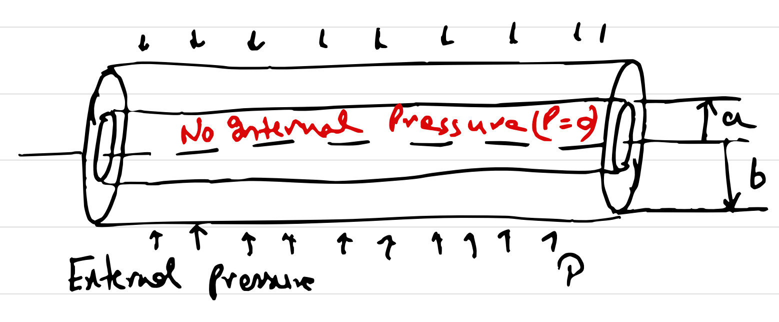
Problem - 2 Sheet - 4



Using Cauchy - Navier Equation

$$\mu \nabla^2 \underline{u} + (\lambda+\mu)\nabla(\nabla\cdot\underline{u}) = 0 \quad \text{— (1)}$$

where $\underline{u}$ is displacement vector
& λ, μ are Levi-Cevita Parameters

Now using identity

$$\nabla(\nabla\cdot\underline{u}) = \nabla\times(\nabla\times\underline{u}) + \nabla^2\underline{u}$$

in which the term $\nabla\times(\nabla\times\underline{u}) = 0$
due to no-rotational displacement

so $$\nabla(\nabla\cdot\underline{u}) = \nabla^2\underline{u} \quad \text{— (2)}$$

Using (1) & (2)

$$\mu\nabla^2 u + (\lambda+\mu)\nabla^2\underline{u} = 0$$

$$\Rightarrow (\lambda+2\mu)\nabla^2\underline{u} = 0 \Rightarrow \boxed{\nabla^2\underline{u} = 0} \quad \text{— (3)}$$

* We will use cylindrical coordinate system
* Due to symmetry of problem we can assume
$\underline{u}(r) = u_r(r)\hat{e}_r$

Therefore the problem reduced to following form of Laplacian operator Eq (3)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} = 0$$

$$\Rightarrow \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r = 0$$

Here we can use an anszat

$$\underline{u}(\underline{r}) = \left(Ar + \frac{B}{r}\right)\hat{e}_r$$

The strain tensor

$$\hat{\mathcal{E}} = (\nabla\underline{u})^s = \begin{bmatrix} A - \frac{B}{r^2} & 0 & 0 \\ 0 & A + \frac{B}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$tr\,\hat{\mathcal{E}} = 2A$$

we can now write stress tensor

$$T = \frac{E}{1+\nu}\left[\hat{\mathcal{E}} + \frac{\nu}{1-2\nu}tr\,\hat{\mathcal{E}}\,\mathbb{1}\right]$$

$$T = \frac{E}{1+\nu}\begin{bmatrix} \frac{A}{1-2\nu} - \frac{B}{r^2} & 0 & 0 \\ 0 & \frac{A}{1-2\nu} + \frac{B}{r^2} & 0 \\ 0 & 0 & \frac{2\nu}{1-2\nu}A \end{bmatrix}$$

Now we will use Boundary Condition to calculate the integration coefficient (A & B)

B.C-(1) $$T\bar{n}\Big|_{r=a} = 0 \qquad (\bar{n} = -\hat{e}_r)$$

BC-(2) $$T.\bar{n}\Big|_{r=b} = -P\hat{e}_r \qquad (\bar{n} = +\hat{e}_r)$$

So applying BC-(1)

$$T\bar{n}\Big|_{r=a} = \frac{E}{1+\nu}\begin{bmatrix} \frac{A}{1-2\nu} - \frac{B}{a^2} & 0 & 0 \\ 0 & \frac{A}{1-2\nu} + \frac{B}{a^2} & 0 \\ 0 & 0 & \frac{2\nu}{1-2\nu}A \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{-\hat{e}_r}$$

$$\Rightarrow \frac{A}{1-2\nu} - \frac{B}{r^2} = 0$$

$$\Rightarrow \boxed{B = \frac{a^2}{1-2\nu}A} \quad \text{— (4)}$$

now applying second B.C (2)

$$T\bar{n}\Big|_{r=b} = \frac{E}{1+\nu}\begin{bmatrix} \frac{A}{1-2\nu} - \frac{B}{b^2} & 0 & 0 \\ 0 & \frac{A}{1-2\nu} + \frac{B}{b^2} & 0 \\ 0 & 0 & \frac{2\nu}{1-2\nu}A \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{+\hat{e}_r}$$

$$= -P\hat{e}_r$$

$$\Rightarrow \frac{E}{1+\nu}\left(\frac{A}{1-2\nu} - \frac{B}{b^2}\right) = -P$$

$$\Rightarrow \frac{A}{1-2\nu} - \frac{B}{b^2} = \frac{-P(1+\nu)}{E}$$

using eqn (4)

$$\frac{A}{1-2\nu} - \frac{a^2 A}{(1-2\nu)b^2} = \frac{-P(1+\nu)}{E}$$

$$\Rightarrow A = \frac{P(1+\nu)(1-2\nu)}{E(a^2/b^2-1)}$$

$$B = \frac{a^2}{\cancel{1-2\nu}} \times \frac{P(1+\nu)\cancel{(1-2\nu)}}{E(a^2/b^2-1)}$$

$$B = \frac{P(1+\nu)}{E(a^2/b^2-1)}a^2$$

So our solution

$$\underline{u}(\underline{r}) = \left(\frac{P(1+\nu)(1-2\nu)}{E\left(\frac{a^2}{b^2}-1\right)}r + \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\frac{a^2}{r}\right)\hat{e}_r$$

$$= \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[(1-2\nu)r + \frac{a^2}{r}\right]\hat{e}_r$$

Now we can write strain tensor again

$$\hat{\mathcal{E}} = \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\begin{bmatrix} (1-2\nu) - \frac{a^2}{r^2} & 0 & 0 \\ 0 & (1-2\nu) + \frac{a^2}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the pipe will thicken if

$$\hat{e}_r\left(\underline{u}(b\hat{e}_r) - \underline{u}(a\hat{e}_r)\right) > 0$$

LHS

$$\Rightarrow \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[(1-2\nu)b + \frac{a^2}{b} - (1-2\nu)a - a\right]\hat{e}_r$$

$$\Rightarrow \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[(1-2\nu)(b-a) + a\left(\frac{a}{b}-1\right)\right]\hat{e}_r$$

$$\Rightarrow \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[(1-2\nu)b\left(1-\frac{a}{b}\right) - a\left(1-\frac{a}{b}\right)\right]\hat{e}_r$$

$$= \frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[\left(1-\frac{a}{b}\right)\left((1-2\nu)b - a\right)\right]\hat{e}_r$$

$$= \underbrace{\frac{P(1+\nu)}{E\left(\frac{a^2}{b^2}-1\right)}\left[\left(\frac{a}{b}-1\right)\right.}_{+ve}\left.\left(a - (1-2\nu)b\right)\right]\hat{e}_r$$

therefore for LHS > 0

$$a - (1-2\nu)b > 0$$

$$\Rightarrow a > (1-2\nu)b$$

or $$\boxed{\frac{a}{b} > 1-2\nu}$$ for thickening of pipe