Problem 1 PS 10 Jerzy Gandzyk

The cylinder is infinite, thus we can assume due to symmetry in θ, and translational symmetry in z that velocity field is:

$$\bar{u} = u_r(r)\,\hat{e}_r$$

a) div in cylindrical coordinates:

$$d(\bar{A} \lrcorner \Omega) = (\operatorname{div}\bar{A})\,\Omega \qquad \Omega = \sqrt{\det g}\, dr \wedge d\theta \wedge dz = r\, dr \wedge d\theta \wedge dz$$

So:

$$d(\bar{A} \lrcorner \Omega) = \left[\partial_r(r A_r) + r\partial_\theta(A_\theta) + r\partial_z(A_z)\right]\Omega$$

So $\operatorname{div}\bar{A} = \frac{1}{r}\partial_r(rA_r) + \partial_\theta(A_\theta) + \partial_z(A_z)$

but if we remember that $\bar{A} = A_r^{\xi}\partial_r + A_\theta^{\xi}\partial_\theta + A_z^{\xi}\partial_z = A_r'\hat{e}_r + A_\theta'\hat{e}_\theta + A_z'\hat{e}_z$ for normalized vector field (normalized $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$) $\underbrace{A_\theta r}_{A_\theta'}\hat{e}_\theta$

We have

$$\operatorname{div}\bar{A} = \frac{1}{r}\partial_r(rA_r') + \frac{1}{r}\partial_\theta A_\theta' + \partial_z A_z'$$

So for our field $\bar{u} = u_r(r)\hat{e}_r$ we have:

$$\bar{\nabla}\cdot\bar{u} = \frac{1}{r}\partial_r(r u_r)$$

b) To find deformation field we need to solve N-C eq.:

$$\bar{f} + \mu\bar{\nabla}^2\bar{u} + (\mu+\lambda)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = 0$$

but for our field $\bar{\nabla}\times\bar{u} = 0 \Rightarrow \bar{\nabla}^2\bar{u} = \bar{\nabla}(\bar{\nabla}\cdot\bar{u})$

and we have body force due to rotation $\bar{f} = +g_0\omega^2 r\,\hat{e}_r$

centrifugal force!

So $(2\mu+\lambda)\,\bar{\nabla}(\bar{\nabla}\cdot\bar{u}) = \rho_0\omega^2 r\,\hat{e}_r$

Thus $(2\mu+\lambda)\,\partial_r\left[\frac{1}{r}\partial_r(r u_r)\right] = -\rho_0\omega^2 r$

$$\partial_r(r u_r) = -\frac{\rho_0\omega^2}{2\mu+\lambda}\,\frac{r^3}{2} + \overbrace{2A}^{\text{const}}\,r$$

$$u_r(r) = -\overbrace{\frac{\rho_0\omega^2}{2\mu+\lambda}\,\frac{r^3}{8}}^{\beta} + Ar + \frac{B}{r}$$

but for a cylinder we also have boundary conditions:
we considered rotating frame of reference with the cylinder,

So: $u_r(0) = 0$ ← finite in the middle $\Rightarrow B = 0$

And outer edge of cylinder should be stress free:

$\hat{T} = 2\mu\hat{\varepsilon} + \lambda\,\mathrm{Tr}\,\hat{\varepsilon}\,\mathbb{1}$, where $\hat{\varepsilon} = (\bar{\nabla}\bar{u})^S = \begin{bmatrix} \partial_r u_r & \\ & \frac{u_r}{r} \end{bmatrix}^S$ ← we can drop z dependence

$$\hat{\varepsilon} = \begin{bmatrix} -3\beta r^2 + A & \\ & -\beta r^2 + A \end{bmatrix}$$

Sanity check:
$\mathrm{div}\,\bar{u} = \cancel{8\lambda} - 4\beta r^2 + 2A$
$\mathrm{Tr}\,\hat{\varepsilon} = -4\beta r^2 + 2A$ ✓

