Krzysztof A. Meissner

Classical Electrodynamics

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FACULTY OF PHYSICS
UNIVERSITY OF WARSAW

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These notes were written for the lectures of Classical Electrodynamics at the Bachelor course at the Faculty of Physics at the University of Warsaw. The notes do not refer to the original sources but they are based mostly on J.D. Jackson "Classical Electrodynamics" John Wiley&Sons (1962) and L.D. Landau, J. Lifszyc ,,The Classical Theory of Fields" but with inclusion of many other scattered sources.

Contents

1 Maxwell Equations

1.1 Introduction

In 1863 James Clerk Maxwell has written down equations that summarized more than a hundred years of developments adding his own absolutely crucial ingredient – displacement current (the last term in 1.4 below). In the classical electrodynamics we have two fields – the electric field $\mathbf{E}(\mathbf{x})$ (vector) and the magnetic field $\mathbf{B}(\mathbf{x})$ (pseudovector). We also have matter described by a charge density $\rho(\mathbf{x})$ (scalar field) and a current $\mathbf{j}(\mathbf{x})$ (vector field).

Neglecting the historical development we start with the Maxwell equations in their full glory:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{1.4}$$

The first and the last are called source equations and the other two sourceless equations. We also have the conservation of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.5}$$

c in these equations is the defined speed of light and the constants ε_0 and μ_0 are experimentally measured but subject to one relation

$$\varepsilon_0 \mu_0 = \frac{1}{c^2} \tag{1.6}$$

The Maxwell equations and this relation show the tight bonds between the electric and magnetic fields – we will discuss it in great detail in the course.

It is crucial that the operator ∇ (in cartesian coordinates)

$$\nabla_i = \frac{\partial}{\partial x^i} \tag{1.7}$$

transforms as a (co)vector under rotations. Therefore if we have a vector field $\mathbf{E}(\mathbf{x})$ then its divergence

$$\nabla \cdot \mathbf{E} = \frac{\partial E^i}{\partial x^i} \tag{1.8}$$

is a scalar field and its rotation

$$(\nabla \times \mathbf{E})^i = \varepsilon^{ijk}\partial_j E_k \tag{1.9}$$

is a (pseudo)vector field. It follows that the equations (1.1-1.4) are true in any coordinate system.

Classical electrodynamics is just an application of these equations in the more and more complicated situations. In the course of these lectures we will start with pure electric field independent of time in presence of more and more complicated matter, then we will discuss pure magnetic field independent of time also in presence of matter and at the end we will discuss time dependent fields in configurations of increasing complexity with radiating charges as the prime example.

We now check the internal consistency of the Maxwell equations. Let us take the divergence of (1.2). The LHS is identically zero while the RHS also vanishes by virtue of (1.3). Taking divergence of (1.4) we get

$$0 = \mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2}\frac{\partial \nabla \cdot \mathbf{E}}{\partial t} \tag{1.10}$$

Then we use (1.1) to get

$$\varepsilon_0\mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2}\frac{\partial \rho}{\partial t} \tag{1.11}$$

Comparing with (1.5) we get (1.6). After writing these equations Maxwell realized the necessity of adding the displacement term in (1.4) not observed experimentally at that time and it was a crucial step in predicting the existence of electromagnetic waves.

For arbitrary vector field $\mathbf{A}$ and 3-manifold $\mathcal{M}_3$ with a 2-dimensional boundary $\partial\mathcal{M}_3$ we have the Gauss equation

$$\int_{\mathcal{M}_3} \nabla \cdot \mathbf{A}\, \mathrm{d}V = \int_{\partial\mathcal{M}_3} \mathbf{A} \cdot \mathrm{d}\sigma \tag{1.12}$$

while for a 2-manifold $\mathcal{M}_2$ with 1-dimensional boundary $\partial\mathcal{M}_2$ we have the Stokes equation

$$\int_{\mathcal{M}_2} (\nabla \times \mathbf{A}) \cdot \mathrm{d}\sigma = \oint_{\partial\mathcal{M}_2} \mathbf{A} \cdot \mathrm{d}\mathbf{s} \tag{1.13}$$

They are a generalization of the one-dimensional result that the integral is the inverse of a differential

$$\int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}\mathrm{d}x = f(b) - f(a) \tag{1.14}$$

Applying these identities to (1.1-1.4) we get the integral form of the Maxwell equations

$$\int_{\partial\mathcal{M}_3} \mathbf{E} \cdot \mathrm{d}\sigma = \frac{Q}{\varepsilon_0} \tag{1.15}$$

$$\oint_{\partial\mathcal{M}_2} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\partial \Phi_{\mathrm{B}}}{\partial t} \tag{1.16}$$

$$\int_{\partial\mathcal{M}_3} \mathbf{B} \cdot \mathrm{d}\sigma = 0 \tag{1.17}$$

$$\oint_{\partial\mathcal{M}_2} \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \mu_0\, I + \frac{1}{c^2}\frac{\partial \Phi_{\mathrm{E}}}{\partial t} \tag{1.18}$$

Q is a total charge in the 3D manifold $\partial\mathcal{M}_3$ with 2D boundary $\partial\mathcal{M}_3$. Φ_{B} (respectively Φ_{E}) is the magnetic (respectively electric) flux through 2D manifold $\mathcal{M}_2$ with 1D boundary $\partial\mathcal{M}_2$.

We also have the integral form of the charge conservation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_{\partial\mathcal{M}_3} \mathbf{j}\cdot\mathrm{d}\sigma \tag{1.19}$$

where Q is the charge inside $\mathcal{M}_3$.

Historically, these equations were very slowly discovered. The first is called the Gauss Law, the second Faraday Law of Induction, the third is the absence of magnetic charges and the fourth Oersted's Law with the Maxwell correction.

1.2 Units in electrodynamics

It is important to be careful about the dimensions of fields and the constants (we will discuss the detailed reasons as we go along). In the SI system of units, changed in 2019, (being now the only allowed system in Europe and adopted in these lectures) the defining unit is the current unit, namely amper A. From the present perspective of particle physics it is just the charge (measured in coulombs C) per second, where the charge of a proton defined as

$$e := 1.602176634\cdot 10^{-19}\ \mathrm{C} \tag{1.20}$$

The speed of light c is in the present SI unit system defined as

$$c := 299\ 792\ 458\ \mathrm{m/s} \tag{1.21}$$

The field dimensions read

$$[\mathbf{E}] = \frac{\mathrm{N}}{\mathrm{C}}, \qquad [\mathbf{B}] = \frac{\mathrm{N}}{\mathrm{A}\cdot\mathrm{m}} \tag{1.22}$$

while the dimensions of densities read

$$[\rho] = \frac{\mathrm{C}}{\mathrm{m}^3}, \qquad [\mathbf{j}] = \frac{\mathrm{A}}{\mathrm{m}^2} \tag{1.23}$$

In the present SI system of units either ε_0 or μ_0 are experimentally measured – μ_0 is (up to ten digits) equal to

$$\mu_0 \approx 4\pi\cdot 10^{-7}\frac{\mathrm{N}}{\mathrm{A}^2} \tag{1.24}$$

Then ε_0 is given by the relation (1.6) and its inverse is numerically given by

$$\frac{1}{4\pi\varepsilon_0} = \frac{\mu_0 c^2}{4\pi} \approx 8.9875517862(14)\cdot 10^9\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}^2} \tag{1.25}$$

where the number in parenthesis gives the error of the last two digits.

Although Quantum Electrodynamics is beyond the scope of this course it is tempting to quote another physical constant of fundamental importance – fine structure constant α:

$$\alpha := \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137.035999177(21)} \tag{1.26}$$

where

$$\hbar = \frac{h}{2\pi} \tag{1.27}$$

and h is the Planck constant. Since the Planck constant h is defined in the SI unit system as

$$h = 6.62607015 \cdot 10^{-34}\ \mathrm{J \cdot s} \tag{1.28}$$

the only measured quantity in α is ε_0.

2 Lorentz and gauge symmetries of electrodynamics

2.1 Lorentz and gauge symmetries of the Maxwell equations

We will now describe the symmetry that plays a fundamental role in the description of electrodynamics and subsequent passing to Quantum Electrodynamics (QED). We consider the sourceless Maxwell equations (1.3) and (1.2) and use the mathematical results described in Lecture 1.

If the manifold has vanishing second Betti number b_2 then

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \tag{2.1}$$

for some vector field $\mathbf{A}$. Then using (1.2) we get

$$\nabla \times \mathbf{E} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t} \Rightarrow \mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.2}$$

for some φ (if the manifold is simply connected, i.e. $b_1 = 0$, φ is globally defined). They are called a vector and a scalar potentials, respectively. We will discuss later on the case (magnetic monopole) when the second Betti number is nonvanishing and one cannot introduce globally defined $\mathbf{A}$.

We can see that the pair of potentials $(\varphi, \mathbf{A})$ is not uniquely defined by these equations. We can perform so called gauge transformation without changing the resulting $\mathbf{E}$ and $\mathbf{B}$:

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\alpha \\ \varphi &\rightarrow \varphi - \frac{\partial \alpha}{\partial t} \end{aligned} \tag{2.3}$$

where $\alpha = \alpha(t, \mathbf{x})$ is an arbitrary globally defined function.

Although at this stage it seems to be just a curiosity, the potentials and the presence of this symmetry is absolutely crucial in explicit realization of Lorentz symmetry inherent in the theory and in defining the lagrangian formulation of the theory as we will see below.

We can formulate classical electrodynamics in terms of 4-dimensional quantities explicitly showing Lorentz symmetry inherently present in the theory (as is well known Lorentz discovered the transformations in 1895 as the ones that leave the Maxwell equations covariant and only 10 years later Einstein developed Special Theory of Relativity providing the physical interpretation)

In this section from now on we put

$$c = \varepsilon_0 = \mu_0 = 1 \tag{2.4}$$

Let us take the pair of potentials $(\varphi, \mathbf{A})$ and try to treat them as one 4-dimensional vector A^μ, $\mu = 0, 1, 2, 3$:

$$A^\mu = (\phi, A^i) \tag{2.5}$$

Then we form a 4-dimensional vector with 2 indices

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.6}$$

We raise and lower indices by the metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{2.7}$$

It is easy to see that the gauge transformations are

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{2.8}$$

and they trivially do not change $F_{\mu\nu}$ defined in (2.6).

Using (2.1) and (2.2) we get

$$F_{0i} = -E^i, \quad F_{ij} = \epsilon_{ijk} B^k \tag{2.9}$$

or in contravariant components

$$F^{0i} = E^i, \quad F^{ij} = \epsilon_{ijk} B^k \tag{2.10}$$

In the matrix notation

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2.11}$$

The sourceless Maxwell equations are satisfied automatically (the very idea of introducing potentials served exactly this purpose) and they follow from the identity

$$\partial_\mu F_{\rho\sigma} + \partial_\sigma F_{\mu\rho} + \partial_\rho F_{\sigma\mu} = 0 \tag{2.12}$$

where we use $(0ij)$ to recover the equation (1.2) and (ijk) to recover (1.3).

To write the other pair of Maxwell equations in the 4-dimensional form we make a 4-vector out of sources ρ and $\mathbf{j}$:

$$j^\mu = (\rho, j^i) \tag{2.13}$$

Then we can write the source equations as

$$\partial_\mu F^{\mu\nu} = -j^\nu \tag{2.14}$$

For $\nu = 0$ we have

$$\partial_i F^{i0} = -\partial_i E^i = -j^0 = -\rho \tag{2.15}$$

i.e. eq. (1.1) while for $\nu = i$ we have

$$\partial_0 F^{0i} + \partial_j F^{ji} = \partial_t E^i + \epsilon_{jik} \partial_j B^k = -j^i \tag{2.16}$$

i.e. eq. (1.4). The 4-dimensional Lorentz symmetry is in this formulation explicit since we use only 4-dimensional scalars, vectors and tensors.

2.2 Maxwell action and the energy-momentum tensor

We have now gathered all necessary ingredients to write down the action for the electric and magnetic fields coupled to external sources in the Lorentz and gauge invariant way:

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right) \tag{2.17}$$

from which, by differentiating with respect to A^ν we recover (2.14).

The equation of motion (2.14) is consistent only if the continuity equation is satisfied

$$\partial_\mu j^\mu = 0 \tag{2.18}$$

The same equation follows when we impose the condition of invariance of (2.17) under gauge transformations (2.8):

$$S' = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu (A_\mu + \partial_\mu \alpha) \right) = S - \int \mathrm{d}^4x \, (\alpha \partial_\mu j^\mu) \tag{2.19}$$

If the trajectories of charges (j_μ) are not fixed i.e. they are not external but react to the electric magnetic fields then one has to include the action for these fields as well – we will discuss it in the following lectures.

The action (2.17) for the pure electromagnetic field has symmetries of time and space translations so we can use the Noether procedure to arrive at the (symmetric) energy momentum tensor for the electromagnetic field:

$$T^{\mu\nu}_{\mathrm{em}} = F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{2.20}$$

This form requires an explanation. If we apply the usual Noether procedure we end up with a non-symmetric energy-momentum tensor. Such a form leads to some problems in the interpretation and people found a way to correct for this by adding a total derivative of the three index tensor to arrive at the symmetric form while still satisfying the basic requirement for the energy-momentum tensor that it has to be conserved in the absence of external currents. On the other hand the symmetric form is automatically obtained if we differentiate the action by the metric tensor since by definition

$$\delta S = \frac{1}{2} \int \mathrm{d}^4x \, T^{\mu\nu} \delta g_{\mu\nu} \tag{2.21}$$

In the calculation one has to be careful – it is $F_{\mu\nu}$ with lower indices that does not involve the metric tensor and

$$\delta g^{\mu\nu} = -g^{\mu\rho} \delta g_{\rho\sigma} g^{\sigma\nu}, \quad \delta \sqrt{-g} = \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} \tag{2.22}$$

The components read

$$\begin{aligned} T^{00}_{\mathrm{em}} &= \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) \\ T^{0i}_{\mathrm{em}} &= (\mathbf{E} \times \mathbf{B})^i \\ T^{ij}_{\mathrm{em}} &= -E^i E^j - B^i B^j + \frac{1}{2} \delta_{ij} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) \end{aligned} \tag{2.23}$$

The 00 component is an energy density of the field and the $0i$ component is the flux of energy in the direction i the so called Poynting vector. The ij components are the so called Maxwell stress tensor.

These formulae will be used in the future so we give the energy density and the flux with explicit ε_0 and μ_0:

$$\begin{aligned} T^{00}_{\text{em}} &= \frac{\varepsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\ T^{0i}_{\text{em}} &= \frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right)^i \end{aligned} \tag{2.24}$$

It is important to note that the energy-momentum tensor of the electromagnetic field satisfies

$$\eta_{\mu\nu}T^{\mu\nu}_{\text{em}} = 0 \tag{2.25}$$

what follows from the symmetry of the action under rescalings

$$x^\mu \to \mathrm{e}^{\sigma}x^\mu, \quad A^\mu \to \mathrm{e}^{-\sigma}A^\mu \tag{2.26}$$

and it is part of a larger conformal symmetry.

