Phase Space Quantum Mechanics Canonical Regime Part 2

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Ordinary description of quantum mechanics

Tensor product of Hilbert spaces

$$L^2(\mathbb{R}^2) = \left(L^2(\mathbb{R})\right)^* \otimes L^2(\mathbb{R}),$$

where

$$(\varphi \otimes \psi)(x, y) = \varphi^*(x)\psi(y),$$
$$\langle \varphi_1 \otimes \psi_1 | \varphi_2 \otimes \psi_2 \rangle_{L^2} = \langle \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2},$$

for $\varphi, \varphi_1, \varphi_2 \in \left(L^2(\mathbb{R})\right)^* \cong L^2(\mathbb{R})$ and $\psi, \psi_1, \psi_2 \in L^2(\mathbb{R})$.

Isomorphisms of $L^2(\mathbb{R}^2)$

The Fourier transform $\mathcal{F}_y$ is an isomorphism of $L^2(\mathbb{R}^2)$. For $\Psi(x, y) \in L^2(\mathbb{R}^2)$, the function

$$\Psi(x, p) = \mathcal{F}_y(\Psi(x, y)) = \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}y \, e^{-\frac{i}{\hbar}py} \Psi(x, y)$$

will be called an (x, p)-representation of $\Psi(x, y)$ and it will be considered as a function on the phase space $M = \mathbb{R}^2$ in the canonical coordinates of position x and momentum p.

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The map

$$T_\sigma \Psi(x,y) := \Psi(x - \bar{\sigma} y, x + \sigma y), \quad \Psi \in L^2(\mathbb{R}^2)$$

is an isomorphism of $L^2(\mathbb{R}^2)$.

(σ, α, β)-twisted tensor product of Hilbert spaces

$$\mathcal{H} = \left(L^2(\mathbb{R})\right)^* \otimes_{\sigma,\alpha,\beta} L^2(\mathbb{R}) := S_{\alpha,\beta}\mathcal{F}_y T_\sigma \left(\left(L^2(\mathbb{R})\right)^* \otimes L^2(\mathbb{R})\right),$$

where

$$\langle \varphi_1 \otimes_{\sigma,\alpha,\beta} \psi_1 | \varphi_2 \otimes_{\sigma,\alpha,\beta} \psi_2 \rangle_{\mathcal{H}} = \langle \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2},$$

for $\varphi_1, \varphi_2 \in \left(L^2(\mathbb{R})\right)^* \cong L^2(\mathbb{R})$ and $\psi_1, \psi_2 \in L^2(\mathbb{R})$.

Generators of $\mathcal{H}$

The generators of $\mathcal{H}$ are of the form

$$\begin{aligned}\Psi^{\sigma,\alpha,\beta}(x,p) &= (\varphi \otimes_{\sigma,\alpha,\beta} \psi)(x,p) \\ &= \frac{1}{(2\pi\hbar)^{3/2}\sqrt{\alpha\beta}} \iiint \mathrm{d}x'\,\mathrm{d}p'\,\mathrm{d}y\, \varphi^*(x' - \bar{\sigma}y)\psi(x' + \sigma y) \\ &\quad \cdot e^{-\frac{1}{2\hbar\alpha}(x-x')^2} e^{-\frac{1}{2\hbar\beta}(p-p')^2} e^{-\frac{i}{\hbar}p'y},\end{aligned}$$

where $\varphi, \psi \in L^2(\mathbb{R})$.

Basis in $\mathcal{H}$

If $\{\varphi_i\}$ is an orthonormal basis in $L^2(\mathbb{R})$, then $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ is an orthonormal basis in $\mathcal{H}$ and for any $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi \in \mathcal{H}$ where $\varphi, \psi \in L^2(\mathbb{R})$:

$$\varphi = \sum_i b_i \varphi_i, \qquad \psi = \sum_j c_j \varphi_j, \qquad \text{for some } b_i, c_i \in \mathbb{C},$$

$$\Psi = \sum_{i,j} a_{ij} \Psi_{ij}, \qquad a_{ij} = b_i^* c_j.$$

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The basis functions Ψ_{ij} are idempotent:

$$\Psi_{ij} \star_{\sigma,\alpha,\beta} \Psi_{kl} = \frac{1}{\sqrt{2\pi\hbar}} \delta_{il} \Psi_{kj}.$$

Theorem

Every pure state $\Psi_{\mathrm{pure}} \in \mathcal{H}$ *is of the form*

$$\Psi_{\mathrm{pure}} = \varphi \otimes_{\sigma,\alpha,\beta} \varphi,$$

for some normalized function $\varphi \in L^2(\mathbb{R})$. *Conversely, every function* $\Psi \in \mathcal{H}$ *of the form* $\varphi \otimes_{\sigma,\alpha,\beta} \varphi$ *is a pure state.*