So $\hat{T} = \mathrm{diag}\left(-\beta r^2(6\mu + 4\lambda) + A(2\mu+2\lambda),\ -\beta r^2(2\mu+4\lambda) + A(2\mu+2\lambda)\right)$

and $T_{rr}(r=R) = 0 \Rightarrow -\beta R^2(\overset{3}{\cancel{6}}\mu + \overset{2}{\cancel{4}}\lambda) + A(\cancel{2}\mu + \cancel{2}\lambda) = 0$

$$\Rightarrow A = +\beta R^2\,\frac{3\mu+2\lambda}{\mu+\lambda} = +\beta R^2\,\frac{3 + \frac{4\nu}{1-2\nu}}{1 + \frac{2\nu}{1-2\nu}} = +\beta R^2(3-2\nu)$$

Problem 1 cd

Jerzy Gendzyk

So we finally get deformation field:

$$u_r = -\beta r^3 + Ar = -\beta r^3 + \beta R^2 r(3-2\nu) =$$

$$= \beta r R^2\left(3-2\nu - \frac{r^2}{R^2}\right) = \frac{\rho\omega^2 r R^2}{8(2\mu+\lambda)}\left(3-2\nu - \frac{r^2}{R^2}\right)$$

$\operatorname{sgn} u_r = \operatorname{sgn}\left(3-2\nu-\frac{r^2}{R^2}\right) = 1 \leftarrow$ the cylinder expands everywhere

(c) Now lets decompose the ~~stress~~ strain tensor:

$$\cancel{T_{rr} = -\beta r^2(6\mu+4\lambda) + A(2\mu+2\lambda)} =$$

$$\varepsilon_{rr} = \partial_r u_r = \beta R^2\left(3-2\nu-\frac{3r^2}{R^2}\right)$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} = \beta R^2\left(3-2\nu-\frac{r^2}{R^2}\right) > 0 \text{ everywhere}$$

So strain in θ corresponds to expansion everywhere

on the other hand

$$\varepsilon_{rr} = 0 \iff r_0 = R\sqrt{1-\frac{2}{3}\nu} \leftarrow \text{Exists } \leftarrow \text{for } \nu > 0$$

what about $\nu < 0$?

Thus strain in $\hat{e}_r$ corresponds to extension close to the center and to contraction further away!

d) If we consider breakage we observe the

$$(\varepsilon_{rr})_{max} = (\varepsilon_{\theta\theta})_{max} = \beta R^2(3-2\nu) \leftarrow \text{at the center } \underline{r=0}$$

we have biggest strains

And in small neighbourhood $\varepsilon_{\theta\theta} > \varepsilon_{rr}$ So breakage will take place in the middle along plane (r,z) for some θ:

b*) Distribution of stress:

$$T_{rr} = -2\beta r^2 (3\mu+2\lambda) + 2\beta R^2 \overset{\frac{3\mu+2\lambda}{\mu+\lambda}}{\overbrace{(3-2\nu)}}(\mu+\lambda) =$$

$$= 2\beta(3\mu+2\lambda)(R^2-r^2)$$

$$T_{\theta\theta} = -2\beta r^2 (\mu+2\lambda) + 2\beta R^2 (3\mu+2\lambda) =$$

$$= 2\beta\left[\mu(3R^2-r^2) + 2\lambda(R^2-r^2)\right]$$

Remarks • $\bar{\nabla}\times\bar{u} = \bar{\nabla}\times(u_r\hat{e}_r) = \underbrace{(\bar{\nabla}u_r)}_{f\hat{e}_r}\times\hat{e}_r + u_r \underbrace{\bar{\nabla}\times\hat{e}_r}_{0} = 0$

• For breakage it will certainly start at $r=0$ but the difference between ε_{rr} and $\varepsilon_{\theta\theta}$ is $\sim\left(\frac{r}{R}\right)^2$ so it is not necessary for it to go in (r,z) plane

- I am not sure

Zad 2 Szymon Kłapiński 440454

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \varepsilon & -\gamma \\ \gamma & -\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon x - \gamma y \\ \gamma x - \varepsilon y \end{pmatrix}$$

$$\bar{\nabla} \cdot \bar{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial}{\partial x}(\varepsilon x - \gamma y) + \frac{\partial}{\partial y}(\gamma x - \varepsilon y) = \varepsilon - \varepsilon = 0 \quad V$$

$$\bar{u} = \bar{\nabla} \times (\psi \bar{e}_z) = \begin{pmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \\ 0 \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix}$$