We can define the energy and momentum of the electromagnetic field (as a 4-vector) by the integration over the whole space

$$P^\mu_{\text{em}} = \int \mathrm{d}^3x T^{0\mu}_{\text{em}} \tag{2.27}$$

The energy and momentum defined in this way depends only on time. The derivative with respect to time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}P^\nu_{\text{em}} = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x T^{0\nu}_{\text{em}} = \int \mathrm{d}^3x \partial_\mu T^{\mu\nu}_{\text{em}} \tag{2.28}$$

where we used the fact that the spatial divergence integrated over the whole space vanishes. We calculate the divergence of $T^{\mu\nu}_{\text{em}}$:

$$\partial_\mu T^{\mu\nu}_{\text{em}} = \left(\partial_\mu F^{\mu\rho}\right)F^\nu{}_\rho + F^{\mu\rho}\left(\partial_\mu F^\nu{}_\rho\right) - \frac{1}{4}\partial^\nu\left(F_{\rho\sigma}F^{\rho\sigma}\right) \tag{2.29}$$

We use (2.12) to calculate

$$F^{\mu\rho}\partial_\mu F^\nu{}_\rho = F^{\mu\rho}\left(-\partial_\rho F_\mu{}^\nu - \partial^\nu F_{\rho\mu}\right) \tag{2.30}$$

so that

$$F^{\mu\rho}\partial_\mu F^\nu{}_\rho = \frac{1}{4}\partial_\nu\left(F_{\mu\rho}F^{\mu\rho}\right) \tag{2.31}$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}P^\nu_{\text{em}} = \int \mathrm{d}^3x \partial_\mu T^{\mu\nu}_{\text{em}} = \int \mathrm{d}^3x j_\mu F^{\mu\nu} \tag{2.32}$$

so the momentum of the field is conserved in the absence of external currents. We will see in the next lecture that the loss (gain) in the energy and momentum of the field is exactly equal to the gain (loss) of the charged particle moving in the field.

There are two Lorentz invariants that can be formed out of $F^{\mu\nu}$:

$$\begin{aligned} I_1 &= F_{\mu\nu}F^{\mu\nu} = -2(\mathbf{E}^2 - \mathbf{B}^2) \\ I_2 &= \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -2\mathbf{E}\cdot\mathbf{B} \end{aligned} \tag{2.33}$$

A special solution corresponds to both invariants, I_1 and I_2, vanishing - it is an electromagnetic wave in vacuum.

2.3 Lorentz transformation properties of the electric and magnetic fields

The transformation properties of the electric and magnetic fields under Lorentz transformations can be read off from the transformation properties of the tensor $F^{\mu\nu}$ i.e.

$$F^{\rho\sigma\prime} = \Lambda^{\rho}{}_{\alpha}F^{\alpha\beta}\Lambda_{\beta}{}^{\sigma} \tag{2.34}$$

where the Lorentz transformations satisfy

$$\Lambda^{\mu}{}_{\rho}\eta^{\rho\nu}\Lambda_{\nu}{}^{\rho} = \eta^{\mu\rho}, \quad \Lambda\eta\Lambda^{T} = \eta \tag{2.35}$$

Explicitly, we have the transformation law in the direction of x axis ($\gamma = 1/\sqrt{1-v^2}$)

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}' = \tag{2.36}$$
$$\begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we can read off the transformation laws for parallel and transverse fields (we reintroduce c for future applications)

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2) - \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}(\mathbf{v}\cdot\mathbf{B}) \end{aligned} \tag{2.37}$$

or

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}) \\ \mathbf{B}'_{\perp} &= \gamma(\mathbf{B}_{\perp} - \mathbf{v}\times\mathbf{E}_{\perp}/c^2) \end{aligned} \tag{2.38}$$

As can be easily seen

$$\begin{aligned}\mathbf{E}'^2 - c^2\mathbf{B}'^2 &= \mathbf{E}'^2_{\parallel} + \mathbf{E}'^2_{\perp} - c^2\mathbf{B}'^2_{\parallel} - c^2\mathbf{B}'^2_{\perp} = \\ &\mathbf{E}^2_{\parallel} + \gamma^2(\mathbf{E}^2_{\perp} + v^2\mathbf{B}^2_{\perp}) - c^2\mathbf{B}^2_{\parallel} - c^2\gamma^2(\mathbf{B}_{\perp} + v^2\mathbf{E}^2_{\perp}/c^4) = \mathbf{E}^2_{\parallel} + \mathbf{E}^2_{\perp} - c^2\mathbf{B}^2_{\parallel} - c^2\mathbf{B}^2_{\perp}\end{aligned} \tag{2.39}$$

Therefore if $E > B$ we can find such a Lorentz transformation that in the new coordinate frame we have purely electric field $B' = 0$. Analogously, if $B > E$ we can find such a Lorentz transformation that in the new coordinate frame we have purely magnetic field $E' = 0$.

3 Charged particles in the electric and magnetic fields

In this lecture we discuss the behavior of charged particles in the external electric and magnetic fields in the fully relativistic description.

We start with the description of charged particles with mass m and charge q in the external 4-potential A^μ. The particle's position and internal time are a 4-vector x^μ and the affine parameter along the trajectory is denoted by τ. Then we write the action that is invariant under Lorentz transformations and also independent of the parametrization $\tau \to f(\tau)$:

$$S = \int \mathrm{d}\tau \left(-m\sqrt{-\dot{x}^\mu \dot{x}_\mu} - q\dot{x}^\mu A_\mu(x^\rho) \right) \tag{3.1}$$

where dot denotes differentiation with respect to τ. We make here an assumption that charges are independent of Lorentz transformations – if they weren't anything composed of positive and negative charges moving with different velocities would exhibit fluctuations of charge and that would be easily detectable.

The equations of motion for x^μ read

$$m\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} = -q\dot{x}_\nu F^{\nu\mu} \tag{3.2}$$

Recall that

$$F^{0i} = E^i, \quad F^{ij} = \varepsilon_{ijk} B^k \tag{3.3}$$

If we now choose the parametrization $\tau = x^0 = t$ then we get for the 0-component

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q\mathbf{v} \cdot \mathbf{E} \tag{3.4}$$

while for the spatial components we get the Lorentz force

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{3.5}$$

where the energy and momentum of the particle form a 4-vector

$$p^\mu = (E, \mathbf{p}) = \left(\frac{m}{\sqrt{1 - v^2/c^2}}, \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) \tag{3.6}$$

To compare these formulae with the densities used for the electric and magnetic fields we introduce here charge density and current density using delta functions.

$$\rho(\mathbf{x}) = q\delta(\mathbf{x} - \mathbf{x}_0(t)), \quad \mathbf{j}(\mathbf{x}) = q\dot{\mathbf{x}}_0(t)\delta(\mathbf{x} - \mathbf{x}_0(t)) \tag{3.7}$$

and they form a 4-vector. Then the equation of motion (3.2) can be written as

$$\frac{\mathrm{d}P^\nu}{\mathrm{d}t} = -\int \mathrm{d}^3x j_\mu F^{\mu\nu} \tag{3.8}$$

If we add the 4-momentum vectors of the particle and of the field then from (2.32) and (3.8) this sum is a constant of motion (as it should):

$$\frac{\mathrm{d}(P^\nu + P^\nu_{\mathrm{em}})}{\mathrm{d}t} = 0 \tag{3.9}$$

or in terms of the energy-momentum tensors

$$\partial_\mu \left(T^{\mu\nu}_{\mathrm{mech}} + T^{\mu\nu}_{\mathrm{em}}\right) = 0 \tag{3.10}$$

3.1 Trajectories in constant fields

We discuss three cases: a charged particle in a constant electric, constant magnetic and constant orthogonal electric and magnetic fields..

For the constant electric field in the, say, x direction we have from (3.5)

$$p_x = qEt \Rightarrow v_x = \frac{qEt}{\sqrt{m^2 + (qEt)^2/c^2}} \tag{3.11}$$

For large times velocity approaches c as it should.

For a constant magnetic field of value B in, say, z direction equations (3.5) read in components

$$\begin{aligned} \frac{\mathrm{d}p_x}{\mathrm{d}t} &= qBv_y \\ \frac{\mathrm{d}p_y}{\mathrm{d}t} &= -qBv_x \\ \frac{\mathrm{d}p_z}{\mathrm{d}t} &= 0 \end{aligned} \tag{3.12}$$

Since the particle moves uniformly in the z direction we may assume $p_z = 0$ and add uniform motion if needed. The total momentum p is a constant of motion (what is seen in (3.4) or by multiplying the first equation by p_x and the second by p_y and adding) and so is the total velocity v and therefore we can rewrite these equations as

$$\begin{aligned} \frac{\mathrm{d}v_x}{\mathrm{d}t} &= -\frac{qB\sqrt{1 - v^2/c^2}}{m} v_y \\ \frac{\mathrm{d}v_x}{\mathrm{d}t} &= \frac{qB\sqrt{1 - v^2/c^2}}{m} v_x \end{aligned} \tag{3.13}$$

It is easy to recognize these equations as the equations for circular oscillator with the solution

$$\begin{aligned} v_x(t) &= v\cos(\omega t) \\ v_y(t) &= v\sin(\omega t) \end{aligned} \tag{3.14}$$

where

$$\omega = \frac{qB\sqrt{1-v^2/c^2}}{m} \tag{3.15}$$

We see that in the nonrelativistic case the frequency ω does not depend on the velocity and it is the basis for construction of cyclotrons. If the velocity starts to be relativistic then one has to adjust the frequency accordingly and it is necessary in synchrotrons (where it is usually done by the adjustment of the magnetic field).

One can integrate once more to get the trajectory

$$\begin{aligned} x(t) &= R\sin(\omega t) \\ y(t) &= -R\cos(\omega t) \end{aligned} \tag{3.16}$$

where the relativistic result for the radius is formally the same as the nonrelativistic one

$$R = \frac{p}{qB} \tag{3.17}$$

For example for LHC in Geneva where $R \approx 2.8$ km (only 17.6 out of 26.7 km of the tunnel are curved and filled with dipole magnets) to keep protons on their orbit for $p = 7$ TeV$/c$ the required magnetic field is 8.3 T.

We consider now a particle moving in both fields $\mathbf{E}$ and $\mathbf{B}$ perpendicular to each other (i.e. $\mathbf{E}\cdot\mathbf{B} = 0$ which is a Lorentz invariant statement). We assume $E_x \neq 0$, $B_y \neq 0$ and $E_x < cB_y$. Then we can ask whether there exists a coordinate system in which the electric field vanishes. Using the relations derived in lecture 3

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}(\mathbf{v}\cdot\mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2) - \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}(\mathbf{v}\cdot\mathbf{B}) \end{aligned} \tag{3.18}$$

we have $\mathbf{E}' = 0$ when (using $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$)

$$\mathbf{v} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{3.19}$$

and then

$$\mathbf{E}' = 0, \quad \mathbf{B}' = \sqrt{1 - E^2/(B^2c^2)}\mathbf{B} \tag{3.20}$$

This equation can be easily inferred from the fact that there is a Lorentz invariant

$$\mathbf{E}^2 - c^2\mathbf{B}^2 = \mathbf{E}'^2 - c^2\mathbf{B}'^2 \tag{3.21}$$

The velocity $\mathbf{v}$ is called a drift. In the new coordinate system there is only a magnetic field so the particle moves along a circle around $\mathbf{B}'$. If $E_x > cB_y$ then analogously there exists a reference frame where there exists only the electric field

$$\mathbf{B}' = 0, \quad \mathbf{E}' = \sqrt{1 - B^2c^2/E^2}\mathbf{E} \tag{3.22}$$

3.2 Free electron laser trajectories

As the last example we consider an ultrarelativistic particle moving in the direction z in a spatially varying magnetic field in the direction y (and z but it will be unimportant in the following) and satisfying sourceless Maxwell equations:

$$B_y = B_0 \sin(k_u z) \cosh(k_u y), \quad B_z = B_0 \cos(k_u z) \sinh(k_u y) \tag{3.23}$$

where

$$k_u = \frac{2\pi}{\lambda_u} \tag{3.24}$$

Such a field satisfies $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ as it should.

We discuss the case where the initial conditions are $y_0 = 0$, $v_y(0) = 0$ so $y = 0$ always (i.e. $B_z = 0$ along the trajectory). We assume that $v_z \sim c$ and $v_x \ll c$. Then

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = q v_z B_y = q v_z B_0 \sin(k_u z) \tag{3.25}$$

Writing p_x as $m\gamma v_x$ (where γ is constant since the magnetic field does not change the energy) we get

$$m\gamma \frac{\mathrm{d}v_x}{\mathrm{d}z} v_z = q v_z B_0 \sin(k_u z) \tag{3.26}$$

so that

$$v_x = -\frac{q B_0}{m\gamma k_u} \cos(k_u z) \tag{3.27}$$

Therefore

$$x(z) \approx -\frac{q B_0}{m\gamma c k_u^2} \sin(k_u z) \tag{3.28}$$

It can be written as

$$x(z) \approx -\frac{\psi}{k_u} \sin(k_u z) \tag{3.29}$$

where ψ is the maximal angle of the trajectory with respect to the z axis and ψ satisfies

$$\psi\gamma := K = \frac{q B_0}{k_u m c} \tag{3.30}$$

We will use these formulae when discussing free electron lasers in one of the last lectures.

4 Electrostatics

4.1 Electric fields from a distribution of charges

In this part we will discuss solutions of the Maxwell equations with vanishing magnetic field but with nontrivial boundary conditions for either the electric field or the electric potential.

We start with the Maxwell equations and assume that $\mathbf{B} = 0$ and all the fields and charge densities are time independent. We are left with two relevant equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{4.1}$$

and

$$\nabla \times \mathbf{E} = 0 \tag{4.2}$$

The second equation allows us to write

$$\mathbf{E} = -\nabla \varphi \tag{4.3}$$

and then the first can be written in the form of Poisson equation

$$\Delta \varphi = -\frac{\rho}{\varepsilon_0} \tag{4.4}$$

It is easy to check that the Coulomb potential

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \tag{4.5}$$

satisfies

$$\Delta \varphi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{x}) \tag{4.6}$$

Another example is an infinite wire charged with line density λ. Then at distance ρ the electric field is given by

$$\mathbf{E} = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{\lambda \rho \mathbf{e}_\rho}{4\pi\varepsilon_0 (x^2+\rho^2)^{3/2}} = \frac{\lambda \mathbf{e}_\rho}{2\pi\varepsilon_0 \rho} \Rightarrow \varphi(\rho) = -\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{\rho}{\rho_0}\right) \tag{4.7}$$

what also directly follows from the Gauss' law.

If we have a charge density $\rho(\mathbf{x}')$ confined to some region $|\mathbf{x}'| < R$ then the potential at some distant $\mathbf{x}$, $|\mathbf{x}| \gg R$ can be expanded in inverse powers of R by means of the following expansion (using the generating function for the Legendre polynomials)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|\mathbf{x}|}\frac{1}{\sqrt{1+a^2-2a\cos\gamma}} = \frac{1}{|\mathbf{x}|}\sum_{l=0}^{\infty} P_l(\cos\gamma)a^l \tag{4.8}$$

where $a = |\mathbf{x}'|/|\mathbf{x}|$ and γ is an angle between $\mathbf{x}$ and $\mathbf{x}'$. Now we use the representation of P_l in terms of spherical harmonics:

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} Y^{\star}_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \tag{4.9}$$

to arrive at the multipole expansion formula for the potential at $\mathbf{x}$

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \tag{4.10}$$

where

$$Q_{lm} = \int \mathrm{d}^3x'\,\rho(\mathbf{x}')r'^l\,Y^{\star}_{lm}(\theta';\phi') \tag{4.11}$$

The expansion in cartesian coordinates can be rewritten as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \frac{3(\mathbf{x}\cdot\mathbf{x}')^2 - r^2r'^2)}{2r^5} + \dots \tag{4.12}$$

and the potential reads

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{Q_{ij}x^ix^j}{r^5} + \dots\right) \tag{4.13}$$

where

$$\begin{aligned} q &= \int \mathrm{d}^3x'\rho(\mathbf{x}') \\ \mathbf{p} &= \int \mathrm{d}^3x'\rho(\mathbf{x}')\mathbf{r}' \\ Q_{ij} &= \frac{1}{2}\int \mathrm{d}^3x'\rho(\mathbf{x}')(3x'^ix'^j - \delta^{ij}r'^2) \end{aligned} \tag{4.14}$$

are the charge, dipole moment and quadrupole moment, respectively.

