# Hydrodynamics and Elasticity 2023/2024

## Sheet 1

One of the problems will be collected and marked.

**Problem 1** For three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, the *volume product* $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ has the intepretation of the volume of a parallelpiped spanned by the three vectors. We can write it as a determinant

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We can also write it as $[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})]_i = \epsilon_{ijk}a_ib_jc_k$ Take now two matrices

$$\boldsymbol{M} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{N} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

composed of elements of vectors $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c},\ \boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z}$ and calculate $\det(\boldsymbol{M}\cdot\boldsymbol{N}^T)$. The relationship holds for any vectors, so choose now the following basis vectors

$$\boldsymbol{a} = \boldsymbol{e}_i,\ \boldsymbol{b} = \boldsymbol{e}_j,\ \boldsymbol{c} = \boldsymbol{e}_k,\ \boldsymbol{x} = \boldsymbol{e}_l,\ \boldsymbol{y} = \boldsymbol{e}_m,\ \boldsymbol{z} = \boldsymbol{e}_n,$$

and the index representation to prove the following formula

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{km}\delta_{jn},$$

and show that it follows from the formula above that

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

and that $\epsilon_{ijk}\epsilon_{ijk} = 6$.

**Problem 2** Prove the following identites for a scalar field $\phi$, vector fields $\boldsymbol{a}$, $\boldsymbol{v}$, $\boldsymbol{u}$ and tensor field $\boldsymbol{T}$
(a) $\nabla\cdot(\phi\boldsymbol{a}) = \phi\nabla\cdot\boldsymbol{a} + \boldsymbol{a}\cdot\nabla\phi$,
(b) $\nabla\times(\nabla\times\boldsymbol{a}) = \nabla(\nabla\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}$,
(c) $\nabla\cdot(\boldsymbol{v}\times\boldsymbol{u}) = \boldsymbol{u}\cdot(\nabla\times\boldsymbol{v}) - \boldsymbol{v}\cdot(\nabla\times\boldsymbol{u})$.
(d) $\mathrm{Div}\,(\phi\boldsymbol{T}) = \boldsymbol{T}\cdot(\nabla\phi) + \phi\,\mathrm{Div}\,\boldsymbol{T}$, where the divergence is defined as $(\mathrm{Div}\,\boldsymbol{T})_i = \frac{\partial T_{ij}}{\partial x_j}$.

**Problem 3** For a given tensor $\boldsymbol{T} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.
(a) Find the symmetric part $\boldsymbol{T}^S$ and the antisymmetric part $\boldsymbol{T}^A$ of $\boldsymbol{T}$.
(b) Find the dual (axial) vector of the antisymmetric part.
(c) Show that for any vector $\boldsymbol{v}$ and any tensor $\boldsymbol{R}$, we have

$$\boldsymbol{v}\cdot\boldsymbol{R}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{R}^S\cdot\boldsymbol{v} \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{R}^A\cdot\boldsymbol{v} = 0$$

**(*) Problem 4** Show that for a volume $V$ enclosed by a surface $S$ with a normal vector $\boldsymbol{n}$, the following identity holds

$$\int_V [(\nabla\times\boldsymbol{a})\times\boldsymbol{b} + (\nabla\times\boldsymbol{b})\times\boldsymbol{a} + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) + \boldsymbol{b}(\nabla\cdot\boldsymbol{a}))]\,\mathrm{d}V = \int_S [\boldsymbol{n}\cdot(\boldsymbol{ab}+\boldsymbol{ba}) - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n}]\,\mathrm{d}S$$

*Maciej Lisicki & Piotr Szymczak*