The radial pressure $p_r = -\sigma_{rr}$ can never become larger than P because we may write

$$\frac{p_r}{P} = \frac{b^2 - r^2}{b^2 - a^2}\frac{a^2}{r^2}, \tag{9.78}$$

which is the product of two factors, both smaller than unity for $a < r < b$. The tangential pressure $p_\phi = -\sigma_{\phi\phi}$ and the longitudinal pressure $p_z = -\sigma_{zz}$ are both negative (tensions), and can become large for thin-walled pipes. The average pressure

$$p = \frac{1}{3}\left(p_r + p_\phi + p_z\right) = -\frac{2}{3}(1+\nu)\frac{a^2}{b^2 - a^2}P \tag{9.79}$$

is also negative and like the longitudinal pressure is constant throughout the material. The average pressure does not vanish at $r = b$, and this confirms the suspicion voiced on page 103 that the pressure behaves differently in a solid with shear stresses than the pressure in a fluid at rest, where it has to be continuous across boundaries in the absence of surface tension.

Blowup: A pipe under pressure blows up if the material is extended beyond a certain limit. Compression does not matter, except for very large pressures. The point where the pipe breaks is primarily determined by the point of maximal local tension. As we have seen, this occurs at the inside of the pipe for $r = a$, where

$$\sigma_{\phi\phi} = \frac{b^2 + a^2}{b^2 - a^2}P. \tag{9.80}$$

When this tension exceeds the tensile strength in a brittle material, a crack will develop where the material has a small weakness, and the pipe blows up from the inside!

Example 9.9 [One-inch water pipe]: A standard American 1-inch iron water pipe has $2a = 0.957$ in. and $2b = 1.315$ in. Taking the yield strength of iron to be 200 MPa = 2,000 bar, the blowup pressure becomes $P = 615$ bar. Even with a safety factor 10, normal water pressures can never blow up such a pipe, as long as corrosion has not thinned the wall too much.

Example 9.10 [Frost bursting]: Broken water pipes in winter are a common phenomenon. The reason is that water expands by about 9% when freezing at 0°C. After freezing it contracts slowly if the temperature continues to drop. The bulk modulus of solid ice is $K = 8.8$ GPa = 88,000 bar. If the water is prevented from expanding along the pipe, for example by being blocked by already frozen regions, it will in principle be able to develop a radial pressure of about 9% of the bulk modulus, or 8,000 bar, which is four times larger than the yield strength of iron. No wonder that pipes burst! The calculation is, however, only an estimate, because other phases of ice exist at high pressures.

Unclamped pipe

In older houses where central heating pipes have been clamped too tight by wall fixtures, major noise problems can arise because no normal fixtures can withstand the large pressures that arise when the water temperature changes and the pipes expand and contract longitudinally. In practice, pipes should always be thought of as being unclamped.

The constancy of the longitudinal tension (9.77c) permits us to solve the case of an unclamped pipe by superposing the above solution with the displacement field for uniform stretching (8.23) on page 133. In the cylindrical basis, the field of uniform stretching becomes (after interchanging x and z)

$$u_r = -\nu r\frac{Q}{E}, \qquad u_z = z\frac{Q}{E}, \tag{9.81}$$

where Q is the tension applied to the ends.

Choosing Q equal to the longitudinal tension (9.77c) in the clamped pipe,

$$Q = 2\nu \frac{a^2}{b^2 - a^2} P, \tag{9.82}$$

and subtracting the stretching field from the clamped pipe field (9.75), we find for the unclamped pipe that

$$u_r = \frac{a^2}{b^2 - a^2} \left((1-\nu)r + (1+\nu)\frac{b^2}{r} \right) \frac{P}{E}, \tag{9.83a}$$

$$u_z = -2\nu \frac{a^2}{b^2 - a^2} z \frac{P}{E}. \tag{9.83b}$$

The strains are likewise obtained from the clamped strains (9.76a) by subtracting the strains for uniform stretching, and we get

$$u_{rr} = \frac{a^2}{b^2 - a^2} \left(1 - \nu - (1+\nu)\frac{b^2}{r^2} \right) \frac{P}{E}, \tag{9.84a}$$

$$u_{\phi\phi} = \frac{a^2}{b^2 - a^2} \left(1 - \nu + (1+\nu)\frac{b^2}{r^2} \right) \frac{P}{E}, \tag{9.84b}$$

$$u_{zz} = -2\nu \frac{a^2}{b^2 - a^2} \frac{P}{E}. \tag{9.84c}$$

The superposition principle guarantees that the radial and tangential stresses are the same as before and given by (9.77), while the longitudinal stress now vanishes, $\sigma_{zz} = 0$.

Thin wall approximation

Most pipes have thin walls relative to their radius. Let us introduce the wall thickness, $d = b - a \ll a$, and put the radial distance, $r = a + sd$ with $0 \leq s \leq 1$. In the leading approximation we get the displacement field for the unclamped pipe:

$$u_r \approx a \frac{a}{d} \frac{P}{E}, \qquad u_z \approx -z\nu \frac{a}{d} \frac{P}{E}. \tag{9.85}$$

The corresponding strains become

$$u_{rr} \approx -\nu \frac{a}{d} \frac{P}{E}, \qquad u_{\phi\phi} \approx \frac{a}{d} \frac{P}{E}, \qquad u_{zz} \approx -\nu \frac{a}{d} \frac{P}{E}. \tag{9.86a}$$

The strains all diverge for $d \to 0$, and the condition for small strains is now $P/E \ll d/a$. Finally, we get the non-vanishing stresses

$$\sigma_{rr} \approx -(1-s)P, \qquad \sigma_{\phi\phi} \approx \frac{a}{d} P. \tag{9.87a}$$

The radial pressure $p_r = -\sigma_{rr}$ varies between 0 and P as it should when s ranges from 0 to d. It is always positive and of order P, whereas the tangential tension $\sigma_{\phi\phi}$ diverges for $d \to 0$. Blowups always happen because the tangential tension becomes too large.

Problems

9.1 Show that Navier's equation of equilibrium (9.2) may be written as

$$\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu} \nabla \nabla \cdot \boldsymbol{u} = -\frac{1}{\mu} \boldsymbol{f},$$

where ν is Poisson's ratio.