The electric field of a monopole (charge) is obviously the Coulomb field

$$\mathbf{E}_0(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}}{r^3} \tag{4.15}$$

while for a dipole it is

$$\mathbf{E}_1(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{p}}{r^5} \tag{4.16}$$

As we see it is not necessary that the charges are strictly confined to some region, it is enough that the density ρ vanishes sufficiently fast at infinity so that all the integrals are finite. On the other hand one has to be careful in using this formula for $r \to 0$ since it is singular and requires proper definition.

The Coulomb force between two charges can be justified by the field energy considerations. The energy of the electric field when two charges are present subtracted from the separate energies of two fields:

$$\Delta W = \frac{\varepsilon_0}{2} \int \mathrm{d}^3 x \left[(\mathbf{E}_1 + \mathbf{E}_2)^2 - \mathbf{E}_1^2 - \mathbf{E}_2^2 \right] = \varepsilon_0 \int \mathrm{d}^3 x \mathbf{E}_1 \cdot \mathbf{E}_2 \tag{4.17}$$

Placing one charge at the origin and the other on the z axis at distance l we get

$$\Delta W = \frac{q_1 q_2}{(4\pi)^2 \varepsilon_0} \int r^2 \mathrm{d}r \mathrm{d}\Omega \frac{\mathbf{r} \cdot (\mathbf{r} - \mathbf{l})}{r^3 (r^2 + l^2 - 2rl\cos\theta)^{3/2}} \tag{4.18}$$

It is easy to integrate if we notice that the integrand is proportional to a derivative over r of $1/(r^2 + l^2 - 2rl\cos\theta)^{1/2}$ and therefore

$$\Delta W = \frac{q_1 q_2}{(4\pi)^2 \varepsilon_0} \frac{4\pi}{l} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{l} \tag{4.19}$$

i.e. the Coulomb force. Therefore one can interpret the Coulomb force as originating from the energy of the fields.

4.2 Green's function

The solution to the Poisson equation depends not only on the charge density inside the region but also on the boundary conditions imposed on φ (or its derivative) on the boundary. In the theory of partial differential equations one introduces the Green's function $G(\mathbf{x}, \mathbf{x}')$ being the solution to the equation

$$\Delta_x G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{4.20}$$

and satisfying either Dirichlet or Neumann boundary conditions. Such a function is in general not known and we will discuss here several methods to solve (4.4) in some interesting cases of large symmetry.

If we know the Green's function we can derive an explicit formula for the potential. We start with the identity (9.20)

$$\int\limits_{\mathcal{M}_3} \nabla \cdot \mathbf{A} \, \mathrm{d}V = \int\limits_{\partial\mathcal{M}_3} \mathbf{A} \cdot \mathrm{d}\mathbf{S} \tag{4.21}$$

and apply it for

$$\mathbf{A}(\mathbf{x}') = \varphi(\mathbf{x}') \nabla' G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}') \nabla' \varphi(\mathbf{x}') \tag{4.22}$$

Using (4.4) and (4.20)

$$\varepsilon_0 \varphi(\mathbf{x}) = -\int\limits_{\mathcal{M}_3} G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\,\mathrm{d}V' + \int\limits_{\partial\mathcal{M}_3} (\varphi(\mathbf{x}')\nabla' G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\nabla'\varphi(\mathbf{x}'))\cdot\mathrm{d}\mathbf{S}' \quad (4.23)$$

If there are no charges inside a given region then the potential inside is fully determined by the potential at the boundary.

If we neglect the boundary contributions we have

$$\varphi(\mathbf{k}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{k})G(\mathbf{k}) \quad (4.24)$$

where the Fourier transforms are defined as

$$G(\mathbf{k}) = \int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'} G(\mathbf{x}') = \frac{2}{k^2} \quad (4.25)$$

if we assume that the Green's function is equal to $1/(4\pi r)$ only inside the ball of radius L then

$$G(\mathbf{k}) = \int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'} G(\mathbf{x}') = 2\frac{\sin^2(kL/2)}{k^2} \quad (4.26)$$

4.3 Electrostatics with conducting surfaces

In this part we will discuss solutions of the Maxwell equations in presence of charges and conducting surfaces. In the case of conducting surfaces the potential has to be identical along the surface since otherwise it would mean some electric field along the surface inducing electric current and the situation couldn't be independent of time. Therefore inside an empty region surrounded by a conducting surface the potential inside should be everywhere the same and equal to the potential of the surface.

4.3.1 Method of images

We start with the Poisson equation (4.4)

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0} \quad (4.27)$$

In the simplest case of no boundaries (and vanishing potential at infinity) we know that charges q_a located at $\mathbf{x}_a$ give the potential

$$\varphi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_a \frac{q_a}{|\mathbf{x}-\mathbf{x}_a|} \quad (4.28)$$

If these charges are surrounded by a surface with some prescribed potential (usually a constant one as in the presence of conducting surface) then we have to modify this solution to match the boundary condition. It is then useful to use the so called method of images. We assume that there are some charges q_b located outside the region at points

$\mathbf{x}_b$ and therefore not contributing to the RHS of (4.27). The potential inside the region being the sum of the two potentials still satisfies (4.27 with only real charges q_a included but when the charges q_b are appropriately chosen this sum can satisfy the boundary conditions imposed. Unfortunately this method works in a simple way only for very simple surfaces (like spheres), otherwise the problem does not get any simpler when formulated as the problem of virtual charges.

As an example let us consider one charge q placed at $\mathbf{x}_0 = (x_0, 0, 0)$ $(x_0 > 0)$ and a sphere of radius R and center at $(0, 0, 0)$ (the sphere surrounds the charge $R > x_0$). Then we try to place an imaginary charge q' outside the sphere so that the sum of the two potentials will vanish on the sphere (although not evident, it turns out that one imaginary charge is enough to satisfy the condition). We try to locate it at $\mathbf{x}_1 = (x_1, 0, 0)$ $(x_1 > R)$ and then the solution inside the sphere (up to a constant) is

$$4\pi\varepsilon_0\varphi(\mathbf{x})_{\text{in}} = \frac{q}{|\mathbf{x} - \mathbf{x}_0|} + \frac{q'}{|\mathbf{x} - \mathbf{x}_1|} \tag{4.29}$$

with the condition of vanishing potential on the sphere

$$0 = \frac{q}{|\mathbf{x} - \mathbf{x}_0|} + \frac{q'}{|\mathbf{x} - \mathbf{x}_1|} \quad \text{for } |\mathbf{x}| = R \tag{4.30}$$

Squaring this identity we get

$$q'^2(R^2 - 2xx_0 + x_0^2) = q^2(R^2 - 2xx_1 + x_1^2) \tag{4.31}$$

It can be satisfied for varying x only if

$$q'^2 x_0 = q^2 x_1 \tag{4.32}$$

and then

$$(x_1 - x_0)(R^2 - x_0 x_1) = 0 \tag{4.33}$$

Therefore the solution is

$$q' = -q\frac{R}{x_0}, \quad x_1 = \frac{R^2}{x_0} \tag{4.34}$$

and with such choice the (4.29) gives the potential inside the sphere.

Using this solution we can write down the Green's function inside the sphere

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') &= \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi R|\mathbf{x} - \mathbf{x}'R^2/x'^2|} = \\ &= \frac{1}{4\pi(x^2 - 2\mathbf{x}\cdot\mathbf{x}' + x'^2)^{1/2}} - \frac{1}{4\pi(x^2x'^2/R^2 - 2\mathbf{x}\cdot\mathbf{x}' + R^2)^{1/2}} \end{aligned} \tag{4.35}$$

It is easy to check that indeed $G(\mathbf{x}, \mathbf{x}') = 0$ for all $\mathbf{x}$ on the sphere.

Having found the solution inside with the vanishing potential on the sphere we have to discuss the form of the solution outside and the value of constants that eventually have to be added. If the sphere is grounded (i.e. kept at zero potential) then the solution

inside is given by 4.29) and the potential outside vanishes. If the sphere is isolated, its total charge has to vanish and the Gauss law says that since there is a charge q inside and the potential is constant on the sphere so the potential outside has to be equal to

$$4\pi\varepsilon_0\varphi_{\text{out}}(\mathbf{x}) = \frac{q}{|\mathbf{x}|} \tag{4.36}$$

Therefore to make the potential continuous on the sphere we have to add a constant to the potential (4.29) and the full solution for the isolated sphere reads

$$\begin{aligned} 4\pi\varepsilon_0\varphi(\mathbf{x}) &= \frac{q}{|\mathbf{x}-\mathbf{x}_0|} - \frac{qR}{x_0|\mathbf{x}-\mathbf{x}_0R^2/x_0^2|} + \frac{q}{R} \quad \text{inside} \\ 4\pi\varepsilon_0\varphi(\mathbf{x}) &= \frac{q}{|\mathbf{x}|} \quad \text{outside} \end{aligned} \tag{4.37}$$

For a system of conductors, each charged with a charge q_a and having a potential V_a we have (due to linearity of electrodynamics)

$$q_a = \sum_b C_{ab}V^b \tag{4.38}$$

The coefficients C_{ab} are called coefficients of capacitance. If one grounds all conductors except one (a) and this choice is fixed then one uses the term capacitance for

$$C = \frac{q_a}{V_a} \tag{4.39}$$

The energy of such a system is equal to

$$W = \frac{1}{2}\sum_{ab} C_{ab}V^aV^b \tag{4.40}$$

Another example is given by a flat surface charged with a surface density ρ (surface 1) and another one at distance d which is grounded i.e. with zero potential (surface 2). Outside the grounded surface there can be no electric field so it must have a charge density $-\rho$ to shield the charged surface. Therefore the electric field between the plates is equal to

$$E_{\text{in}} = \frac{\rho}{\varepsilon_0} \tag{4.41}$$

and outside (on both sides) it vanishes. The potential of the charged surface is equal to

$$V_1 = \frac{\rho d}{\varepsilon_0} \tag{4.42}$$

The capacitance

$$C_{11} = \frac{Q}{V} = \frac{\varepsilon_0 S}{d} \tag{4.43}$$

and

$$C_{21} = -\frac{\varepsilon_0 S}{d} \tag{4.44}$$

5 Dielectrics

In this lecture we will discuss solutions of the Maxwell equations in presence of dielectric media.

We start with the Maxwell equations and as before we assume that $\mathbf{B} = 0$ and all the fields and charge densities are time independent. We are left with two relevant equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{5.1}$$

and

$$\nabla \times \mathbf{E} = 0 \tag{5.2}$$

As before the second equation allows us to write

$$\mathbf{E} = -\nabla \phi \tag{5.3}$$

and then the first can be written in the form of Poisson equation (4.4)

$$\Delta \phi = -\frac{\rho}{\varepsilon_0} \tag{5.4}$$

In the presence of dielectric media the density ρ can be divided into parts – one so called free charges (sum of delta functions), the second dipole charges (sum of derivatives of delta functions) and so on. For discrete charges, dipoles and so on one can write

$$\rho(\mathbf{x}) = \sum_a q_a \delta(\mathbf{x} - \mathbf{x}_a) - \sum_b \mathbf{p}_b \cdot \nabla \delta(\mathbf{x} - \mathbf{x}_b) + \dots \tag{5.5}$$

where $\dots$ stand for higher derivatives of the delta functions. The reason for this expansion is clear – integrating $\rho(\mathbf{x})$ over a small region surrounding $\mathbf{x}_a$ we get q_a while integrating $\rho(\mathbf{x})(\mathbf{x} - \mathbf{x}_b)$ over a small region surrounding $\mathbf{x}_b$ we get $\mathbf{p}_b$.

Neglecting all higher derivatives (i.e. quadrupoles, octupoles and so on which is usually a very good approximation) and introducing continuous densities we write

$$\rho(\mathbf{x}) = \int \mathrm{d}V' \left(\rho_0(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') - \mathbf{P}(\mathbf{x}') \cdot \nabla_x \delta(\mathbf{x} - \mathbf{x}') \right) \tag{5.6}$$

Integrating by parts and neglecting the boundary terms we see that ρ can then be effectively described by

$$\rho(\mathbf{x}) = \rho_0(\mathbf{x}) - \nabla \cdot \mathbf{P}(\mathbf{x}) \tag{5.7}$$

Returning to equation (5.1) and rearranging we get

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_0 \tag{5.8}$$

where on the RHS we have only free charges. The quantity on the LHS is called $\mathbf{D}(\mathbf{x})$.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{5.9}$$

It should be emphasized that $\mathbf{D}(x)$ is only an auxiliary vector and on the microscopic scale we should use the electric field. We can therefore write

$$\nabla \cdot \mathbf{D} = \rho_0 \tag{5.10}$$

If we have a polarized medium confined by a boundary then besides a volume density of bound charges given by

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{5.11}$$

we have also a surface bound charge density (coming from the integration by parts)

$$\sigma_b = \mathbf{P} \cdot \mathbf{n} \tag{5.12}$$

The bound charges should be conserved. The conservation law reads

$$\frac{\partial \rho_b}{\partial t} + \nabla \cdot \mathbf{j}_b = 0 \tag{5.13}$$

Using (5.11) we get

$$\mathbf{j}_b = \frac{\partial \mathbf{P}}{\partial t} \tag{5.14}$$

We can therefore write down the source Maxwell equations with the bound charges present

$$\nabla \cdot \mathbf{D} = \rho_0 \tag{5.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \right) \tag{5.16}$$

where neither ρ_0 nor $\mathbf{j}$ contain bound (polarization) charges – they are fully contained in $\mathbf{D}$.

The question arises what do we know about the polarization field $\mathbf{P}(\mathbf{x})$ in the real media? The most frequent case is that the polarization is induced when the external electric field is applied and then in the lowest approximation it is proportional to the external electric field

$$\mathbf{P}(\mathbf{x}) = (\varepsilon - 1)\varepsilon_0 \mathbf{E}(\mathbf{x}) \tag{5.17}$$

with some constant ε (that varies with frequency for fields dependent on time but in our case it is just a constant). The dimension of $\mathbf{P}$ is $[\mathrm{C/m^2}]$.

5.1 Molecular polarizability

There are two models of the molecular polarizability – either the electric field induces polarization in the initially unpolarized molecules or it orders the directions of existing electric dipoles.

