To solve the first part of the problem we will show that for

$$\Gamma(r,t) = 2\pi r u_\theta(r,t)$$

the equation

$$\frac{\partial \Gamma}{\partial t} = \nu \left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r} \right), \tag{1}$$

is equivalent to the second Navier-Stokes equation[1]

$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u}. \tag{2}$$

The θ component equation 2 reads

$$\frac{\partial u_\theta}{\partial t} = \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} \right). \tag{3}$$

Substituting

$$\Gamma(r,t) = 2\pi r u_\theta(r,t)$$

into 3 we obtain

$$r\frac{\partial u_\theta}{\partial t} = \nu \left(2\frac{\partial u_\theta}{\partial r} + r\frac{\partial^2 u_\theta}{\partial r^2} - \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad \Longleftrightarrow \quad \frac{\partial u_\theta}{\partial t} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right).$$

On the other hand, using

$$\nabla^2 u_\theta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r} \right),$$

[1] Under the assumption of incompressability

equation 3 becomes

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) \quad\iff\quad \frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

Thus, the evolution equation 1 is satisfied.

To solve the evolution equation we propose that

$$\Gamma = f(\eta), \quad \eta = \frac{r}{\sqrt{\nu t}}.$$

Substituting it into the evolution equation we obtain

$$-\frac{1}{2}\frac{\mathrm{d}\Gamma}{\mathrm{d}\eta}\frac{r}{\sqrt{\nu t}^3} = -\frac{1}{r}\frac{\mathrm{d}\Gamma}{\mathrm{d}\eta}\frac{1}{\sqrt{\nu t}} + \frac{1}{\nu t}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}\eta^2}.$$

Multiplying both sides by $r\sqrt{\nu t}$ we obtain

$$-\frac{1}{2}\frac{\mathrm{d}\Gamma}{\mathrm{d}\eta}\frac{r^2}{\nu t} = -\frac{\mathrm{d}\Gamma}{\mathrm{d}\eta} + \frac{r}{\sqrt{\nu t}}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}\eta^2}. \quad\iff\quad \frac{\eta^2}{2}\frac{\mathrm{d}\Gamma}{\mathrm{d}\eta} = \frac{\mathrm{d}\Gamma}{\mathrm{d}\eta} - \eta\frac{\mathrm{d}^2\Gamma}{\mathrm{d}\eta^2}.$$

Denoting $\xi := \frac{\mathrm{d}\Gamma}{\mathrm{d}\eta}$ we obtain

$$\eta\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = -\left(\frac{\eta^2}{2} - 1\right)\xi \quad\iff\quad \frac{\mathrm{d}\xi}{\mathrm{d}\eta} = -\left(\frac{\eta}{2} - \frac{1}{\eta}\right)\xi.$$

Integration by parts gives us

$$\log\xi = -\frac{\eta^2}{4} + \log\eta + C, \quad C \in \mathbb{R},$$

thus

$$\xi = \frac{\mathrm{d}\Gamma}{\mathrm{d}\eta} = \eta e^C \exp\left(-\frac{\eta^2}{4}\right).$$

Integrating with respect to η gives

$$\Gamma = C'\exp\left(-\frac{\eta^2}{4}\right) + D = C'\exp\left(-\frac{r^2}{4\nu t}\right) + D$$

where C', D are some constants. Using conditions $\Gamma(r,0) = \Gamma_0$, $\Gamma(0,t) = 0$ for $t > 0$ we obtain $D = \Gamma_0$, $C' = -D = -\Gamma_0$. Thus the complete solution

$$\Gamma(r,t) = \Gamma_0\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right], \quad u_\theta(r,t) = \frac{\Gamma_0}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$

The vorticity is equal to

$$\omega = \frac{1}{r}\frac{\partial r u_\theta}{\partial r} = \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} = \frac{\Gamma_0}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right).$$

At $t = 0$ it is just a Dirac delta, then it's just a gaussian function with $\sigma = \sqrt{2\nu t}$.

For $r \ll \sqrt{\nu t}$ we can expand $\exp(x) = 1 + x + \mathcal{O}(x)$ to obtain

$$u_\theta(r,t) = \frac{\Gamma_0 r}{8\pi\nu t}.$$