Observables $A \in \mathcal{A}_Q$ can be considered as operators on $\mathcal{H}$ given by

$$A \star_{\sigma,\alpha,\beta} = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}).$$

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$$\hat{\Psi} = \sqrt{2\pi\hbar}\Psi \star_{\sigma,\alpha,\beta} .$$

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Note, that for $\Psi = \sum_{i,j} c_{ij}\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j$ there holds

$$\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho},$$

where $\hat{\rho} = \sum_{i,j} c_{ij} \left|\varphi_j\right\rangle \left\langle\varphi_i\right|$.

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In particular, for $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ there holds

$$\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho},$$

where $\hat{\rho} = \left|\psi\right\rangle \left\langle\varphi\right|$.

If $\Psi_{\mathrm{pure}} = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$ is a pure state then $\hat{\Psi}_{\mathrm{pure}} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_{\mathrm{pure}}$ where $\hat{\rho}_{\mathrm{pure}} = |\varphi\rangle \langle\varphi|$. Moreover, operators $\hat{\rho}_{\mathrm{pure}}$ satisfy

1. $\hat{\rho}_{\mathrm{pure}} = \hat{\rho}^{\dagger}_{\mathrm{pure}}$ (hermiticity),
2. $\hat{\rho}^{2}_{\mathrm{pure}} = \hat{\rho}_{\mathrm{pure}}$ (idempotence),
3. $\operatorname{tr} \hat{\rho}_{\mathrm{pure}} = 1$ (normalization).

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If $\Psi_{\mathrm{mix}} = \sum_{\lambda} p_{\lambda} \Psi^{(\lambda)}_{\mathrm{pure}}$ is a mixed state then $\hat{\Psi}_{\mathrm{mix}} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_{\mathrm{mix}}$ where

$$\hat{\rho}_{\mathrm{mix}} = \sum_{\lambda} p_{\lambda} \hat{\rho}^{(\lambda)}_{\mathrm{pure}}.$$

Theorem

Let $A \in \mathcal{A}_Q$ and $\Psi \in \mathcal{H}$ be such that $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ for $\varphi, \psi \in L^2(\mathbb{R})$, then

$$A_L \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta})\Psi = \varphi \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\psi,$$
$$A_R \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta})\Psi = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\varphi \otimes_{\sigma,\alpha,\beta} \psi,$$

if $\psi \in D(A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}))$ and $\varphi \in D(A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}))$, where $\hat{q} = x$, $\hat{p} = -i\hbar\partial_x$ and $D(\hat{A})$ denotes a domain of an operator $\hat{A}$.

From previous theorem it follows that operators $A\star_{\sigma,\alpha,\beta}$ can be written as

$$A\star_{\sigma,\alpha,\beta} = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}) = \hat{1} \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}).$$

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Moreover, the action of observables $A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta})$ on states $\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}$ reads

$$A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta})\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\hat{\rho},$$
$$\hat{\Psi} A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}) = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}).$$

Theorem

Every solution of the $\star_{\sigma,\alpha,\beta}$-genvalue equation

$$A \star_{\sigma,\alpha,\beta} \Psi = a\Psi$$

for $A \in \mathcal{A}_Q$ and $a \in \mathbb{C}$ is of the form

$$\Psi = \sum_i \varphi_i \otimes_{\sigma,\alpha,\beta} \psi_i,$$

where $\varphi_i \in L^2(\mathbb{R})$ are arbitrary and $\psi_i \in L^2(\mathbb{R})$ are the eigenvectors of the operator $A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ corresponding to the eigenvalue a spanning the subspace of all eigenvectors, i.e. ψ_i satisfy the eigenvalue equation

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\psi_i = a\psi_i.$$

Theorem

Every solution of the $\star_{\sigma,\alpha,\beta}$*-genvalue equation*

$$\Psi \star_{\sigma,\alpha,\beta} B = b\Psi$$

for $B \in \mathcal{A}_Q$ *and* $b \in \mathbb{C}$ *is of the form*

$$\Psi = \sum_i \psi_i \otimes_{\sigma,\alpha,\beta} \varphi_i,$$

where $\varphi_i \in L^2(\mathbb{R})$ *are arbitrary and* $\psi_i \in L^2(\mathbb{R})$ *are the eigenvectors of the operator* $B^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ *corresponding to the eigenvalue* b^* *spanning the subspace of all eigenvectors, i.e.* ψ_i *satisfy the eigenvalue equation*

$$B^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\psi_i = b^*\psi_i.$$

In the nondegenerate case the solution Ψ to the following pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$A \star_{\sigma,\alpha,\beta} \Psi = a\Psi, \qquad \Psi \star_{\sigma,\alpha,\beta} B = b\Psi,$$

is unique up to a multiplication constant and is of the form $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$, where $\varphi, \psi \in L^2(\mathbb{R})$ satisfy the following eigenvalue equations