In the first case we can use a classical picture of an electron bound to an atom with some dissipation and the external force (we temporarily assume time dependence of the electric field):

$$m\ddot{z} + m\gamma\dot{z} + m\omega_0^2 z = eE\mathrm{e}^{-\mathrm{i}\omega t} \tag{5.18}$$

with the solution

$$z(t) = \frac{eE\mathrm{e}^{-\mathrm{i}\omega t}}{m(\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega)} + \text{eigenmodes} \tag{5.19}$$

Hence the polarizability is for low frequencies independent of frequency

$$\mathbf{p} = \frac{e^2}{m\omega_0^2}\mathbf{E} \tag{5.20}$$

but for larger frequencies, close to the eigenfrequencies of the system, the dependence is very strong and the real part changes sign when crossing ω_0.

The result can be checked by the statistical mechanics approach. Hamiltonian for the electron in the constant field E is given by

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega_0^2}{2}\mathbf{x}^2 - eEz \tag{5.21}$$

Introducing

$$\mathbf{x}' = \mathbf{x} - eE\mathbf{e}_z/(m\omega_0^2) \tag{5.22}$$

we get

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega_0^2}{2}\mathbf{x}'^2 - \frac{e^2E^2}{2m\omega_0^2} \tag{5.23}$$

Therefore average dipole moment is given by ($\beta = 1/(kT)$)

$$\langle\mathbf{p}\rangle = \frac{\int \mathrm{d}^3p\mathrm{d}^3x(ez\mathbf{e}_z)\mathrm{e}^{-\beta H}}{\int \mathrm{d}^3p\mathrm{d}^3x\mathrm{e}^{-\beta H}} = \frac{\int \mathrm{d}^3p\mathrm{d}^3x\, e(z' + eE/(m\omega_0^2))\mathbf{e}_z)\mathrm{e}^{-\beta H}}{\int \mathrm{d}^3p\mathrm{d}^3x\mathrm{e}^{-\beta H}} = \frac{e^2}{m\omega_0^2}\mathbf{E} \tag{5.24}$$

i.e. the thermal motion of particles does not influence the polarizability.

To account for different reactions of different electrons in an atom one introduces "oscillator strengths" f_i and then

$$\mathbf{p} = \sum \frac{f_i e^2/m}{\omega_i^2 - \omega^2 - \mathrm{i}\gamma_i\omega}\mathbf{E} \tag{5.25}$$

These oscillator strengths can be calculated in the framework of quantum mechanics.

We can estimate the orders of magnitude involved. If we have a density N then

$$\varepsilon - 1 = \frac{Ne^2}{\varepsilon_0 m\omega_0^2} \tag{5.26}$$

For gases in normal conditions we have $N_A = 6.023 \cdot 10^{23}$ molecules in 22.4 liters so $N \sim 2.7 \cdot 10^{25}/\mathrm{m}^3$. Binding energy of electrons in atoms is of the order of the visible light 3 eV (0.3 μ) i.e. $\omega_0 \sim 6 \cdot 10^{15}$ 1/s. Therefore

$$\varepsilon - 1 \sim 0.003 \tag{5.27}$$

and for air it is 0.00054, for methyl alcohol 0.0057 and for helium 0.000068 (in this case the lowest lying level is about 21 eV so it gives a factor of 50 less than the estimate above). For dense media N is roughly 10^3 times bigger and $\varepsilon - 1 \sim O(1)$. Obviously this model is very approximate since a molecule has many eigenfrequencies and ε changes when we approach any of them.

In the second case when the particles have their own dipole moment p_0 (for example water or hydrochloric acid HCl) we are forced to use statistical mechanics since otherwise we would get an obviously wrong result that for arbitrarily small field all dipoles point in the direction of the field. To describe the system statistically we start with the hamiltonian

$$H = H_0 - \mathbf{p}_0 \cdot \mathbf{E} \tag{5.28}$$

We assume here that the electric field is small $\beta p_0 E \ll 1$ i.e. for $T = 300$ K ($k = 1.38065 \cdot 10^{-23}$ J/K)

$$E \ll \frac{kT}{p_0} \sim \frac{4 \cdot 10^{-21}}{2 \cdot 1.6 \cdot 10^{-19} \cdot 10^{-10}} \mathrm{V/m} = 1.2 \cdot 10^8 \mathrm{V/m} \tag{5.29}$$

where we estimated the atomic dipole moment as 3 D (debye)

$$p_0 \sim 3D = 3 \cdot 0.208 \cdot 1.6 \cdot 10^{-19} \cdot 10^{-10} \mathrm{Cm} \sim 10^{-29} \mathrm{Cm} \tag{5.30}$$

(the highest dipole moment known in chemistry is 11.5 D). Then the average dipole moment per molecule is given by

$$\langle \mathbf{p} \rangle = \frac{\int \mathrm{d}\theta \sin\theta \, p_0 \cos\theta \mathbf{e}_z \mathrm{e}^{\beta p_0 E \cos\theta}}{\int \mathrm{d}\theta \sin\theta \, \mathrm{e}^{\beta p_0 E \cos\theta}} \sim p_0 \mathbf{e}_z \frac{\int \mathrm{d}w \, w(1 + \beta p_0 E w)}{\int \mathrm{d}w} \sim \frac{1}{3} \beta p_0^2 \mathbf{E} \tag{5.31}$$

Therefore in general

$$\varepsilon - 1 = \frac{Ne^2}{\varepsilon_0 m \omega_0^2} + \frac{N p_0^2}{3 \varepsilon_0 kT} \tag{5.32}$$

The temperature dependence allows to distinguish both contributions (i.e. from induced and internal polarizability). For room temperatures we can estimate $p_0 \sim 10^{-29}$ Cm and $kT \sim 1.38 \cdot 10^{-23} \cdot 300 \sim 4 \cdot 10^{-21}$ J. Then the the second contribution gives

$$\frac{N p_0^2}{3 \varepsilon_0 kT} \sim \frac{2.7 \cdot 10^{25} \cdot 10^{-58}}{3 \cdot 0.88 \cdot 10^{-11} \cdot 4 \cdot 10^{-21}} \sim 0.03 \tag{5.33}$$

so the contribution for gases from dipole orientation is slightly bigger than from the induced dipole moment.

5.2 Clausius-Mossotti equation

We start from the observation that an average electric field inside a given region is related to a total dipole moment inside the region. The relation for an arbitrary region is in general complicated and involves also higher moments of the distribution (quadrupoles and so on) but it turns out that if the region is a ball the calculation is easy and the relation involves only the dipole moment.

For a system of charges located at $\mathbf{r}_a$ we can write

$$\int\limits_{B(0,R)} \mathrm{d}V\, \mathbf{E} = -\int\limits_{B(0,R)} \mathrm{d}V\, \nabla\varphi(\mathbf{x}) = -\sum_a \int\limits_{S(0,R)} \mathrm{d}\mathbf{S}\, \frac{q_a}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}_a|} \tag{5.34}$$

where the field $\mathbf{E}$ is from the charges enclosed in the ball only. We now calculate the z component this equation. We use the expansion of $1/|\mathbf{r}-\mathbf{r}_a|$ in Legendre polynomials and we notice that $\mathrm{d}\mathbf{S}$ contribution is proportional to $\cos\theta$ i.e. the first Legendre polynomial. Therefore, because of orthogonality of the Legendre polynomials we keep only $l = 1$ term in the expansion (the integration over the $l = 0$ term vanishes) and we get

$$\int\limits_{B(0,R)} \mathrm{d}V\, E_z = -\sum_a \int \sin\theta \mathrm{d}\theta \mathrm{d}\phi\, \cos^2\theta \frac{q_a z_a}{4\pi\varepsilon_0} \tag{5.35}$$

where one $\cos(\theta)$ comes from projecting $\mathrm{d}\mathbf{S}$ on the z axis and the other from $\mathbf{r}_a \cdot \mathbf{r}$ in the expansion of the denominator $|\mathbf{r} - \mathbf{r}_a|$. Dividing both sides by the volume of the ball and using the fact that the z axis was arbitrary we get

$$\langle \mathbf{E} \rangle = -\frac{1}{3\varepsilon_0}\mathbf{P} \tag{5.36}$$

where $\mathbf{P}$ is the polarization density.

The dielectric constant describes the coefficient of proportionality between the electric field and the polarization – at the microscopic level it is given by

$$\mathbf{P} = (\varepsilon - 1)\varepsilon_0 \left(\mathbf{E} + \mathbf{E}_{\mathrm{loc}} - \frac{1}{3\varepsilon_0}\mathbf{P}\right) \tag{5.37}$$

where $\mathbf{E}_{\mathrm{loc}}$ is a contribution from microscopically close charges – there are arguments that for high symmetry cases it vanishes. Solving for $\mathbf{P}$ and dividing by the density of molecules N we get a molecular polarizability

$$\gamma = \frac{P}{NE} = \frac{3\varepsilon_0(\varepsilon - 1)}{N(\varepsilon + 2)} \tag{5.38}$$

This equation is called the Clausius-Mossotti equation.

In the form

$$\frac{n^2 - 1}{n^2 + 2} = \frac{N\gamma}{3\varepsilon_0} \tag{5.39}$$

for the refractive indices n, polarizability γ and a number of particles in a volume N it was discovered independently by Hendrik Lorentz and Ludvig Lorenz and is known as the Lorentz-Lorenz equation.

5.3 Energy of fields in the dielectric media

In the empty space the energy of the electric field is equal to

$$W = \frac{\varepsilon_0}{2} \int \mathrm{d}^3 x \mathbf{E}^2 \tag{5.40}$$

or in terms of charges and potentials (we assume that the potentials are induced by charges hence $\frac{1}{2}$) as

$$W = \frac{1}{2} \int \mathrm{d}^3 x \rho(\mathbf{x}) \varphi(\mathbf{x}) = \frac{1}{2} \int \mathrm{d}^3 x \mathbf{E}(\mathbf{x}) \mathbf{D}(\mathbf{x}) \tag{5.41}$$

The difference between initial and final situations when we put some dielectric media but keep free charges in the original positions is given by

$$\begin{aligned} \delta W &= \frac{1}{2} \int \mathrm{d}^3 x (\mathbf{E} \cdot \mathbf{D} - \mathbf{E}_0 \cdot \mathbf{D}_0) = \\ &= \frac{1}{2} \int \mathrm{d}^3 x (\mathbf{E} \cdot (\mathbf{D} - \mathbf{D}_0) + \mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 (\mathbf{D}_0 - \mathbf{D}) - \mathbf{E}_0 \cdot \mathbf{D}) \\ &= \frac{1}{2} \int \mathrm{d}^3 x (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}) \end{aligned} \tag{5.42}$$

since

$$\int \mathrm{d}^3 x \mathbf{E} \cdot (\mathbf{D} - \mathbf{D}_0) = \int \mathrm{d}^3 x \varphi \cdot \nabla \cdot (\mathbf{D} - \mathbf{D}_0) = 0 \tag{5.43}$$

by the assumption that we do not change the free charges (and similarly for the other term). Therefore for

$$\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{5.44}$$

we have

$$\delta W = -\frac{1}{2} \int \mathrm{d}^3 x \mathbf{P}(\mathbf{x}) \mathbf{E}_0(\mathbf{x}) \tag{5.45}$$

Hence the dielectric is pulled to the regions where the electric field is bigger.

It is important to note that in the dielectric medium of dielectric constant ε the electric field with given charges is ε times smaller than in the vacuum and therefore a capacitance of a given system is ε times bigger.

It is important to note that the method of images is a very useful tool to satisfy the boundary conditions in solving the problems with media with different dielectric constants.

5.4 Example

To illustrate the application of boundary conditions we now discuss the case of a dielectric ball of dielectric constant ε and radius R in a constant electric field $\mathbf{E}$ (to be definite

in the z direction) in vacuum. We use the expansion (9.60) separately inside the ball and outside the ball

$$\begin{aligned} \varphi_{\text{in}}(\mathbf{x}) &= \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ \varphi_{\text{out}}(\mathbf{x}) &= \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) - Er\cos(\theta) \end{aligned} \tag{5.46}$$

where we have used the condition that the potential is finite at $r \to 0$ and that at infinity we have constant electric field in the z direction.

The equations

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot (\varepsilon \mathbf{E}) &= 0 \end{aligned} \tag{5.47}$$

show that on the boundary we should impose continuity on the tangent electric field and perpendicular $\varepsilon\mathbf{E}$. Therefore we write

$$\begin{aligned} \frac{\partial \varphi_{\text{in}}}{\partial \theta}(R) &= \frac{\partial \varphi_{\text{out}}}{\partial \theta}(R) \\ \frac{\partial \varepsilon \varphi_{\text{in}}}{\partial r}(R) &= \frac{\partial \varphi_{\text{out}}}{\partial r}(R) \end{aligned} \tag{5.48}$$

Substituting (5.46) and using the linear independence of Legendre polynomials we get for $l \geq 2$

$$\begin{aligned} A_l R^l &= B_1 R^{-l-1} \\ l\varepsilon A_l R^{l-1} &= -(l+1) B_l R^{-l-2} \end{aligned} \tag{5.49}$$

The only solutions to these equations is $A_l = B_l = 0$. For $l = 1$ we have

$$\begin{aligned} A_1 R &= B_1 R^{-2} - ER \\ \varepsilon A_1 &= -2B_1 R^{-3} - E \end{aligned} \tag{5.50}$$

hence

$$A_1 = -\frac{3E}{\varepsilon + 2}, \quad B_1 = \frac{(\varepsilon - 1) E R^3}{\varepsilon + 2} \tag{5.51}$$

The field inside is along the z axis, smaller than at infinity, and with the polarization

$$P_z = \frac{3\varepsilon_0 E(\varepsilon - 1)}{\varepsilon + 2} \tag{5.52}$$

Since it constant inside the ball there is no volume bound charge density and the surface bound charge density is equal to

$$\sigma_b = \mathbf{P} \cdot \mathbf{n} = \frac{3E\varepsilon_0(\varepsilon - 1)}{\varepsilon + 2} \cos\theta \tag{5.53}$$

what effectively screens the inside of the ball (but not entirely as in the case of a conducting ball). The outside field is equivalent to a field of a dipole with

$$p = 4\pi\varepsilon_0 \frac{(\varepsilon - 1)ER^3}{\varepsilon + 2} \tag{5.54}$$

Dividing by the volume of the ball we recover, what is not surprising, the Clausius-Mossotti equation (5.38).

6 Magnetostatics

In this lecture we will discuss solutions of the Maxwell equations in presence of currents and an induced magnetic field.

We start with the Maxwell equations and we assume that $\mathbf{E} = 0$ and all the fields and current densities are time independent. We are left with two relevant equations

$$\nabla \cdot \mathbf{B} = 0 \tag{6.1}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{6.2}$$

which of course implies the continuity equation

$$\nabla \cdot \mathbf{j} = 0 \tag{6.3}$$

The integral form of (6.2) is the so called Ampère's law:

$$\oint_{\partial \mathcal{M}} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I \tag{6.4}$$

where I is a current flowing through $\mathcal{M}_{\in}$.

As before the first equation (6.1) allows us to write

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.5}$$

so if $\nabla \cdot \mathbf{A} = 0$ we get the Poisson equation for $\mathbf{A}$

$$\Delta \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{j}(\mathbf{r}) \Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 x' \, \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{6.6}$$

Since $\nabla \cdot \mathbf{j} = 0$ this solution indeed satisfies $\nabla \cdot \mathbf{A} = 0$.

