Hydrodynamics and Elasticity
Test I
27 November 2023

1. **(A)** Determine the deformation of an elastic spherical shell of outer radius b and inner radius a, inside of which there is vacuum, and outside of which there is external pressure p. The material which the shell is made of is characterised by the Lamé constants λ and μ. **(B)** Calculate the change of the thickness of the shell resulting from this external pressure for $b/a = 2$. Do such elastic materials exists, for which the thickness of a compressed shell would increase? Justify your answer with a calculation. **(C)** How does the T_{rr} component of the stress tensor change in the limits of (i) $\nu \to 1/2$, and (ii) $\nu \to -1$?

Notes:

- To analyse deformations and stresses in **(B)** and **(C)** after finding a solution for u_r that satisfies suitable boundary conditions, it is convenient to switch to a description of elastic properties of the material by the Young modulus E and Poisson ratio ν, that are related with the Lamé constants by

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

- You can (but you do not have to!) use the general form of a gradient of a bmtor field in spherical coordinates

$$[\boldsymbol{\nabla v}] = \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi} - \frac{v_\varphi \cot\theta}{r} \\ \frac{\partial v_\varphi}{\partial r} & \frac{1}{r}\frac{\partial v_\varphi}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r} \end{pmatrix}$$

Solution We consider a purely radial deformation:

$$\boldsymbol{u} = u_r(r)\widehat{\boldsymbol{e}}_r.$$

Note that we may write $\boldsymbol{u} = \nabla\psi(r)$, where $\psi(r) = \int u_r(r)\mathrm{d}r$. This is convenient, since then

$$\nabla^2\boldsymbol{u} = \nabla(\nabla^2\psi)$$

and the Navier-Cauchy equation in the absence of body forces

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = 0$$

can be written as

$$(2\mu+\lambda)\nabla(\nabla\cdot\boldsymbol{u}) = 0.$$

Using the Gauss' law for a sphere of radius r we can find the form of this operator under such deformation. We have

$$\oint_{\partial V} \boldsymbol{u}\cdot\mathbf{d}\boldsymbol{S} = \pi r^2 u_r = \int_V (\nabla\cdot\boldsymbol{u})\mathrm{d}V = \int_0^r \mathrm{d}r'\, 4\pi r'^2 \mathrm{d}r,$$

and, differentiating both sides with respect to r, we find

$$\frac{\mathrm{d}r^2 u_r}{\mathrm{d}r} = (\nabla\cdot\boldsymbol{u})r^2.$$

We thus find the form of the divergence

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2u_r)\right] = 0,$$

which has a general solution

$$u_r = Ar + \frac{B}{r^2}.$$

The components of the strain tensor are found from the known form of the gradient of the deformation field by symmetrisation

$$\widehat{E} = (\mathrm{Grad}\,\boldsymbol{u})^{sym}.$$

Thus

$$E_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \qquad E_{\theta\theta,\phi\phi} = \frac{u_r}{r}$$

We now find the components of the stress tensor from

$$\widehat{T} = 2\mu\widehat{E} + \lambda(\mathrm{Tr}\widehat{E})\mathbf{1}.$$

We get

$$T_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + 2\lambda\frac{u_r}{r}, \qquad T_{\theta\theta,\phi\phi} = \lambda\frac{\mathrm{d}u_r}{\mathrm{d}r} + 2(\mu+\lambda)\frac{u_r}{r}.$$

Therefore

$$E_{rr} = A - \frac{2B}{r^3}, \qquad E_{tt} = A + \frac{B}{r^3}$$

where t is any direction perpendicular to the radial one. For stresses, we have

$$T_{rr} = (2\mu + 3\lambda)A - 4\mu\frac{B}{r^3}, \qquad T_{tt} = (2\mu + 3\lambda)A + 2\mu\frac{B}{r^3}$$

The boundary conditions in the problem translate to the following

$$T_{rr}|_{r=b} = -p, \qquad T_{rr}|_{r=a} = 0,$$

from which we find the constants

$$A = -\frac{b^3}{b^3 - a^3}\frac{p}{2\mu + 3\lambda}, \qquad B = -\frac{a^3 b^3}{b^3 - a^3}\frac{p}{4\mu}.$$

Note that both constants are negative.

The deformation field reads:

$$u_r = -\frac{b^3}{b^3 - a^3}\left(\frac{r}{3\lambda + 2\mu} + \frac{a^3}{4\mu r^2}\right)p$$

It is now useful to transit from Lamé constants to the Young's modulus E and Poisson ratio ν. We then have

$$u_r = -\frac{b^3}{b^3 - a^3}\left((1-2\nu)r + (1+\nu)\frac{a^3}{2r^2}\right)\frac{p}{E}.$$

Since $-\frac{1}{2} \le \nu \le 1$, the deformation is negative, which is expected for compression.