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In particular, a pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$A_L \star_{\sigma,\alpha,\beta} \Psi = a\Psi, \qquad (A_R \star_{\sigma,\alpha,\beta})^{\dagger}\Psi = a^{*}\Psi$$

have a solution Ψ in the form of a pure state $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$, where $\varphi \in L^2(\mathbb{R})$ is a solution to the eigenvalue equation

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\varphi = a\varphi.$$

Theorem

Let $A \in \mathcal{A}_Q$, $\Psi = \sum_\lambda p_\lambda \Psi^{(\lambda)}_{\mathrm{pure}} = \sum_\lambda p_\lambda (\varphi^{(\lambda)} \otimes_{\sigma,\alpha,\beta} \varphi^{(\lambda)}) \in \mathcal{H}$ *be some mixed state and* $\hat{\rho} = \sum_\lambda p_\lambda |\varphi^{(\lambda)}\rangle \langle \varphi^{(\lambda)}|$ *the corresponding density operator. Then there holds*

$$\langle A \rangle_\Psi = \sum_\lambda p_\lambda \langle \varphi^{(\lambda)} | A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) \varphi^{(\lambda)} \rangle_{L^2} = \mathrm{tr}(\hat{\rho} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})).$$

Time evolution of density operators

The time evolution of states represented as operators on the Hilbert space $L^2(\mathbb{R})$ is governed by a Hermitian operator $H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ corresponding to the Hamiltonian H:

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} - [H_{\sigma,\alpha,\beta}(\hat{q},\hat{p}), \hat{\rho}] = 0.$$

Stationary states of the harmonic oscillator

The Hamiltonian of the harmonic oscillator:

$$H(x,p) = \frac{1}{2}\left(p^2 + \omega^2 x^2\right).$$

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The stationary pure states of the harmonic oscillator are precisely the solutions of the following pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$H \star_{\sigma,\alpha,\beta} \Psi = E\Psi,$$
$$\Psi \star_{\sigma,\alpha,\beta} H = E\Psi,$$

for $E \in \mathbb{R}$.

Lets introduce new coordinates called *holomorphic coordinates*

$$a(x,p) = \frac{\omega x + ip}{\sqrt{2\hbar\omega}}, \qquad \bar{a}(x,p) = \frac{\omega x - ip}{\sqrt{2\hbar\omega}}.$$

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In this new coordinates the Hamiltonian H takes the form

$$H(a,\bar{a}) = \hbar\omega\left(\bar{a} \star_{\sigma,\alpha,\beta} a + \bar{\lambda}\right) = \hbar\omega\left(a \star_{\sigma,\alpha,\beta} \bar{a} - \lambda\right),$$

where $\lambda = \frac{1}{2}(1 + \omega\alpha + \omega^{-1}\beta)$ and $\bar{\lambda} := 1 - \lambda = \frac{1}{2}(1 - \omega\alpha - \omega^{-1}\beta)$.

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The $\star_{\sigma,\alpha,\beta}$-genvalues of H are equal

$$E_n = (n + \bar{\lambda})\hbar\omega.$$

Assume that $\sigma = \frac{1}{2}$ and $\beta = \omega^2\alpha$.

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$$\Psi_n = \frac{1}{\sqrt{2\pi\hbar\lambda}}(-1)^n \left(\frac{\bar{\lambda}}{\lambda}\right)^n L_n\left(\frac{H}{\hbar\omega\lambda\bar{\lambda}}\right) \exp\left(-\frac{H}{\hbar\omega\lambda}\right) \quad \text{for } \lambda \neq 0, 1,$$

$$\Psi_n = \frac{1}{\sqrt{2\pi\hbar n!}} \left(\frac{H}{\hbar\omega}\right)^n \exp\left(-\frac{H}{\hbar\omega}\right) \quad \text{for } \lambda = 1.$$

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$$\Psi_n = \frac{1}{\sqrt{2\pi\hbar}n!}\left(\frac{H}{\hbar\omega}\right)^n \exp\left(-\frac{H}{\hbar\omega}\right) \quad \text{for } \lambda = 1.$$

Ψ_n converges to a classical pure state $(x = 0, p = 0)$ in the limit $\hbar \to 0^+$:

$$\lim_{\hbar \to 0^+} \rho_n(x, p) = \delta(x)\delta(p),$$

where $\rho_n = \frac{1}{\sqrt{2\pi\hbar}}\Psi_n$.