Applying $\nabla \times$ to (6.2) and using (6.1) we get the Poisson equation directly for $\mathbf{B}$:

$$\Delta \mathbf{B} = -\mu_0 \nabla \times \mathbf{j} \tag{6.7}$$

One has to be careful when applying this equation in coordinate system different from the cartesian one – in general by definition $\Delta \mathbf{C} = \nabla(\nabla \cdot \mathbf{C}) - \nabla \times (\nabla \times \mathbf{C})$.

If there are no boundaries the solution to (6.7) is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 x' \, \frac{\nabla' \times \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 x' \, \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{6.8}$$

where we integrated by parts. This equation is known as the Biot-Savart Law.

Let us calculate the magnetic field in several examples.

For a straight, long wire with current I it is straightforward to calculate$\mathbf{B}$ using (6.4). If $\mathcal{M}_2$ is a disc of radius R then we immediately get

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\mathbf{e}_\phi \tag{6.9}$$

where $\mathbf{e}_\phi$ is a unit vector surrounding the wire.

For a long coil with N turns per meter and a current I we can use the Ampère's law to get (far from the ends of the coil)

$$Bl = \mu_0 NlI \Rightarrow B = \mu_0 NI \tag{6.10}$$

The field is parallel to the coil and homogeneous inside the coil.

For an arbitrary coil with N turns per meter and a current I we can get an expression for the magnetic field on the axis at a distance z from the middle of the coil by the Biot-Savart law

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{6.11}$$

The field is in the z direction and we can easily integrate the projection on the z direction

$$B(z) = \frac{\mu_0}{4\pi}\int_{-L/2}^{L/2}\mathrm{d}z' \frac{INR^2 2\pi}{(R^2+(z'-z)^2)^{3/2}} = \frac{\mu_0 IN}{2}\left.\frac{z'-z}{\sqrt{R^2+(z'-z)^2}}\right|_{-L/2}^{L/2} = \frac{\mu_0 IN}{2}(\sin\theta_1 + \sin\theta_2) \tag{6.12}$$

where θ_1, θ_2 are angles at which we see the ends of the coil (for a very long coil $\theta_1 = \pi/2$ and $\theta_2 = \pi/2$ and we reproduce the result from the previous paragraph). We see that at the end of the long coil ($\theta_1 = 0$ and $\theta_2 = \pi/2$) the field is only half of the field in the middle.

In the last example we want to show that the flux through the end of the long coil is exactly half of the flux through the middle of the coil i.e. half of the flux "leaks out" through the sides (the usual drawing of the field of the coil where all the flux goes through the end is wrong).

To calculate the flux through the endcap of a long coil of radius R with N turns per meter and a current I we use the relation

$$\Phi = \int_{B_2(R)} \mathbf{B}\cdot\mathrm{d}S = \oint_{S_1(R)} \mathbf{A}\cdot\mathrm{d}\mathbf{l} \tag{6.13}$$

where $B_2(R)$ is the disc of the endcap and S_1 its boundary (coinciding with the coil's end). To calculate $\mathbf{A}$ we use (6.6) in the cylindrical coordinates

$$\mathbf{A} = \frac{\mu_0 NIR\mathbf{e}_\phi}{4\pi}\int_{-\infty}^{0}\mathrm{d}z'\int_0^{2\pi}\mathrm{d}\phi\,\cos\phi\frac{1}{\sqrt{2R^2(1-\cos\phi)+z'^2}} \tag{6.14}$$

We integrate by parts over $\mathrm{d}\phi$

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 NIR\mathbf{e}_\phi}{4\pi}\int\limits_{-\infty}^{0}\mathrm{d}z'\int\limits_0^{2\pi}\mathrm{d}\phi\left(\frac{\partial}{\partial\phi}\sin\phi\right)\frac{1}{\sqrt{2R^2(1-\cos\phi)+z'^2}} = \\ &= \frac{\mu_0 NIR\mathbf{e}_\phi}{4\pi}\int\limits_{-\infty}^{0}\mathrm{d}z'\int\limits_0^{2\pi}\mathrm{d}\phi\sin\phi\frac{R^2\sin\phi}{(2R^2(1-\cos\phi)+z'^2)^{3/2}} = \\ &= \frac{\mu_0 NIR\mathbf{e}_\phi}{4\pi}\int\limits_0^{2\pi}\mathrm{d}\phi\cos^2(\phi/2) = \frac{\mu_0 NIR\mathbf{e}_\phi}{4} \end{aligned} \tag{6.15}$$

where we used the integral

$$\int\mathrm{d}z\frac{1}{(z^2+a^2)^{3/2}} = \frac{z}{a^2\sqrt{z^2+a^2}} \tag{6.16}$$

Therefore indeed

$$\Phi = \int\limits_{B_2(R)}\mathbf{B}\cdot\mathrm{d}S = \oint\limits_{S_1(R)}\mathbf{A}\cdot\mathrm{dl} = \frac{\mu_0 NI\pi R^2}{2} \tag{6.17}$$

i.e. the flux is twice smaller at the end than in the middle of the long coil.

6.1 Multipole expansion

Before we start let us note several identities that follow from (6.3).

$$0 = \int\mathrm{d}^3x\,(\partial_i j^i)x^k = -\int\mathrm{d}^3x\,j^k \Rightarrow \int\mathrm{d}^3x\,\mathbf{j}(\mathbf{x}) = 0 \tag{6.18}$$

where we integrated by parts and used the assumption that $\mathbf{j}(\mathbf{x})$ has compact support. The second one

$$0 = \int\mathrm{d}^3x\,(\partial_i j^i)x^k x^l = -\int\mathrm{d}^3x\,(x^k j^l + x^l j^k) \Rightarrow \int\mathrm{d}^3x\,x^k j^l = \frac{1}{2}\varepsilon^{klm}\int\mathrm{d}^3x(\mathbf{r}\times\mathbf{j})^m \tag{6.19}$$

We will also use

$$\begin{aligned}\nabla\times(\psi\mathbf{C}) &= \nabla\psi\times\mathbf{C} + \psi\nabla\times\mathbf{C} \\ \nabla\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\nabla\cdot\mathbf{C}) - \mathbf{C}(\nabla\cdot\mathbf{B}) + (\mathbf{C}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{C}\end{aligned} \tag{6.20}$$

We now expand (6.6) in powers of $1/r$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\left(\int\mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{r}')}{r} + \int\mathrm{d}^3x'\,\frac{\mathbf{j}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}')}{r^3} + \ldots\right) \tag{6.21}$$

The first term vanishes because of (6.18) and the second can be rewritten using (6.19) as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} + \ldots \tag{6.22}$$

where

$$\mathbf{m} = \frac{1}{2}\int \mathrm{d}^3x'\, \mathbf{r}' \times \mathbf{j}(\mathbf{r}') \tag{6.23}$$

is the so called total magnetization of the system. For the flat current loop it is equal to

$$|\mathbf{m}| = IS \tag{6.24}$$

where I is the current and S is the area of the loop.

The magnetic field of the dipole is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5} \tag{6.25}$$

The magnetic moment is related to the angular momentum. If the current is given by a movement of particles with charges q_a and masses m_a then we can write

$$\mathbf{j} = \sum_a q_a \mathbf{v}_a \delta(\mathbf{x} - \mathbf{x}_a) \tag{6.26}$$

The magnetic moment is on the other hand given by

$$\mathbf{m} = \frac{1}{2}\int \mathrm{d}^3x'\, \mathbf{r}' \times \mathbf{j}(\mathbf{r}') = \sum_a \frac{q_a}{2m_a}\mathbf{L}_a \tag{6.27}$$

where the angular momentum

$$\mathbf{L}_a = m_a \mathbf{r}_a \times \mathbf{v}_a \tag{6.28}$$

If the ratio q_a/m_a is the same for all particles then

$$\mathbf{m} = \frac{q}{2m}\mathbf{L} \tag{6.29}$$

where $\mathbf{L}$ is the total angular momentum. The classical factor $\frac{1}{2}$ is replaced by 1 for spins in the lowest order relativistic theory of massive fermions (Dirac equation) and this 1 receives small corrections in the full Quantum Electrodynamics (the first one is $\alpha/(2\pi)$ where α is the fine structure constant $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \sim 1/137$).

6.2 Forces in the magnetic field

We start with the Lorentz force acting on a charge q in an external magnetic field $\mathbf{B}$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} \tag{6.30}$$

One can show that the magnetic force is intimately connected with the electric (Coulomb) force by the following reasoning. If a charge q is moving with velocity v at distance d with respect to a wire of area A with a current I then it sees a (approximately at rest) certain density ρ of positive ions and the same density of electrons moving with some velocity v along the wire.

$$\rho_{\mathrm{ion}} = \rho_e = \rho \tag{6.31}$$

since the wire is uncharged. There is magnetic force acting on the charge

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B} = \frac{qv^2\mu_0 e\rho S\mathbf{e}_\rho}{2\pi r} \tag{6.32}$$

where $e > 0$ is equal to minus the electron charge. If we now change the frame to the one moving at velocity $\mathbf{v}$ then the charge is at rest and the magnetic field from the positive ions (that now move with velocity $-\mathbf{v}$) does not exert any force on the charge. However the densities change because of the Lorentz contraction

$$\rho'_{\mathrm{ion}} = \frac{\rho}{\sqrt{1-v^2/c^2}}, \qquad \rho'_e = \rho\sqrt{1-v^2/c^2} \tag{6.33}$$

Therefore in this frame the wire is charged with the linear density

$$\rho_{eff} = \rho'_{\mathrm{ion}} - \rho'_e = \frac{\rho v^2}{c^2\sqrt{1-v^2/c^2}} \tag{6.34}$$

Therefore there is an electric force

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t'} = q\mathbf{E} = \frac{qv^2 e\rho S\mathbf{e}_\rho}{\epsilon_0 c^2 2\pi r\sqrt{1-v^2/c^2}} \tag{6.35}$$

Using

$$\mathrm{d}t' = \sqrt{1-v^2/c^2}\mathrm{d}t \tag{6.36}$$

we get the same result in both frames if

$$\mu_0 = \frac{1}{\epsilon_0 c^2} \tag{6.37}$$

what shows the unification of magnetic and electric interactions as seen by different observers,

A force acting on a short element of wire with current I in an external magnetic field $\mathbf{B}$ is given by

$$\mathrm{d}\mathbf{F} = I\mathrm{d}\mathbf{l}\times\mathbf{B} \tag{6.38}$$

Therefore for two long wires at a distance d we have the force (using (6.9)

$$F = \frac{\mu_0 I_1 I_2}{2\pi d} \tag{6.39}$$

The definition of ampere is that the it is the current which in two long wires at a distance of 1 m produces the force $2\cdot 10^{-7}$ N/m (hence by definition $\mu_0 = 4\pi\cdot 10^{-7}$ N/A^2).

In the presence of an electric field the force acting on a localized distribution of charge is given by

$$\mathbf{F} = \int \mathrm{d}^3x\,\rho(\mathbf{r})\mathbf{E}(\mathbf{r}) = \int \mathrm{d}^3x\,\rho(\mathbf{r})(\mathbf{E}(0) + \mathbf{r}\cdot\nabla\mathbf{E}(0) + \ldots) = Q\mathbf{E}(0) + (\mathbf{p}\cdot\nabla)\mathbf{E}(0) + \ldots \tag{6.40}$$

while the leading term for the torque (moment of force) reads

$$\mathbf{N} = \int \mathrm{d}^3x\, \mathbf{r} \times \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) = \mathbf{p} \times \mathbf{E} \tag{6.41}$$

In the presence of a magnetic field the total force acting on some localized distribution of currents is given by

$$\mathbf{F} = \int \mathrm{d}^3x\, \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) = \int \mathrm{d}^3x\, \mathbf{j}(\mathbf{r}) \times (\mathbf{B}(0) + (\mathbf{r}\cdot\nabla)\mathbf{B}(0) + \ldots) \tag{6.42}$$

The first term vanishes because of (6.18). We write the second term in components

$$F_i = \varepsilon_{ijk} \int \mathrm{d}^3x\, \mathbf{j}_j x_l (\partial_l B_k)(0) \tag{6.43}$$

and using (6.19) we write

$$F_i = \frac{1}{2}\varepsilon_{ljm}\varepsilon_{ijk} \int \mathrm{d}^3x\, (\mathbf{r}\times\mathbf{j})_m (\partial_l B_k)(0) = \partial_i(\mathbf{m}\cdot\mathbf{B}) \tag{6.44}$$

where we used $\nabla\cdot\mathbf{B} = 0$. Therefore

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \tag{6.45}$$

The potential energy of a dipole in the magnetic field (but not the full energy!) is therefore given by

$$U = -\mathbf{m}\cdot\mathbf{B} \tag{6.46}$$

The torque in the magnetic field is given by the leading term

$$\mathbf{N} = \int \mathrm{d}^3x\, \mathbf{r} \times (\mathbf{j}\times\mathbf{B}(0)) = \int \mathrm{d}^3x\, (\mathbf{j}(\mathbf{r}\cdot\mathbf{B}(0)) - \mathbf{B}(0)(\mathbf{r}\cdot\mathbf{j})) \tag{6.47}$$

The second term vanishes because of (6.19) and the first (also using (6.19)) gives

$$\mathbf{N} = \mathbf{m}\times\mathbf{B}(0) \tag{6.48}$$

7 Magnetization

In a similar way to the discussion of dielectrics we can divide any current into free currents $\mathbf{j}_0$ and bound currents

$$\mathbf{j}(\mathbf{x}) = \sum_a \mathbf{j}_0(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_a) + \sum_b \varepsilon_{ijk} M_j \partial_k \delta(\mathbf{x}-\mathbf{x}_b) + \dots \tag{7.1}$$

where $\nabla \cdot \mathbf{j}_0 = 0$ and the second term is written in such a way to satisfy this condition automatically and $\dots$ stand for higher derivatives of the delta function. Neglecting all higher derivatives and introducing continuous current densities we write

$$\mathbf{j}(\mathbf{x}) = \mathbf{j}_0(\mathbf{x}) + \nabla \times \mathbf{M}(\mathbf{x}) \tag{7.2}$$

Returning to equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{7.3}$$

and rearranging we get

$$\nabla \times (\mathbf{B} - \mu_0 \mathbf{M}) = \mu_0 \mathbf{j}_0 \tag{7.4}$$

where on the RHS we have only free currents. We introduce

$$\mathbf{H} := \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{7.5}$$

and then

$$\nabla \times \mathbf{H} = \mathbf{j}_0 \tag{7.6}$$

The field $\mathbf{H}$ is very useful in actual applications since most often we control the external currents (in electrostatics we usually control potentials and therefore the analogous field $\mathbf{D}$ is not so useful) but we have to remember that at the microscopic level we should use only the field $\mathbf{B}$.

We can now write down Maxwell equations in the presence of media

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{j}_0 + \frac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{7.7}$$

and it is in this form that Maxwell wrote originally his equations. Only later it became clear that the fields $\mathbf{D}$ and $\mathbf{H}$ are secondary and at the microscopic level everything can (and should) be described by the field $\mathbf{E}$ and $\mathbf{B}$ only.

The question arises what is the relation between $\mathbf{H}$ (or $\mathbf{M}$) and $\mathbf{B}$ in the magnetically active media? As with dielectrics we have two different situations – either the magnetization is induced by the external magnetic field or the molecules or atoms have their own magnetic moment. The difference in this case is much more dramatic than in the dielectric case since the interaction of internal magnetic moments can be so strong to totally order these magnetic moments in large domains (in ferromagnetic media below the Curie temperature).