1) $\frac{\partial \psi}{\partial y} = \varepsilon x - \gamma y$ 2) $\frac{\partial \psi}{\partial x} = \varepsilon y - \gamma x$

1) $\psi = \varepsilon x y - \frac{1}{2}\gamma y^2 + f(x)$

2) $\psi = \varepsilon x y - \frac{1}{2}\gamma x^2 + f(y)$

$$\Rightarrow \quad \psi = -\frac{1}{2}\gamma(x^2+y^2) + \varepsilon x y \quad (+\text{const}) \quad (*)$$

Checking the formula :

$$(\varepsilon-\gamma)(x^2+2xy+y^2) - (\varepsilon+\gamma)(x^2-2xy+y^2) = C$$

$$\varepsilon(2xy+2xy) - \gamma(2x^2+2y^2) = C$$

$$4\varepsilon x y - 2\gamma(x^2+y^2) = C$$

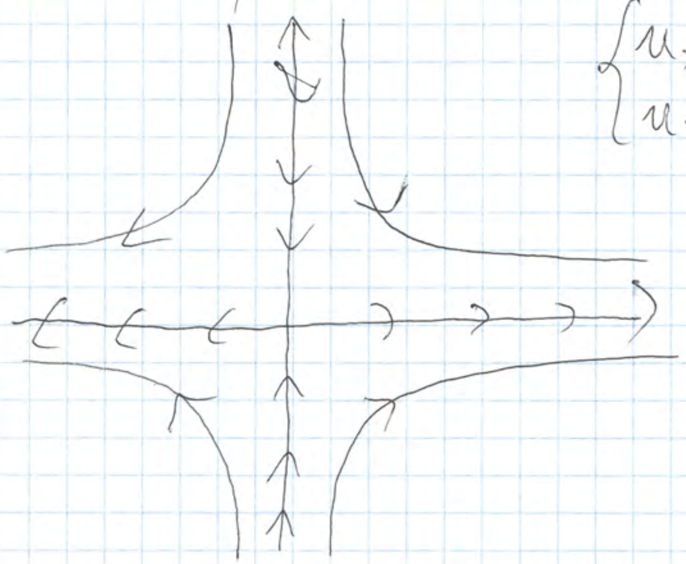
$$\varepsilon x y - \frac{1}{2}\gamma(x^2+y^2) = \frac{C}{4} = \text{const}$$

Is the same as $(*)$, so it's OK

Streamlines $\Rightarrow \psi = \text{const} = C$

1) $\varepsilon = 1, \ \gamma = 0 \Rightarrow (x+y)^2 - (x-y)^2 = \text{const}$

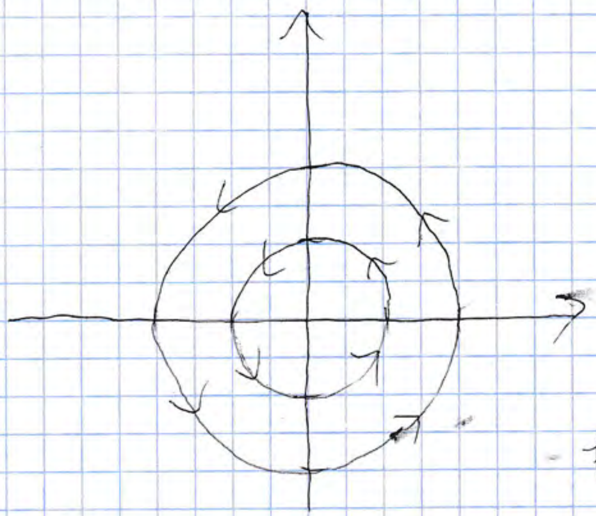
$\Rightarrow \quad xy = \text{const}$



$$\begin{cases} u_x = x \\ u_y = -y \end{cases}$$

further away there are higher speeds, In $(0,0)$ $v = 0$

2) $\varepsilon = 0$, $\gamma = 1$ $\Rightarrow$ $x^2 + y^2 = \text{const}$



$$\begin{cases} u_x = -y \\ u_y = x \end{cases}$$

Vortex!