Let us now calculate the change in the thickness of the shell $d = b - a$ under this deformation

$$\delta d = u_r(b) - u_r(a) = -\frac{b^3}{b^2 + ab + a^2}\left(1 - 2\nu - (1+\nu)\frac{a(a+b)}{2b^2}\right)\frac{p}{E}.$$

The sign of it is determined by the expression in brackets. For $b/a = 2$ we have

$$1 - 2\nu - 3(1+\nu)/8 = 5/8 - 19\nu/8.$$

For $\nu > 5/19$ the shell thickens upon compression, and otherwise it becomes thinner.

In the language of E, ν the strain components read

$$u_{rr} = -\frac{b^3}{b^3 - a^3}\left(1 - 2\nu - (1+\nu)\frac{a^3}{r^3}\right)\frac{p}{E}$$

$$u_{tt} = -\frac{b^3}{b^3 - a^3}\left(1 - 2\nu + (1+\nu)\frac{a^3}{2r^3}\right)\frac{p}{E}$$

From the properties of ν it follows, that the material of the shell is always compressed in the azimuthal directionw ($u_{tt} < 0$). On the other hand, in the radial direction the material is always stretched for $r = a$ and it even can be stretched for $r = b$.

The stresses take the form

$$T_{rr} = -\frac{b^3}{b^3 - a^3}\left(1 - \frac{a^3}{r^3}\right)p$$

$$T_{tt} = -\frac{b^3}{b^3 - a^3}\left(1 + \frac{a^3}{2r^3}\right)p$$

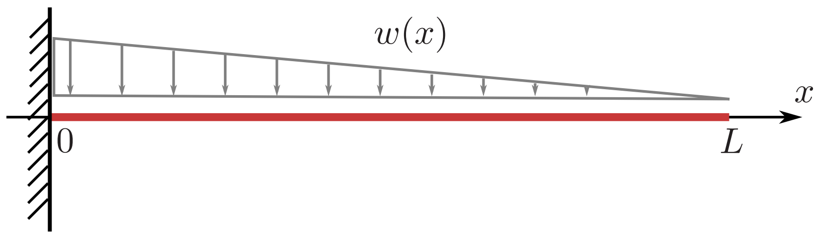
Another surprise – the stresses are independent of the elastic properties of the shell material.

2. Find the deflection of a horizontal cantilever beam of length L and a circular cross section (with radius a), that has one end fixed to a wall (see drawing), and deforms under a distributed load that varies linearly with the distance from the wall, $w(x) = \alpha(L - x)$. The Young's modulus of the beam is E. Neglect the weight of the beam.

(a) Find the function $y(x)$ describing the shape of the deformed beam.

(b) Find the deflection of the beam $y_{\max}$ at the point of maximal displacement.

(c) Find the reaction force $\boldsymbol{R}$ and torque $\boldsymbol{M}_0$ at the fixed end of the beam.



Solution The beam satisfies a fourth-order equation we derived during lectures and used in classes:

$$EI\frac{\mathrm{d}^4 y(x)}{\mathrm{d}x^4} = q \equiv w(x),$$

with four boundary conditions. The fixed end at $x = 0$ implies

$$y(0) = 0, \qquad y'(0) = 0,$$

while the free end at $x = L$ requires

$$y''(L) = 0, \qquad y'''(L) = 0.$$

The vanishing third derivative means that there cannot be a force acting on the terminal cross-section, and the second derivative equal to zero guarantees no net torque at that point. Thus, the deflection reads

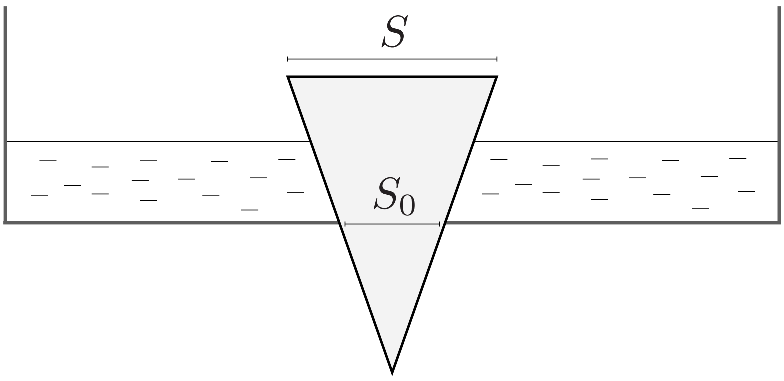
$$y(x) = \frac{\alpha x^2}{120EI}\left(10L^3 - 10L^2x + 5Lx^2 - x^3\right),$$