In the first case when the magnetization is induced when the external magnetic field is applied and then in the lowest approximation it is proportional to the external magnetic field

$$\mathbf{B}(\mathbf{x}) = \mu\mu_0\mathbf{H}(\mathbf{x}) \tag{7.8}$$

with some constant μ (that varies with frequency for fields dependent on time but in our case it is just a constant). In principle one should write this relation for $\mathbf{H}$ as a function of $\mathbf{B}$ (since $\mathbf{B}$ is a fundamental field and $\mathbf{H}$ is an auxiliary field) but first, this form is traditionally used, second $\mathbf{H}$ is what is directly related to the external currents and third it it is easier to operate with very large μ (for ferromagnetic materials) than with very small $1/\mu$.

The linear relation between $\mathbf{B}$ and $\mathbf{H}$ is rather well satisfied in the case when $\mu \sim 1$ but for $\mu \gg 1$ it is not satisfied and moreover the actual value of $\mathbf{B}$ for a given $\mathbf{H}$ depends on the history of the sample (the phenomenon of hysteresis).

We start now with $\mu \sim 1$. If $\mu < 1$ the substance is called *diamagnetic* if $\mu > 1$ it is *paramagnetic* (note that for dielectrics we have $\epsilon > 1$ – it is connected with the definition $\mathbf{D} \sim \epsilon\mathbf{E}$ and here $\mathbf{B} \sim \mu\mathbf{H}$). The diamagnetic properties exist when the substance does not have any internal magnetic moment and paramagnetic (and ferromagnetic) when it does. The difference with the electrostatic case (where we have only dielectric substances and there are no paraelectric ones) lies in the different reaction of magnetic dipole moments from the electric dipole moments.

If we have a substance without any internal magnetic moment (either orbital or spin) then in the magnetic field the carriers (electrons) start to move on circular orbits and the induced magnetic field from such current loops decreases the magnetic field (i.e. $\mu < 1$). The substances with such behavior are water, carbon, lead, silver and the most diamagnetic element known is bismuth. The effect is very weak and it turns out that classically it should vanish (it is purely quantum mechanical because of discrete Landau levels). There exist however ideal diamagnets (zero magnetic field inside) namely superconductors of the first kind (Meissner effect) but the origin of the effect is entirely different and purely quantum mechanical.

If molecules or atoms have their internal magnetic moment then the external magnetic field tends to orient the directions along the external magnetic field. There are two distinct situations - either the interaction between the neighboring atoms is weak and the orientation is separate for each of the molecule (paramagnet) or the interaction is strong and the atoms are oriented independently of the external field (ferro- or antiferromagnet).

In the first, paramagnetic case we can use a similar reasoning to calculate an average

magnetization as a function of temperature. If atoms have their own magnetic dipole moment m_0 and they do not interact with one another (this assumption is obviously wrong in the ferromagnetic case) then the hamiltonian is given by

$$H = H_0 - \mathbf{m}_0 \cdot \mathbf{B} \tag{7.9}$$

The magnetic moment is of the order of Bohr magneton

$$\mu_B = \frac{e\hbar}{2m_\mathrm{e}} = 0.92 \cdot 10^{-23}\,\mathrm{J/T} \tag{7.10}$$

since $\mathbf{m}_0 = \mu_B \mathbf{S}$ where $\mathbf{S}$ is a spin operator. Then the average dipole moment per molecule is given by ($\beta = 1/(kT)$)

$$\langle \mathbf{m} \rangle = \frac{\int \mathrm{d}\theta \sin\theta\, \mu_B \cos\theta \mathbf{e}_z \exp^{\beta\mu_B B\cos\theta}}{\int \mathrm{d}\theta \sin\theta \exp^{\beta\mu_B B\cos\theta}} = \mu_B \left(\coth(\beta\mu_B B) - \frac{1}{\beta\mu_B B} \right) \tag{7.11}$$

For small B we have

$$\langle \mathbf{m} \rangle \sim \frac{1}{3}\beta\mu_B^2\, \mathbf{B} \tag{7.12}$$

what is the case (for $T = 300$ K ($k = 1.38065 \cdot 10^{-23}$ J/K)) when

$$B \ll \frac{kT}{\mu_B} \sim \frac{4 \cdot 10^{-21}}{0.92 \cdot 10^{-23}}\mathrm{T} \sim 400\mathrm{T} \tag{7.13}$$

Then

$$\mu - 1 = \frac{\mu_0 N \mu_B^2}{3kT} \tag{7.14}$$

For gases at room temperatures we can estimate ($N \sim 2.7 \cdot 10^{25}$ m^{-3}) that $\mu - 1$ is around 10^{-5} and therefore it is very small. In dense substances the total contribution from spins almost cancels (because of the Pauli principle) and only those close to the Fermi surface can be ordered and therefore the paramagnetic properties are not multiplied by $\sim 10^3$ as would be suggested by the density factor but remain at the level 10^{-5} (Pauli paramagnetism).

Before we discuss ferromagnetic substances let us derive the expression for the energy density in the magnetic field. In the vacuum we have discussed already that the 00 component of the energy momentum tensor (we assume vanishing electric field) reads

$$T_{00} = \frac{1}{2\mu_0}\mathbf{B}^2 \tag{7.15}$$

In the presence of magnetic substances we write the total energy as

$$W = \int \mathrm{d}^3x\, \frac{1}{2}\mathbf{H} \cdot \mathbf{B} \tag{7.16}$$

If we keep currents fixed and we introduce some magnetically active substance the

$$\delta W = \int \mathrm{d}^3x\, \frac{1}{2}(\mathbf{H} \cdot \mathbf{B} - \mathbf{H}_0 \cdot \mathbf{B}_0) = \int \mathrm{d}^3x\, \frac{1}{2}(\mathbf{H} \cdot \mathbf{B} - \mathbf{H}_0 \cdot \mathbf{B}_0) \tag{7.17}$$

The first term can be written as

$$\int \mathrm{d}^3x \frac{1}{2}\mathbf{H}\cdot\mathbf{B} = \int \mathrm{d}^3x \frac{1}{2}((\mathbf{H}-\mathbf{H}_0)\cdot(\nabla\times\mathbf{A}) + \mathbf{H}_0\cdot\mathbf{B}) \tag{7.18}$$

Using the equation

$$\nabla\cdot(\mathbf{C}\times\mathbf{D}) = \mathbf{D}\cdot(\nabla\times\mathbf{C}) - \mathbf{C}\cdot(\nabla\times\mathbf{D}) \tag{7.19}$$

we integrate by parts the first term and use the fact that we don't change the free currents (i.e.$\nabla\times\mathbf{H} = \nabla\times\mathbf{H}_0$) to show that it vanishes. We apply the same procedure to the second term in (7.17) to arrive at the general result

$$\delta W = \int \mathrm{d}^3x \frac{1}{2}(\mathbf{H}_0\cdot\mathbf{B} - \mathbf{H}\cdot\mathbf{B}_0) \tag{7.20}$$

For a linear relation between $\mathbf{B}$ and $\mathbf{H}$ and the initial $\mathbf{B}_0$ in the vacuum we have

$$\delta W = \int \mathrm{d}^3x \frac{\mu_0}{2}(\mu-1)\mathbf{H}\cdot\mathbf{H}_0 = \int \mathrm{d}^3x \frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{H}_0 \tag{7.21}$$

The difference in sign with respect to the previously used expression $-\mathbf{m}\cdot\mathbf{B}$ (apart from the obvious $\frac{1}{2}$) comes from the fact that now it is the full change of energy including also the work needed to keep the currents constant while before this work was not accounted for.

7.1 Ferromagnetic materials

In contradistinction to the polarization where in matter we do not encounter very large permanent electric dipole moments, the internal magnetic moments in matter can be very large. The reason for this is the fact that spins contribute to the magnetic moment and the mutual interaction of spins can lead to a spontaneous ordering of large regions. In classical electrodynamics two magnetic moments placed close to each other try to orient themselves in an antiparallel way so the net field should be close to 0. In the quantum world there is a Pauli exclusion principle that in partially filled shells can lead to unbalanced spins (one of Hund's rules says that electrons tend to have parallel spins if they start to fill a shell). There are only very few materials that have large enough spin-orbital and spin-spin interaction to allow for the spontaneous ordering – iron, nickel, cobalt, some rare earth materials like neodymium and most of alloys of these metals. If the shells are filled or almost filled the total orbital momentum and the total spin (almost) vanish.

In the statistical description there are two opposing effects – the temperature tries to disorder the system and the interaction tries to order it. As it turns out in ferromagnetic substances there exists a temperature (Curie temperature) below which there is a spontaneous symmetry breaking and the material reaches its maximal possible magnetization. Then on small scales (domains) the material is fully magnetized but on larger

scales the domains can still be disoriented so that the average macroscopic magnetization vanishes. If one switches the magnetic field the domains start to point in the direction of the field (much like in the case of individual magnetic moments but since the internal field of the domains is very large the effect can be very large as well). When the magnetic field is switched off then it costs energy (at the boundary of the domains) to disorient again the domains so they stay (partially) oriented. It requires a magnetic field in the opposite direction to bring the magnetization back to 0. This phenomenon of hysteresis is at the origin of permanent magnets but it makes the solution of actual problems very difficult much more difficult than in the electrostatic case.

The dependence of the magnetic permeability on temperature above the Curie temperature T_C (when the ferromagnetic substance is paramagnetic) is given by the Curie-Weiss' law

$$\mu = \frac{\kappa}{T - T_C} \tag{7.22}$$

7.2 Methods of solving the problems

The equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{j} \tag{7.23}$$

impose the boundary conditions on surfaces separating two media without free surface currents:

$$\mathbf{B}_{\perp}^{(1)} = \mathbf{B}_{\perp}^{(2)}, \qquad \mathbf{H}_{||}^{(1)} = \mathbf{H}_{||}^{(2)} \tag{7.24}$$

In general since $\mathbf{H}(\mathbf{B})$ is a complicated function depending on the history of a sample it is very difficult to give any simple procedure that works in every case and in the real cases one resorts to numerical simulation. We will give below some methods that work in special situations.

- If the relation between $\mathbf{B}$ and $\mathbf{H}$ is linear $\mathbf{B} = \mu\mu_0\mathbf{H}$ then we can write

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{A}\right) = \mu_0 \mathbf{J} \tag{7.25}$$

  In the regions where μ is constant and we impose the Coulomb condition $\nabla \cdot \mathbf{A} = 0$ we get the Poisson's equation

$$-\Delta \mathbf{A} = \mu\mu_0 \mathbf{J} \tag{7.26}$$

  Solving this equation separately in different regions with different μ's we have to impose the continuity on $\mathbf{A}$ across the boundaries.

- if in some region there are no free currents then

$$\nabla \times \mathbf{H} = 0 \tag{7.27}$$

  and we can introduce the magnetic potential

$$\mathbf{H} = -\nabla \Phi_M \tag{7.28}$$

The equation we have to solve is then

$$\nabla \cdot (\mu \nabla \Phi_M) = 0 \tag{7.29}$$

If μ is constant it boils down to the Laplace equation – again we have to glue together the solutions to ensure the continuity of Φ_M.

- if we have permanent magnets with fixed magnetization $\mathbf{M}(\mathbf{r})$ (and no free currents) then we can write

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0 \tag{7.30}$$

Hence

$$\Delta \Phi_M = -\rho_M, \qquad \rho_M = -\nabla \cdot \mathbf{M} \tag{7.31}$$

If there are no external boundaries we write the solution as

$$\Phi_M(\mathbf{r}) = -\frac{1}{4\pi} \int \mathrm{d}^3 x' \frac{\nabla' \cdot \mathbf{M}'}{|\mathbf{r} - \mathbf{r}'|} = -\frac{1}{4\pi} \nabla \cdot \int \mathrm{d}^3 x' \frac{\mathbf{M}'}{|\mathbf{r} - \mathbf{r}'|} \tag{7.32}$$

For large $\mathbf{r}$ we recover the dipole potential

$$\Phi_M = \frac{1}{4\pi} \frac{\mathbf{m} \cdot \mathbf{r}}{r^3} \tag{7.33}$$

- For a given permanent magnetization $\mathbf{M}(\mathbf{r})$ confined to some closed region we can use directly the equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{7.34}$$

where the current $\mathbf{j}$ is a sum of free and bound currents. The bulk bound currents are given by

$$\mathbf{j}_b = \nabla \times \mathbf{M} \tag{7.35}$$

and the surface bound currents are given by

$$\mathbf{j}_{bS} = \mathbf{M} \times \mathbf{n} \tag{7.36}$$

where $\mathbf{n}$ is a vector normal to the surface (outward).

To illustrate the application of boundary conditions we now discuss the case of a magnetized ball of constant magnetization M (to be definite in the z direction) and radius R in vacuum. We use the usual expansion into spherical harmonics separately inside the ball and outside the ball

$$\begin{aligned} \Phi_{M\ \mathrm{in}}(\mathbf{x}) &= \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) \\ \Phi_{M\ \mathrm{out}}(\mathbf{x}) &= \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta) \end{aligned} \tag{7.37}$$

where we have used the condition that the potential should be finite at $r \to 0$ and at $r \to \infty$ and that $\nabla \cdot \mathbf{M}$ is different from zero only on the surface of the ball.

The field $\mathbf{H}_{||}$ is continuous across the surface while the continuous perpendicular field is $\mathbf{H}_{\perp} + \mathbf{M}_{\perp}$. Therefore we write

$$\begin{aligned} -\frac{\partial \Phi_{M \text{ in}}}{\partial \theta}(R) &= -\frac{\partial \Phi_{M \text{ out}}}{\partial \theta}(R) \\ -\frac{\partial \Phi_{M \text{ in}}}{\partial r}(R) + M\cos\theta &= -\frac{\partial \Phi_{M \text{ out}}}{\partial r}(R) \end{aligned} \tag{7.38}$$

Substituting (7.37) and using the linear independence of Legendre polynomials we get for $l \geq 2$

$$\begin{aligned} a_l R^l &= b_1 R^{-l-1} \\ -l a_l R^{l-1} &= (l+1) b_l R^{-l-2} \end{aligned} \tag{7.39}$$

The only solutions to these equations is $a_l = b_l = 0$. For $l = 1$ we have

$$\begin{aligned} -a_1 R &= -b_1 R^{-2} \\ -a_1 + M &= 2 b_1 R^{-3} \end{aligned} \tag{7.40}$$

whence

$$a_1 = \frac{M}{3}, \quad b_1 = \frac{M R^3}{3} \tag{7.41}$$

Therefore the field $\mathbf{B}$ is equal to

$$\mathbf{B}_{\text{in}} = \mu_0(\mathbf{H} + \mathbf{M}) = \frac{2\mu_0}{3}\mathbf{M}. \quad \mathbf{B}_{\text{out}} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{m}}{r^5} \tag{7.42}$$

where $\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}$. Hence the field inside is homogeneous (and smaller than $\mu_0\mathbf{M}$) and it is purely dipole outside with the natural value of the dipole.

If we have a cylinder made of a magnetic material with permeability μ immersed in a constant magnetic field then it is in general difficult to give exact expressions for the $\mathbf{M}$ inside and outside. There are however two limiting cases when it is relatively straightforward.