Coherent states of the harmonic oscillator

Coherent states of the harmonic oscillator are functions $\Psi_z \in \mathcal{H}$ which satisfy the following $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$a_L \star_{\sigma,\alpha,\beta} \Psi_z = z\Psi_z,$$
$$\bar{a}_R \star_{\sigma,\alpha,\beta} \Psi_z = z^*\Psi_z,$$

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where $z \in \mathbb{C}$.

Above system of equations for $z = (\omega x_0 + ip_0)/\sqrt{2\hbar\omega}$ is equivalent to the following system of differential equations

$$(\omega(x - x_0) + i(p - p_0))\Psi_z + \hbar((\bar{\sigma} + \omega\alpha)\partial_x + i(\sigma\omega + \beta)\partial_p)\Psi_z = 0,$$
$$(\omega(x - x_0) - i(p - p_0))\Psi_z + \hbar((\sigma + \omega\alpha)\partial_x - i(\bar{\sigma}\omega + \beta)\partial_p)\Psi_z = 0.$$

The solution to the previous system of differential equations for $\alpha = \beta = 0$ reads

$$\Psi_z(x,p) = \frac{1}{\sqrt{\pi\hbar\omega(\bar{\sigma}^2+\sigma^2)}} \exp\left(-\frac{\omega^2(x-x_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \exp\left(-\frac{(p-p_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \cdot \exp\left(i\frac{2(2\sigma-1)\omega(x-x_0)(p-p_0)}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right).$$

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A quantum distribution function induced by Ψ_z is then given by

$$\begin{aligned}\rho(x,p) &= \frac{1}{\sqrt{2\pi\hbar}}\Psi_z(x,p) \\ &= \frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2+\sigma^2)}} \exp\left(-\frac{\omega^2(x-x_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \exp\left(-\frac{(p-p_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \\ &\quad \cdot \exp\left(i\frac{2(2\sigma-1)\omega(x-x_0)(p-p_0)}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right).\end{aligned}$$

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Note, that the expectation values of position and momentum in a coherent state Ψ_z are equal

$$\langle x \rangle_{\Psi_z} = x_0, \quad \langle p \rangle_{\Psi_z} = p_0.$$

To find the time evolution of the coherent states it is necessary to solve the time evolution equation

$$i\hbar\frac{\partial\rho}{\partial t} - [H,\rho] = 0,$$

where $H(x,p) = \frac{1}{2}(\omega^2 x^2 + p^2)$.

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Above equation is equivalent to the following equation

$$\frac{\partial\rho}{\partial t} - \omega^2 x\frac{\partial\rho}{\partial p} + p\frac{\partial\rho}{\partial x} - i\hbar\omega^2\frac{1}{2}(2\sigma - 1)\frac{\partial^2\rho}{\partial p^2} + i\hbar\frac{1}{2}(2\sigma - 1)\frac{\partial^2\rho}{\partial x^2} = 0.$$

The solution of the previous equation initially in a coherent state reads

$$\rho(x,p,t) = \frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2+\sigma^2)}} \exp\left(-\frac{\omega^2(x-\bar{x}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \exp\left(-\frac{(p-\bar{p}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)$$
$$\cdot \exp\left(i\frac{2(2\sigma-1)\omega(x-\bar{x}(t))(p-\bar{p}(t))}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right),$$

where

$$\bar{x}(t) = x_0\cos\omega t + \frac{p_0}{\omega}\sin\omega t,$$
$$\bar{p}(t) = -\omega x_0\sin\omega t + p_0\cos\omega t,$$

are the expectation values of position and momentum.

The solution of the previous equation initially in a coherent state reads

$$\rho(x,p,t) = \frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2+\sigma^2)}} \exp\left(-\frac{\omega^2(x-\bar{x}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right) \exp\left(-\frac{(p-\bar{p}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)$$
$$\cdot \exp\left(i\frac{2(2\sigma-1)\omega(x-\bar{x}(t))(p-\bar{p}(t))}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right),$$

where

$$\bar{x}(t) = x_0 \cos\omega t + \frac{p_0}{\omega}\sin\omega t,$$
$$\bar{p}(t) = -\omega x_0 \sin\omega t + p_0 \cos\omega t,$$

are the expectation values of position and momentum.

Coherent states, in the limit $\hbar \to 0^+$, converge to the classical pure states of the harmonic oscillator $(\bar{x}(t), \bar{p}(t))$

$$\lim_{\hbar\to 0^+} \rho(x,p,t) = \delta(x-\bar{x}(t))\delta(p-\bar{p}(t)).$$

The end
of
Part 2