and the maximal deflection at the end is

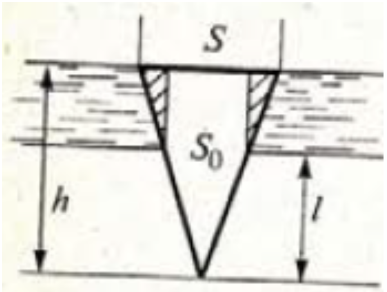
$$y_{\max} = \frac{\alpha L^5}{30EI}.$$

The force and torque at the base can be calculated from $M_x = -EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $F = \frac{\mathrm{d}M}{\mathrm{d}x}$. The reaction force and torque must be equal and opposite to the internal force and torque. Thus we find that the toruqe is $\boldsymbol{M_0} = (\alpha L^3/3)\boldsymbol{e}_x$, and its direction must be along the positive z axis (with x pointing along the beam and y pointing upwards), while the reaction force is $\boldsymbol{R} = (\alpha L^2/2)\boldsymbol{e}_y$. The directions can be figured out from basic mechanics – the reaction force must be equal to the weight of the rod, which is equal to $\int_0^L w(x)\mathrm{d}x = \alpha L^2/2$, and must act upwards to support the beam. The torque should couteract the gravitational torque of the beam, equal to $\int_0^L xw(x)\mathrm{d}x = \alpha L^3/6$.

3. A circular hole in the bottom of a vessel is closed by a conical plug with a cross-section S of the base (see figure below). What is the greatest density of the material of the plug for which it is possible to make the plug float by adding water of density ρ_w? The area of the hole is S_0.



Solution The condition for a plug to float up is the equality of its gravity and maximum buoyant force. The buoyant force is maximum when the water reaches the top of the cork, and it is equal to the weight of the water in the shaded volume of the cork (see drawing).



The volume of the cone is $hS/3$. Thus

$$\rho g h \frac{S}{3} = \rho_w g \left(\frac{hS}{3} - \frac{lS_0}{3} - (h-l)S_0 \right).$$

Therefore

$$\frac{hS}{3}(\rho_w - \rho) - \frac{lS_0}{3}\rho_w g - (h-l)S_0\rho_w g = 0.$$

We now also have the geometric relationship

$$\frac{l^2}{h^2} = \frac{S_0}{S},$$

using which we find

$$\rho = \rho_w \left[1 + 2\left(\frac{S_0}{S}\right)^{3/2} - 3\frac{S_0}{S} \right].$$

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Basis transformation

$$A'_{ij} = Q_{ai} Q_{bj} A_{ab}$$

Vector Laplacian

$$\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u})$$

Navier-Cauchy equation

$$\boldsymbol{f} + \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = 0$$

Equilibrium condition

$$\boldsymbol{f} + \mathrm{Div}\, \hat{T} = 0$$

Strain tensor

$$\hat{\epsilon} = (\nabla \boldsymbol{u})^S$$

$$\hat{\epsilon} = \frac{E}{1+\nu}\left((1+\nu)\hat{T} - \nu \mathrm{Tr}(\hat{T}) \cdot \mathbf{1}\right)$$

$$\hat{\epsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\lambda}{3\lambda + 2\mu}\mathrm{Tr}(\hat{T}) \cdot \mathbf{1}\right)$$

Stress tensor

$$\hat{T} = \frac{E}{1+\nu}\left(\hat{\epsilon} + \frac{\nu}{1-2\nu}\mathrm{Tr}(\hat{\epsilon}) \cdot \mathbf{1}\right)$$

$$\hat{T} = (2\mu\hat{\epsilon} + \lambda \mathrm{Tr}(\hat{\epsilon}) \cdot \mathbf{1})$$

Transformation of coefficients

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

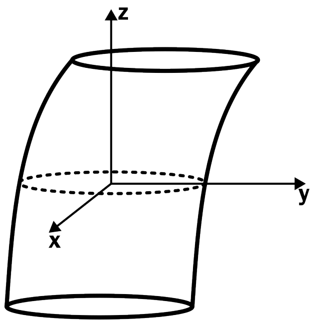
$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

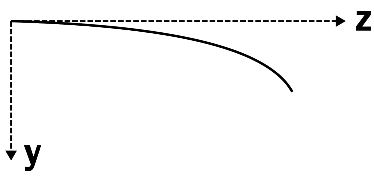
Euler-Bernoulli law

$$M_b = EI\kappa, \qquad I = \int\int x^2 \mathrm{d}S$$

Bending of slender beams

$$M_x = -EI\frac{d^2 y}{dz^2}$$





$$\frac{d}{dz}\begin{pmatrix} M_x \\ F_z \\ F_y \end{pmatrix} = \begin{pmatrix} F_y - \frac{dy}{dz}F_z \\ -K_y \\ -K_z \end{pmatrix}$$

$$T_{zz} = -Ey\kappa$$

is the only nonzero T_{ij}.

$$\partial_z E_b = \frac{1}{2}EI\kappa^2$$

Twisting

$$M_t = \mu J \tau, \qquad J = \int\int \left(x^2 + y^2\right) \mathrm{d}S$$

$$\partial_z E_t = \frac{1}{2}\mu J \tau^2$$