Hydrodynamics and Elasticity 2023/2024

Sheet 8

One of the problems will be handed in and marked.

Problem 1 An ideal fluid is rotating in a gravitational field with a constant angular velocity $\boldsymbol{\Omega}$, so that the fluid velocity in the lab frame is $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$. Let us find the surfaces of constant pressure, which for a particular choice of $p = p_{\text{atm}}$ will give us the shape of the free surface. According to Bernoulli's law the quantity $p/\rho + \frac{1}{2}u^2 + gz$ is constant, so the surface of constant pressure satisfies the equation

$$z = \text{const} - \frac{\Omega^2}{2g}(x^2 + y^2).$$

But this means that the water level is the highest in the middle of a spinning bucket. What is wrong here? What is the true equation for the constant pressure surface and why?

Problem 2 An infinite cylinder of radius a is inserted into an asymptotically uniform flow with velocity U along the x-axis. Use the velocity potential method to find the resulting flow and the drag force on the cylinder.

Problem 3 Consider the purely two-dimensional steady flow of an inviscid incompressible constant density fluid in the absence of body forces. For velocity $\boldsymbol{u}$, the vorticity is $\boldsymbol{\omega} = (0, 0, \omega)$. Let p denote the pressure and ρ the density of the fluid. We define the streamfunction $\boldsymbol{\Psi}(x, y) = \psi(x, y)\boldsymbol{e}_z$ such that $\boldsymbol{u}(x, y) = \nabla \times \boldsymbol{\Psi}$. Show that if ω is a constant both in space and time, then

$$\frac{|\boldsymbol{u}|^2}{2} + \omega\psi + \frac{p}{\rho} = C,$$

where C is a constant.

Now consider a fluid with constant (in both space and time) vorticity ω in the cylindrical annular region $a < r < 2a$. The streamlines are concentric circles, with the fluid speed zero on $r = 2a$ and $V > 0$ on $r = a$. Calculate the velocity field, and hence show that

$$\omega = -\frac{2V}{3a}.$$

Deduce that the pressure difference between the outer and inner edges of the annular region is

$$\Delta p = \left(\frac{15 - 16\ln 2}{18}\right)\rho V^2$$

(*) Problem 4: Blasius theorem Prove that if $w(z)$ is the complex potential describing a steady 2D flow of an ideal, incompressible fluid around a body with a contour C, then the force components acting on the body are given by

$$F_x - iF_y = \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z.$$

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