(I think this is „bathtube Vortex")

- further away - higher speeds $\sim r$
where r = ~~radius~~ radius

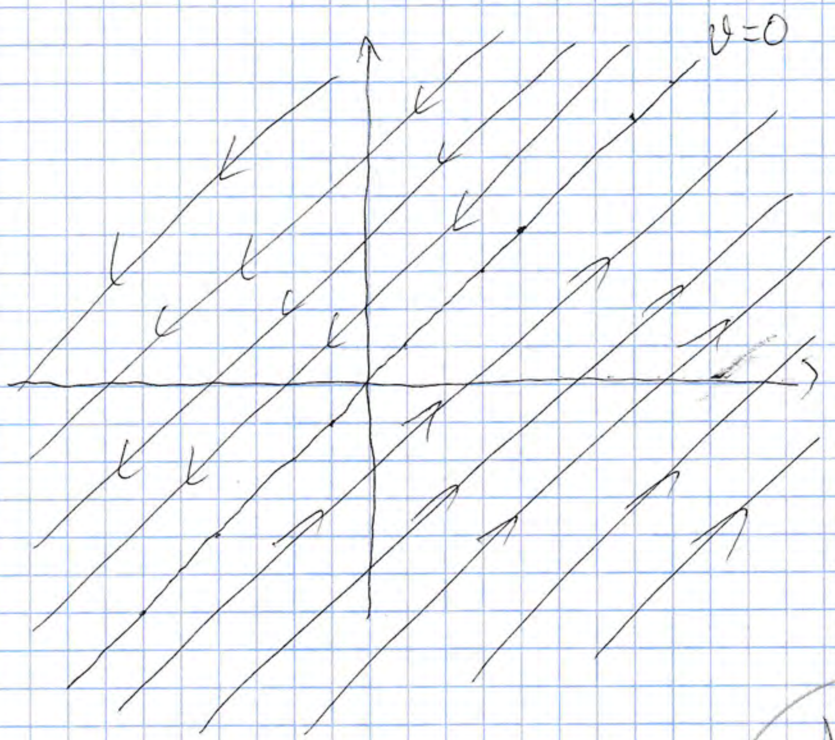
3) $\varepsilon = 1$, $\gamma = 1$ $\Rightarrow$

$$-\frac{1}{2}(x^2+y^2) + xy = \text{const} \quad / \cdot (-2)$$

$$x^2 + y^2 - 2xy = \text{const}$$

$$(x-y)^2 = \text{const}$$

$$y = x \pm \text{const}$$



$$\begin{cases} u_x = x - y \\ u_y = x - y \end{cases}$$

$v = 0$ on line $x = y$

far from line $x = y$
we get higher speeds

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Good job

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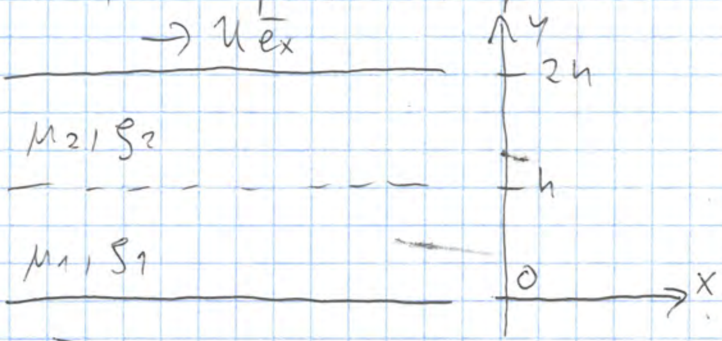

$\bar{v}(x,y) = v_x(y)\,\bar{e}_x$

$$\cancel{\frac{\partial \bar{v}}{\partial t}} + \cancel{(\bar{v}\cdot\nabla)\bar{v}} = \cancel{-\frac{1}{\rho}\nabla p} + \frac{\mu}{\rho}\nabla^2\bar{v} + \cancel{\bar{f}}$$

no time dependence, unidirectional flow ($\bar{v} = v_x(y)\,\bar{e}_x$)

no body forces, no pressure gradient mentioned

So, we are left with $0 = \frac{\mu}{\rho}\nabla^2\bar{v}$

We can multiply by $\frac{\rho}{\mu}$, and we get:

$$\nabla^2\bar{v} = 0 \quad \Rightarrow \quad \frac{\partial^2 v_x}{\partial y^2} = 0$$