If the cylinder is very thin then it is the $\mathbf{B}$ field that is continuous across the base and therefore

$$\mathbf{B} = \mu_0 \mathbf{H}_0 \Rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \mathbf{H}_0 - (\mu - 1)\mathbf{H} \Rightarrow \mathbf{H} = \frac{\mathbf{H}_0}{\mu} \tag{7.43}$$

If the cylinder is very long then it is the tangent component of $\mathbf{H}_0$ that is continuous across the sides and therefore

$$\mathbf{H} = \mathbf{H}_0 \Rightarrow \mathbf{B} = \mu\mu_0 \mathbf{H}_0 \tag{7.44}$$

8 Alternating currents

In the rest of the lectures we will discuss time dependent densities and currents.

We recall the Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{8.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{8.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{8.4}$$

8.1 Circuits

We start with the traditional application of electrodynamics to circuits at small frequencies when we don't have to consider radiation.

The conductors have a simple relation between the current and the voltage. Since

$$\mathbf{j} = \sigma \mathbf{E} \tag{8.5}$$

then we have the Ohm's law

$$V = RI \tag{8.6}$$

where

$$R = \frac{l}{\sigma S} \tag{8.7}$$

For the capacitors we have

$$Q = CV \tag{8.8}$$

so that differentiating with respect to time we have

$$I = C\frac{\mathrm{d}V}{\mathrm{d}t} \tag{8.9}$$

For the inductors the equation (8.2) integrated over some surface gives

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi}{\partial t} \tag{8.10}$$

so that the voltage difference at the ends od a circuit is proportional to the time derivative of a magnetic flux through this circuit. Since

$$\Phi = LI \tag{8.11}$$

where L is the inductance therefore we have

$$V = -L\frac{\partial I}{\partial t} \tag{8.12}$$

For a circuit of a resistor, a capacitor and an inductor connected in series and attached to a battery with alternating current we can write

$$\mathcal{E}\sin\omega t = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C} \tag{8.13}$$

Differentiating with respect to time we have

$$\omega\mathcal{E}\cos\omega t = R\frac{\mathrm{d}I}{\mathrm{d}t} + L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + \frac{I}{C} \tag{8.14}$$

The general solution is the sum the solution to the homogeneous equation and the special solution to inhomogeneous equation. The first solution is called a transient current and the second a stationary current. Constraining ourselves to the stationary current we write

$$I(t) = A\sin(\omega t - \phi) = A(\sin\omega t\cos\phi - \cos\omega t\sin\phi) \tag{8.15}$$

Plugging it to (8.14) we get

$$\begin{aligned} 0 &= \omega R\sin\phi - \omega^2 L\cos\phi + \frac{\cos\phi}{C} \\ \omega\mathcal{E} &= A\left(\omega R\cos\phi + \omega^2 L\sin\phi - \frac{\sin\phi}{C}\right) \end{aligned} \tag{8.16}$$

We get the phase shift

$$\tan\phi = \frac{L\omega - \frac{1}{C\omega}}{R} \tag{8.17}$$

and the amplitude

$$A = \frac{\mathcal{E}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\mathcal{E}}{R}\cos\phi \tag{8.18}$$

Therefore the special solution is equal to

$$I_s(t) = \frac{\mathcal{E}}{R}\cos\phi\sin(\omega t - \phi) \tag{8.19}$$

with ϕ given by (8.17).

The maximum is in the resonance

$$\omega_0 = \sqrt{\frac{1}{LC}} \tag{8.20}$$

and the resistance is purely ohmic there.

We can calculate the average power dissipated by the circuit:

$$P = \frac{1}{T}\int_0^T V(t)I(t)\mathrm{d}t = \frac{\mathcal{E}^2}{2R}\cos^2\phi \tag{8.21}$$

If $\phi \to \pm\frac{\pi}{2}$ there is no power dissipated by the circuit

Let us calculate the full solution with the transient currents is given by he sum of the homogeneous and the special solutions and is equal to

$$I_g(t) = \mathrm{e}^{-\frac{Rt}{2L}}(A_1\cos(\omega_f t) + A_2\sin(\omega_f t)) + \frac{\mathcal{E}}{R}\cos\phi\sin(\omega - \phi) \tag{8.22}$$

where (we assume that the expression below is real)

$$\omega_f = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{8.23}$$

If we assume that $I(0) = 0$ and $\dot{I}(0) = 0$ then we get

$$\begin{aligned} A_1 &= \frac{\mathcal{E}}{R}\sin\phi\cos\phi \\ A_2 &= -\frac{\mathcal{E}\omega}{R\omega_f}\cos^2\phi + \frac{\mathcal{E}}{2L\omega_f}\cos\phi\sin\phi \end{aligned} \tag{8.24}$$

8.2 Faraday's law

As an application of the Faraday's law let us calculate the potential between the pole and the equator induced in a (poorly) conducting sphere of radius R rotating with angular velocity ω in a magnetic field B.

The contribution to $\mathrm{d}\mathcal{E}$ from an element of coordinates θ, ϕ is

$$\mathrm{d}\mathcal{E} = (\mathbf{v}\times\mathbf{B})\cdot\mathrm{dl} = R^2\omega B\cos\theta\sin\theta\mathrm{d}\theta \tag{8.25}$$

Hence integrating from 0 to $\pi/2$ we get

$$\Delta\mathcal{E} = \frac{1}{2}R^2\omega B \tag{8.26}$$

This example shows that we can add the electromotive force piecewise and not only apply it for the whole closed loops. Of course the full EMF vanishes since there is no flux through the loop.

8.3 Conductivity

Let us recall the formula for dielectric constant as a function of frequency

$$\varepsilon(\omega) = 1 + \frac{Ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j} \tag{8.27}$$

where the oscillator strengths satisfy

$$\sum_j f_j = Z \tag{8.28}$$

In the limit of small frequencies there is a crucial difference whether the lowest eigenfrequency vanishes (i.e there are free electrons) or not. In the conducting case there is $f_0 > 0$ electrons that are free and then we separate the contribution from the free electrons from the rest and we write (in the limit $\omega \ll \omega_1$)

$$\varepsilon(\omega) = \varepsilon + \mathrm{i}\frac{Ne^2 f_0}{\varepsilon_0 m\omega(\Gamma - \mathrm{i}\omega)} \tag{8.29}$$

Let us compare this formula with the law

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{8.30}$$

where $\mathbf{j}$ denotes free currents only. For a medium satisfying the Ohm's law

$$\mathbf{j} = \sigma \mathbf{E} \tag{8.31}$$

and the dielectric constant ε we can write (assuming harmonic dependence on time)

$$\nabla \times \mathbf{H} = -\mathrm{i}\omega \left(\varepsilon_0 \varepsilon + \mathrm{i}\frac{\sigma}{\omega}\right) \mathbf{E} \tag{8.32}$$

On the other hand the quantity on the RHS is the full dielectric constant $\varepsilon(\omega)$ so that comparing wih (8.29) the conductivity at $\omega = 0$ is given by

$$\sigma = \frac{Ne^2 f_0}{m\Gamma} \tag{8.33}$$

In the limit of large frequencies (larger than any ω_j with nonnegligible oscillator strength) we have

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p = \sqrt{\frac{Ne^2 Z}{\varepsilon_0 m}} \tag{8.34}$$

where ω_p is called the plasma frequency.

8.4 Kramers-Kronig relations and causality

For a general relation between the $\mathbf{D}$ field at time t (i.e. the reaction of the medium, assumed isotropic and homogeneous) and the electric field $\mathbf{E}$ at the same point we have

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon_0 (\mathbf{E}(t, \mathbf{x}) + \int_0^\infty \mathrm{d}\tau G(\tau) \mathbf{E}(t - \tau, \mathbf{x})) \tag{8.35}$$

where $G(\tau)$ is some real function. Putting the lower limit of the integration to 0 we have used a fundamental concept of causality i.e. the effect cannot occur earlier than the cause. This simple observation leads to very powerful statements in many branches of physics known as dispersion relations. In this case we have for the Fourier transform (i.e. all fields dependent on time as $e^{-i\omega t}$)

$$\varepsilon(\omega) = 1 + \int_0^\infty d\tau G(\tau) e^{i\omega\tau} \tag{8.36}$$

We assume that $G(\tau)$ is continuous i.e. $G(0) = 0$. For nonvanishing conductivity we can use

$$G_\sigma(\tau) = \frac{\sigma}{\varepsilon_0}(1 - \exp(-\Gamma\tau)) \tag{8.37}$$

and then

$$\varepsilon(\omega) = \frac{i\sigma\Gamma}{\varepsilon_0\omega(\Gamma - i\omega)} + 1 + \int_0^\infty d\tau \tilde{G}(\tau) e^{i\omega\tau} \tag{8.38}$$

where

$$\tilde{G}(\tau) = G(\tau) - G_\sigma(\tau) \tag{8.39}$$

tends to 0 for $\tau \to \infty$.

To get the expansion for large ω

$$\varepsilon(\omega) - 1 - \frac{i\sigma\Gamma}{\varepsilon_0\omega(\Gamma - i\omega)} = \int_0^\infty d\tau \tilde{G}(\tau) \frac{1}{i\omega} \frac{d}{d\tau} e^{i\omega\tau} = -\int_0^\infty d\tau \frac{\tilde{G}'(\tau)}{i\omega} e^{i\omega\tau} \tag{8.40}$$

where we used $\tilde{G}(0) = 0$. Continuing this procedure we get the expansion

$$\varepsilon(\omega) \xrightarrow[\omega\to\infty]{} 1 - \frac{\sigma\Gamma}{\varepsilon_0\omega^2} - \frac{\tilde{G}'(0)}{\omega^2} + \dots \tag{8.41}$$

We now continue ω to complex values. We see that because in (8.35) the integral runs from $\tau = 0$ the integral is well defined for the whole upper halfplane in ω. The relation shows that we can extend the range of arguments for negative values and we have

$$\varepsilon(-z) = \varepsilon^\star(z^\star) \tag{8.42}$$

If we introduce

$$\tilde{\varepsilon}(z) := \varepsilon(z) - 1 + \frac{\sigma\Gamma}{\varepsilon_0 z(z + i\Gamma)} \tag{8.43}$$

we can write the Cauchy equation when $z = \omega$ is on the real axis

$$0 = \frac{1}{2\pi i} \oint_C \frac{\tilde{\varepsilon}(\omega')}{\omega' - \omega} d\omega' \tag{8.44}$$

where the contour runs along the real axis, surrounds ω by a small semicircle above (in the clockwise sense) and comes back by the large semicircle in the upper halfplane in the counterclockwise sense. There are no poles inside the contour hence 0 on the LHS. The contribution from the semicircle vanishes for large radius and we are left with

$$0 = \frac{1}{2\pi \mathrm{i}} P \int_{-\infty}^{\infty} \frac{\tilde{\varepsilon}(\omega')}{\omega' - \omega} \mathrm{d}\omega' - \frac{\pi \mathrm{i}}{2\pi \mathrm{i}} \tilde{\varepsilon}(\omega) \tag{8.45}$$

to arrive at

$$\tilde{\varepsilon}(\omega) = \frac{1}{\mathrm{i}\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\tilde{\varepsilon}(\omega')}{\omega' - \omega} \mathrm{d}\omega' \tag{8.46}$$

We can divide it into the real and imaginary parts

$$\begin{aligned} \operatorname{Re} \tilde{\varepsilon}(\omega) &= \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \tilde{\varepsilon}(\omega')}{\omega' - \omega} \mathrm{d}\omega' \\ \operatorname{Im} \tilde{\varepsilon}(\omega) &= -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\operatorname{Re} \tilde{\varepsilon}(\omega')}{\omega' - \omega} \mathrm{d}\omega' \end{aligned} \tag{8.47}$$

and these are the Kramers-Kronig relations relating absorptive and dispersive parts (often called dispersion relations in other branches of physics). One should note that these relations are very general based only on causality and therefore have very broad spectrum of applications.

If we have a narrow absorption line in the absorptive part (we have to add a second term because of (8.42))

$$\operatorname{Im} \varepsilon(\omega) = \frac{\pi K_1 \omega_1^2}{2\omega_1} (\delta(\omega - \omega_1) - \delta(\omega + \omega_1)) \tag{8.48}$$

then far away from the other absorption frequencies

$$\operatorname{Re} \varepsilon(\omega) = \epsilon_\infty + \frac{K_1 \omega_1^2}{\omega_1^2 - \omega^2} \tag{8.49}$$

i.e. it is a rapidly varying function around the absorptive peak. In this model

$$\begin{aligned} \operatorname{Re} \varepsilon(\omega) &= 1 - \frac{\sigma \Gamma}{\varepsilon_0 (\omega^2 + \Gamma^2)} + \sum_j \frac{K_j \omega_j^2}{\omega_j^2 - \omega^2} \\ \operatorname{Im} \varepsilon(\omega) &= \frac{\sigma \Gamma^2}{\varepsilon_0 \omega (\omega^2 + \Gamma^2)} + \sum_j \frac{\pi K_j \omega_j^2}{2\omega_j} (\delta(\omega - \omega_j) - \delta(\omega + \omega_j)) \end{aligned} \tag{8.50}$$

9 Appendix

In the notation used in these lectures indices $i, j, k \ldots$ will denote $1, 2, 3$ i.e spatial dimensions (Greek indices $\mu, \nu \ldots = 0, 1, 2, 3$ will denote 4-dimensional quantities). The summation over repeated indices will always be implicitly assumed. The derivative with respect to time will be denoted by a dot and with respect to (cartesian) spatial directions by

$$\nabla_i := \frac{\partial}{\partial x^i} \equiv \partial_i \tag{9.1}$$

This operator has well defined properties under rotations and transforms tensors into tensors.

We introduce a scalar product of two vectors

$$\mathbf{A} \cdot \mathbf{B} := A_i B^i \tag{9.2}$$

with a number as a result and a vector product

$$(\mathbf{A} \times \mathbf{B})_i := \varepsilon_{ijk} A^j B^k \tag{9.3}$$

with a vector (in 3 dimensions) as a result – ε_{ijk} is a fully antisymmetric tensor with $\epsilon_{123} = 1$ (in 4 dimensions we choose the convention $\varepsilon^{0123} = 1$).

We will often use the identity

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{9.4}$$

Therefore, for example

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{9.5}$$

If we have a vector field $\mathbf{E}(\mathbf{x})$ then its divergence

$$\nabla \cdot \mathbf{E} = \frac{\partial E^i}{\partial x^i} \tag{9.6}$$

is a scalar field and its rotation

$$(\nabla \times \mathbf{E})^i = \varepsilon^{ijk} \partial_j E_k \tag{9.7}$$

is a (pseudo)vector field.

