Hydrodynamics and Elasticity 2023/2024

Sheet 7

One of the problems will be handed in and marked.

Problem 1 Consider a model of a universe, which is a generalization of the model analyzed in class, in which, except for an ordinary matter, there is also dark energy of density ρ_0. We assume that this density is constant in time and homogeneous in space. In this case the formula for energy, derived in class will take the form

$$E = \frac{1}{2}\dot{a}^2 - G\frac{M + M_0}{a} = \frac{4\pi}{3}Ga^2(\rho_c - (\rho + \rho_0)),$$

where $M_0 = (4/3)\pi\rho_0 a^3$ and ρ is the density of an ordinary matter (which obeys the continuity equation). Finally, ρ_c is a critical density, as defined in class.

(a) Find the equation describing the time evolution of the energy, $\dot{E}$, remembering that ρ_c is not constant. Next show that the condition of constancy of energy leads to the following dynamic equation

$$\dot{H} + H^2 = -\frac{4\pi}{3}G(\rho - 2\rho_0)$$

(b) Can such a universe be stationary (with time-independent scale factor)? Under which conditions this can happen? How would you interpret the situation in which $2\rho_0 > \rho$?

(c) Next, consider a situation in which $\rho_0 < \rho_c/3$. Show that if, at a certain moment of time, $2\rho_0 < \rho < \rho_c$, then initially the expansion of the universe will decelerate with time ($\ddot{a} < 0$), but a the certain moment the expansion speed will begin to increase. Recent observations seem to indicate that the cosmic expansion is in fact accelerating and that it may have been decelerating in the past.

Problem 2 A spherical cavity with radius a is suddenly formed in an incompressible fluid of density ρ filling all space. Determine the time taken for the cavity to be filled with fluid. Assume that far from the cavity the pressure is equal to p_0. (Rayleigh, 1917)

Problem 3 Waste water flows into a tank at $Q = 10^{-4}$ m^{3}s^{-1} and out of a short exit pipe of cross-section $A = 4 \times 10^{-5}$ m^2 into the air. In steady state, estimate how high above the pipe is the water in the tank?

(*) Problem 4 Consider a universe in which matter is created everywhere at a constant rate, J, per unit of volume and time. Show that this allows for a steady-state cosmological solution with constant mass density and Hubble constant, even though the distance between any given two galaxies is changing. Determine the rate of mass creation needed to realize such a scenario.

This is so-called Bondi-Gold-Hoyle model of a steady-state universe. This model was inspired at least in part by the 1945 movie entitled Dead of Night. The movie had four parts and a circular structure such that at the end the movie was the same as at the beginning. After seeing this movie in 1946, Thomas Gold and Hermann Bondi wondered if the universe might not be constructed the same way.

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