$\Rightarrow v_x(y) = Ay + B$ We have 2 fluids:

$v_x^1(y) = Ay + B \qquad v_x^2(y) = Cy + D$

Boundary conditions:

1) $v_x^1(0) = 0$

2) $\bar{v}_x^2(2h) = U\bar{e}_x$

3) $v_x^1(h) = v_x^2(h)$

4) $\mu_1 \frac{\partial v_x^1}{\partial y}\Big|_{y=h} = \mu_2 \frac{\partial v_x^2}{\partial y}\Big|_{y=h}$

1) $\Rightarrow v_x^1 = Ay$; $\underline{B = 0}$

2) $\Rightarrow 2hC + D = U \Rightarrow D = U - 2hC$

3) $\Rightarrow Ah = Ch + U - 2Ch$

4) $\Rightarrow \mu_1 A = \mu_2 C \Rightarrow A = \frac{\mu_2}{\mu_1} C$

3) $\Rightarrow \frac{\mu_2}{\mu_1} Ch = Ch - 2Ch + U \Rightarrow C = \frac{U}{h\left(\frac{\mu_2}{\mu_1} + 1\right)}$

$\Rightarrow A = \frac{U}{h\left(\frac{\mu_1}{\mu_2} + 1\right)}$ So we get:

On the next page

$$\boxed{\begin{cases} v_x^1(y) = \dfrac{U}{h\left(\frac{\mu_1}{\mu_2}+1\right)}\, y \\[2ex] v_x^2(y) = \dfrac{U}{h\left(\frac{\mu_2}{\mu_1}+1\right)}(y-2h) + U \end{cases}}$$

All the boundary conditions are satisfied (I checked)

no dependence on ξ in fact.

Stress tensor:

$$\hat{T}' = 2\mu\hat{D} \qquad \hat{D} = \frac{1}{2}(\nabla\vec{v})^S$$

$$\hat{D} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial v_x}{\partial y} \\ \frac{1}{2}\frac{\partial v_x}{\partial y} & 0 \end{pmatrix}$$

$$\frac{\partial v_x^1}{\partial y} = \frac{U}{h\left(\frac{\mu_1}{\mu_2}+1\right)} \qquad \frac{\partial v_x^2}{\partial y} = \frac{U}{h\left(\frac{\mu_2}{\mu_1}+1\right)}$$

$$t_{xy}^1 = 2\mu_1\cdot\frac{1}{2}\frac{U}{h\left(\frac{\mu_1}{\mu_2}+1\right)} = \frac{U}{h\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)}$$

$$t_{xy}^2 = 2\mu_2\cdot\frac{1}{2}\frac{U}{h\left(\frac{\mu_2}{\mu_1}+1\right)} = \frac{U}{h\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)}$$

In fact, $t_{xy}^1 = t_{xy}^2 = \text{const}$ in whole fluid.

10/10

bravo

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The bubble experiences a force due to displaced amount of fluid (if a would be gravity, this would be buoyancy force).

$$F_1 = V_b \rho_f a$$

Another force comes from the drag the bubble experiences in the ~~unstable flo~~ unsteady flow. The effective mass for a ball is given by $M_{eff} = M + \frac{M'}{2}$ where M' is the mass of displaced fluid. This drag comes from the difference in accelerations and is connected with the addition to effective mass.

$$F_2 = -\frac{M'}{2}(a_b - a) = -\frac{V_b \rho_f}{2}(a_b - a)$$

Both forces contribute to the total force acting on the bubble

$$m_b a_b = F_1 + F_2 = V_b \rho_f a - \frac{V_b \rho_f}{2}(a_b - a) = \frac{3}{2} V_b \rho_f a - \frac{V_b \rho_f}{2} a_b$$

$$\text{"} V_b \rho_b a_b$$

$$a_b\left(\rho_b + \frac{\rho_f}{2}\right) = \frac{3}{2}\rho_f a$$

$\rho_b \ll \rho_f$

$$a_b = \frac{3\rho_f}{\rho_f + 2\rho_b} a = \frac{3}{1 + 2\frac{\rho_b}{\rho_f}} a \approx 3a$$