We have for example

$$\begin{aligned} \nabla \cdot (f\mathbf{A}) &= \nabla f \cdot \mathbf{A} + f \nabla \cdot \mathbf{A} \\ \nabla \times (f\mathbf{A}) &= \nabla f \times \mathbf{A} + f \nabla \times \mathbf{A} \\ \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} \end{aligned} \tag{9.8}$$

For an arbitrary scalar field ϕ we have an important identity

$$\nabla \times (\nabla \phi) = 0 \tag{9.9}$$

Also for an arbitrary vector field $\mathbf{E}$ we have

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 \tag{9.10}$$

These equations are better expressed in the language of differential forms. If we have an n-form A_n then an exterior derivative denoted by d acting on A_n produces $(n+1)$-form. If the manifold is metric then there exists also an operation $\star$ that produces $(D-n)$-form where D is the dimension of the manifold. There is also a functor $\delta := \star \mathrm{d} \star$ that acting on A_n produces $(n-1)$- form. These equations then follow from the basic equation

$$\mathrm{dd} = 0 \tag{9.11}$$

when acting on a 0-form, as in (9.9), or 1-form, as in (9.10). A crucial role is played by a laplacian Δ defined as

$$\Delta = \mathrm{d}\delta + \delta \mathrm{d} \tag{9.12}$$

An arbitrary form A_n on a any manifold can be written as a sum of three forms (Hodge decomposition)

$$A_n = \mathrm{d}A_{n-1} + \delta A_{n+1} + H_n \tag{9.13}$$

for some globally defined A_{n-1}, A_{n+1} and so called harmonic form H_n satisfying

$$\mathrm{d}H_n = \delta H_n = 0 \tag{9.14}$$

so it satisfies also the Laplace equation

$$\Delta H_n = 0 \tag{9.15}$$

The number of linearly independent harmonic n-forms (Betti number b_n) is a very important characterization of a manifold (the harmonic forms belong to the so called nth cohomology class dual to the n-th homology class). If the n-th cohomology class for a given manifold is empty then

$$\mathrm{d}A_n = 0 \Rightarrow A_n = \mathrm{d}A_{n-1} \tag{9.16}$$

for some globally defined A_{n-1}. For example the manifolds with $b_1 = 0$ are called simply-connected and then $\mathrm{d}A_1 = 0 \Rightarrow A_1 = \mathrm{d}\phi$.

In the form language the gradient is an action on 0-form producing a one-form (vector):

$$f \to \mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i, \tag{9.17}$$

divergence is an action on a one-form producing a 0-form (scalar)

$$V_1 \to \star\mathrm{d}(\star V_1) = V_0, \tag{9.18}$$

rotation is an action on a one-form producing $(n-2)$-form (in 3 dimensions a pseudovector)

$$V_1 \to \star\mathrm{d}V_1 = V_{n-2} \tag{9.19}$$

For arbitrary vector field $\mathbf{A}$ and 3-manifold $\mathcal{M}_3$ with a 2-dimensional boundary $\partial\mathcal{M}_3$ we have the Gauss equation

$$\int_{\mathcal{M}_3} \nabla\cdot\mathbf{A}\,\mathrm{d}V = \int_{\partial\mathcal{M}_3} \mathbf{A}\cdot\mathrm{d}\mathbf{S} \tag{9.20}$$

while for a 2-manifold $\mathcal{M}_2$ with 1-dimensional boundary $\partial\mathcal{M}_2$ we have the Stokes equation

$$\int_{\mathcal{M}_2} (\nabla\times\mathbf{A})\cdot\mathrm{d}\sigma = \oint_{\partial\mathcal{M}_2} \mathbf{A}\cdot\mathrm{d}\mathbf{s} \tag{9.21}$$

9.1 Coordinate systems

In physical problems it is very often the case that we have some kind of symmetry (spherical, axial). It is then advantageous to use the coordinate system adapted to the symmetry so that the dependence on one or more coordinates drops out.

There exist general expressions for the gradient, divergence, rotation and Laplace operator for a manifold endowed with a metric g_{ij} (especially easy if the metric is diagonal i.e. in the Lamé form) but we will not develop the general theory here quoting only the Laplace operator

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial\xi^j}\right) \tag{9.22}$$

We give below expressions for differential operators only in cylindrical and spherical frames where the metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2 + \mathrm{d}z^2 \tag{9.23}$$

and

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \tag{9.24}$$

9.1.1 Cylindrical coordinates

In the cylindrical frame we define (ρ, ϕ, z) as

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z \tag{9.25}$$

so that

$$\frac{\partial}{\partial\rho} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial\phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{9.26}$$

Then the unit vectors are

$$\mathbf{e}_\rho = \frac{1}{\sqrt{x^2+y^2}}(x, y, 0), \quad \mathbf{e}_\phi = \frac{1}{\sqrt{x^2+y^2}}(-y, x, 0), \quad \mathbf{e}_z = (0, 0, 1) \tag{9.27}$$

so that

$$\nabla\cdot\mathbf{e}_\rho = \frac{1}{\rho}, \quad \nabla\times\mathbf{e}_\rho = 0, \quad \nabla\cdot\mathbf{e}_\phi = 0, \quad \nabla\times\mathbf{e}_\phi = \frac{1}{\rho}\mathbf{e}_z, \quad \nabla\cdot\mathbf{e}_z = 0, \quad \nabla\times\mathbf{e}_z = 0 \tag{9.28}$$

Then

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\psi}{\partial z}\mathbf{e}_z \tag{9.29}$$

since for example

$$\mathbf{e}_\phi\cdot\nabla\psi = \frac{1}{\rho}(-y, x, 0)\cdot\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right) = \frac{1}{\rho}\frac{\partial}{\partial\phi} \tag{9.30}$$

For a vector $\mathbf{A}$

$$\mathbf{A} = A_\rho\mathbf{e}_\rho + A_\phi\mathbf{e}_\phi + A_z\mathbf{e}_z \tag{9.31}$$

we have

$$\begin{aligned} \nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \qquad (9.32)\\ \nabla\times\mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\mathbf{e}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)\mathbf{e}_z \end{aligned}$$

The Laplace operator is given by

$$\Delta = \nabla\cdot(\nabla) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \tag{9.33}$$

9.2 Solutions to the Laplace equation in cylindrical coordinates

It is very useful to have a general solution of the Laplace equation. In 3 dimensions the solution of $\Delta\varphi = 0$ is easy to get by separation of variables.

In cylindrical coordinates we write

$$\psi(\mathbf{x}) = R(\rho)\Phi(\phi)Z(z) \tag{9.34}$$

then the Laplace equation using (9.56) can be written as

$$\Delta\psi = 0 \Rightarrow \frac{1}{\rho R(\rho)}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right) + \frac{1}{\rho^2\Phi(\phi)}\frac{\mathrm{d}^2\Phi(\phi)}{\mathrm{d}\phi^2} + \frac{1}{Z(z)}\frac{\mathrm{d}^2Z(z)}{\mathrm{d}z^2} \tag{9.35}$$

To satisfy this equation the double derivatives of Φ and Z must be equal to a constants, equal respectively to $-m^2$ and k^2 so that

$$\psi(\mathbf{x}) = \sum_m \sum_k R_{mk}(\rho)\mathrm{e}^{\mathrm{i}m\phi}\mathrm{e}^{kz} \tag{9.36}$$

The index m runs usually over integer numbers because of periodicity in ϕ but the range of summation over k (and whether k is real or imaginary) depends on the problem. The function $R_{mk}(\rho)$ therefore satisfies

$$\frac{\mathrm{d}^2R_{mk}}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R_{mk}}{\mathrm{d}\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R_{mk} = 0 \tag{9.37}$$

i.e. the Bessel equation. Therefore in cylindrical coordinates the general solution reads

$$\psi(\mathbf{x}) = \sum_m \sum_k \left(A_m J_m(k\rho) + B_m N_m(k\rho)\right)\mathrm{e}^{\mathrm{i}m\phi}\mathrm{e}^{kz} \tag{9.38}$$

where J are the Bessel functions and N are the Neumann functions (or Bessel functions of the second kind)

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \tag{9.39}$$

The Bessel functions have the Taylor expansion

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)}\left(\frac{x}{2}\right)^{2j} \tag{9.40}$$

For half-integer indices they are given by elementary functions.

9.3 Spherical Bessel and Hankel functions

The spherical Bessel and Hankel functions are defined as

$$\begin{aligned} j_l(x) &= \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x) \\ h_l^{(1)}(x) &= \sqrt{\frac{\pi}{2x}}H^{(1)}_{l+\frac{1}{2}}(x) \end{aligned} \tag{9.41}$$

They are elementary functions that can be obtained from the formulae

$$\begin{aligned} j_l(x) &= (-x)^l\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\left(\frac{\sin x}{x}\right) \\ h_l^{(1)}(x) &= -\mathrm{i}(-x)^l\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\left(\frac{\mathrm{e}^{\mathrm{i}x}}{x}\right) \end{aligned} \tag{9.42}$$

and they both satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x f_l(x)\right) + \left(1 - \frac{l(l+1)}{x^2}\right) x f_l(x) = 0 \tag{9.43}$$

We will need the expansions

$$\begin{aligned} j_l(x) &\xrightarrow[x\to 0]{} \frac{x^l}{(2l+1)!!}(1+O(x^2)) \\ h_l^{(1)}(x) &\xrightarrow[x\to\infty]{} (-\mathrm{i})^{l+1}\frac{\mathrm{e}^{\mathrm{i}x}}{x}\left((1+O\left(\frac{1}{x}\right)\right) \end{aligned} \tag{9.44}$$

The expansion in terms of spherical waves

$$\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \mathrm{i}k\sum_{l=0}^{\infty}(2l+1)h_l^{(1)}(kr)j_l(kr')P_l(\cos\theta) \tag{9.45}$$

and

$$\mathrm{e}^{-\mathrm{i}xt} = \sum_{l=0}^{\infty}(2l+1)(-\mathrm{i})^l j_l(x)P_l(t) \tag{9.46}$$

9.4 Spherical coordinates

In the spherical frame we define (r,θ,ϕ) as

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \tag{9.47}$$

so that

$$\begin{aligned} \frac{\partial}{\partial r} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= \frac{xz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} - \sqrt{x^2+y^2}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned} \tag{9.48}$$

Then the unit vectors are

$$\begin{aligned} \mathbf{e}_r &= \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) \\ \mathbf{e}_\theta &= \frac{1}{\sqrt{x^2+y^2+z^2}}(\frac{xz}{\sqrt{x^2+y^2}}, \frac{yz}{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2}) \\ \mathbf{e}_\phi &= \frac{1}{\sqrt{x^2+y^2}}(-y,x,0) \end{aligned} \tag{9.49}$$

so that

$$\nabla\cdot\mathbf{e}_r = \frac{2}{r}, \quad \nabla\times\mathbf{e}_r = 0, \quad \nabla\cdot\mathbf{e}_\theta = \frac{\mathrm{ctg}\theta}{r}, \quad \nabla\times\mathbf{e}_\theta = \frac{1}{r}\mathbf{e}_z, \quad \nabla\cdot\mathbf{e}_\phi = 0, \quad \nabla\times\mathbf{e}_\phi = -\frac{1}{r}\mathbf{e}_\theta \tag{9.50}$$

Then

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi \tag{9.51}$$

and for a vector $\mathbf{A}$

$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi \tag{9.52}$$

we have

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{9.53}$$

$$\begin{aligned}\nabla\times\mathbf{A} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right)\mathbf{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\mathbf{e}_\phi\end{aligned}$$

The Laplace operator is given by

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{9.54}$$

9.5 Solutions to the Laplace equation in spherical coordinates

In spherical coordinates we write

$$\psi(\mathbf{x}) = \frac{R(r)}{r}\Theta(\theta)\Phi(\phi) \tag{9.55}$$

then the Laplace equation using (9.54) can be written as

$$\Delta\psi = 0 \Rightarrow \frac{r^2}{R(r)}\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{1}{\Theta(\theta)\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2\theta\Phi(\phi)}\frac{\mathrm{d}^2\Phi(\phi)}{\mathrm{d}\phi^2} \tag{9.56}$$

To satisfy this equation double derivatives of R and Φ must be equal to constants, equal respectively to $l(l+1)$ (with solutions r^{l+1} an r^{-l}) and $-m^2$ so that

$$\psi(\mathbf{x}) = \sum_l\sum_m\left(A_{lm}r^l + B_{lm}r^{-l-1}\right)\Theta_{lm}(\theta)\mathrm{e}^{\mathrm{i}m\phi} \tag{9.57}$$

The index m runs over integer numbers because of periodicity in ϕ and it turns out that because of restricted interval of $\theta\in[0,\pi]$ requiring non singular solutions at the ends of this interval range of summation over l is restricted to non-negative integers $\geq|m|$. The function $\Theta_{lm}(\theta)$ satisfies

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta_{lm}}{\mathrm{d}\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta_{lm} = 0 \tag{9.58}$$

i.e. the Legendre equation. Therefore in spherical coordinates the general solution reads

$$\psi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + B_{lm}r^{-l-1}\right)P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi} \tag{9.59}$$

where P_l^m are associated Legendre functions. If additionally the solution has an axial symmetry then the solution doesn't depend on ϕ

$$\psi(\mathbf{x}) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1} \right) P_l(\cos\theta) \tag{9.60}$$

where $P_l(x)$ are Legendre polynomials given by

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left(x^2 - 1 \right)^l \tag{9.61}$$

First 3 polynomials read

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2 - 1}{2} \tag{9.62}$$

The generating function for the Legendre polynomials is given by

$$\frac{1}{\sqrt{1 + t^2 - 2xt}} = \sum_{l=0}^{\infty} P_l(x) t^l \tag{9.63}$$

The Legendre polynomials form a complete set on the interval $[-1, 1]$:

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x') \tag{9.64}$$

The associated Legendre functions are given by the formula

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}} \left(x^2 - 1 \right)^l \tag{9.65}$$

We have a relation between positive and negative m:

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{9.66}$$

and for a given m

$$\int_{-1}^{1} \mathrm{d}x \, P_l^m(x) P_{l'}^{-m}(x) = (-1)^m \frac{2}{2l+1} \delta_{ll'} \tag{9.67}$$

We introduce spherical harmonic functions as

$$Y_{lm}(\theta, \phi) := \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi} \tag{9.68}$$

hence

$$Y_{l\,-m}(\theta, \phi) := (-1)^m Y_{lm}^\star(\theta, \phi) \tag{9.69}$$

They are orthogonal on the sphere

$$\int_0^{2\pi} \int_0^{\pi} \mathrm{d}\theta \sin\theta \, Y_{lm}^\star(\theta, \phi) Y_{l'm'}^\star(\theta, \phi) = \frac{4\pi}{2l+1} \delta_{ll'} \delta_{mm'} \tag{9.70}$$

We use here Y_{lm} which are differently normalized than the usual convention

$$Y_{lm} = \sqrt{\frac{4\pi}{2l+1}} Y_{lm} \tag{9.71}$$

what appears to be much more convenient.

The lowest $Y_{lm}(\theta, \phi)$ are given by

$$\begin{aligned} Y_{00} &= 1 \\ Y_{10} &= \cos\theta \\ Y_{11} &= -\sqrt{\frac{1}{2}} \sin\theta \, \mathrm{e}^{\mathrm{i}\phi} \\ Y_{20} &= \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) \\ Y_{21} &= -\sqrt{\frac{3}{2}} \sin\theta \cos\theta \, \mathrm{e}^{\mathrm{i}\phi} \\ Y_{22} &= \sqrt{\frac{3}{8}} \sin^2\theta \, \mathrm{e}^{2\mathrm{i}\phi} \end{aligned} \tag{9.72}$$

An arbitrary function $g(\theta, \phi)$ can be expanded in spherical harmonics

$$g(\theta, \phi) = \sum_{l,m} g_{lm} \, Y_{lm}(\theta, \phi) \tag{9.73}$$

where

$$g_{lm} = \frac{2l+1}{4\pi} \int \mathrm{d}\Omega \, Y^{\star}_{lm}(\theta, \phi) \, g(\theta, \phi) \tag{9.74}$$