{(i)} = \det \mathrm{II}^{(i)}/\det \mathrm{I}^{(i)}$, $i = 1, 2$. We have $K^{(1)} = -\rho'/(\rho - \sigma)^2\sigma'$, $K^{(2)} = -\sigma'/(\rho - \sigma)^2\rho'$.

It is convenient to choose

$$\kappa^{(i)} = \frac{1}{\sqrt{|K^{(i)}|}},$$

and

$$\gamma^{(i)} = \|\operatorname{grad}^{(i)} \kappa^{(i)}\|^{(i)} = \sqrt{I^{(i)}(\operatorname{grad}^{(i)} \kappa^{(i)}, \operatorname{grad}^{(i)} \kappa^{(i)})}\,.$$

**Proposition.** Under the condition $\gamma^{(1)} + \gamma^{(2)} \neq 0$, a Weingarten surface belongs to the integrable class iff

$$\gamma^{(1)}\gamma^{(2)} = \text{const}\,.$$